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Ultraproducts of real interpolation spaces between L^p-spaces

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Abstract. Let $\{(L^{p_0}(\Omega_d, \mu_d), L^{p_1}(\Omega_d, \mu_d)),\ d \in \mathfrak{D}\}$, $1 \leq p_0 < p_1 < \infty$, be a family of compatible couples of L^p-spaces. We show that, given a countably incomplete ultrafilter $\mathcal{U}$ in $\mathfrak{D}$, the ultraproduct $((L^{p_0}(\Omega_d, \mu_d), L^{p_1}(\Omega_d, \mu_d))_{\theta,q})_{\mathcal{U}}$, $0 < \theta < 1$, $1 \leq q < \infty$ of interpolation spaces defined by the real method is isomorphic to the direct sum of an interpolation space of type $(L^{p_0}(\Omega_1, \nu_1), L^{p_1}(\Omega_1, \nu_1))_{\theta,q}$, an intermediate Köthe space between $\ell^{p_0}(\Omega_2, \nu_2)$ and $\ell^{p_1}(\Omega_2, \nu_2)$, (Ω_2, ν_2) being a purely atomic measure space, and a Köthe function space $K(\Omega_3)$ defined on some purely non atomic measure space (Ω_3, ν_3) in such a way that $\Omega_2 \cup \Omega_3 \neq \emptyset$.

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1 Introduction

Ultraproducts were introduced in Banach space theory by Dacunha-Castelle Krivine in [4] and are very suitable for the study of local theory of Banach spaces and ideals of operators acting between them (see [5] and [18]). An important problem in ultraproducts theory is the study of permanence properties of factor spaces in ultraproducts. The more easy result of this type is the known fact that an ultraproduct of L^p spaces is again an L^p space, $1 \leq p < \infty$. The book [8] is almost entirely devoted to study the structure of ultraproducts of Lebesgue-Bochner spaces $L^p[X]$. The long paper [20] deals with this question when the factor spaces are interpolation spaces of Banach lattices defined by the Calderón-Lozanowski method. In [19] it is shown that an ultraproduct of Köthe function spaces with a non trivial concavity is again a Köthe function space,

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although the resulting measure space has a very abstract character and has no easy connection with the starting measure spaces.

In order to obtain more concrete descriptions, we have studied in [17] the structure of ultraproducts of interpolation spaces of type $(L^{p_0}(\Omega), L^{p_1}(\Omega))_{\theta,q}$, $0 < \theta < 1,\ 1 \leq q < \infty$. Unfortunately our results are based on some assumptions on factor spaces and moreover they do not give a complete description of the full resulting ultraproduct. *In this paper, and using different methods, we improve the results of* [*17*] *by removing all assumptions and getting a more accurate description of the full ultraproduct.*

Section 2 of the paper introduces the basic specific notation in order to implement our methods and some preparatory technical results. Section 3 contains the main theorems of the paper. General notation is standard. To specify the space where a norm $\|.\|$ is computed, we write $\|.\|_E$. If $(\Omega, \mathcal{M}, \mu)$ is a measure space and f is a $\mathcal{M}$-measurable scalar function, the support of f is defined, except by a set of zero μ-measure as $Supp(f) := \{t \in \Omega \mid f(t) \neq 0\}$. $L^0(\Omega, \mathcal{M}, \mu)$ will denote the set of classes of $\mathcal{M}$-measurable real functions defined on Ω modulo equality μ-almost everywhere. Symbols $\mathcal{M}$ or μ can be omitted if there is no risk of confusion. The restriction to a measurable set A of the measure μ will be denoted by μ also.

We use in an essential way results of Banach lattice theory and hence *we shall deal only with real Banach spaces.* A general reference for this material is [1]. We set some notation. Given a Banach lattice E and $x \in E$, the band generated by x will be denoted by B_x. If E is order complete and B is a band in E, P_B will be the band projection associated to B which is given by the formula (see [1])

$$\forall\, x \in E,\ x \geq 0, \quad P_B(x) = \sup\{y \in B \ / \ 0 \leq y \leq x\}.$$

In particular, in the case of the band B_x generated by a single element x (a principal band), to simplify we will write P_x instead of P_{B_x} and the equality

$$\forall\, z \in E,\ z \geq 0,\ \forall\, x \in E\ x \geq 0 \quad P_z(x) = \sup_{n \in \mathbb{N}} x \wedge n\, z. \tag{1}$$

holds.

Given a lattice E and $x \in E, x > 0$, every element $y \in E$ such that $y > 0$ and $y \wedge (x - y) = 0$ is said to be a *component* of x. If E is order complete, the set of all components of x is Boole algebra (see theorem 3.15 in [1]) which will be denoted by $C(x)$. We recall that an *atom* in E is an element $x > 0$ such that $y \wedge z = 0$ and $0 \leq y \leq x, 0 \leq z \leq x$ imply either $y = 0$ or $z = 0$.

A Banach lattice E of measurable real functions defined on the measure space $(\Omega, \mathcal{M}, \mu)$ which is also a solid set in $L^0(\Omega, \mathcal{M}, \mu)$ (i.e $|f| \leq |g|$ and $g \in E$ implies $f \in E$) is called a Köthe function space (defined) on $(\Omega, \mathcal{M}, \mu)$.

Concerning interpolation spaces, we refer the reader to [2] for basic facts. Given a compatible couple (A_0, A_1) of Banach spaces and numbers $0 < \theta < 1$ and $1 \leq q < \infty$, in the interpolation space $(A_0, A_1)_{\theta,q}$ we shall consider two equivalent norms. The first one is

$$\|x\| := \inf \max_{j=0,1} \left(\sum_{n \in \mathbb{Z}} e^{(j-\theta)nq} \|x_n^j\|_{A_j}^q \right)^{\frac{1}{q}},$$

taking the infimum over all sequences $\{x_n^j\}_{n\in\mathbb{Z}}$, $j = 0, 1$ such that $x = x_n^0 + x_n^1$, $x_n^0 \in A_0$ and $x_n^1 \in A_1$ for every $n \in \mathbb{Z}$. The second one is

$$|||x||| := \inf \max_{j=0,1} \left(\sum_{n \in \mathbb{Z}} e^{(j-\theta)nq} \|x_n\|_{A_j}^q \right)^{\frac{1}{q}},$$

taking the infimum over all sequences $\{x_n\}_{n\in\mathbb{Z}}$ such that $x_n \in A_0 \cap A_1$ for every $n \in \mathbb{Z}$ and $x = \sum_{n\in\mathbb{Z}} x_n$ in the space $A_0 + A_1$.

With respect to ultraproducts, our main references are [4] and [9]. We give a brief account of the more relevant definitions and general facts. Given a non void index set $\mathfrak{D}$ and a Banach space E_d for every $d \in \mathfrak{D}$, we denote

$$\ell^\infty((E_d)) := \left\{ (x_d) \in \Pi_{d\in\mathfrak{D}} E_d \mid \sup_{d\in\mathfrak{D}} \|x_d\| < \infty \right\}.$$

Given an ultrafilter $\mathcal{U}$ on $\mathfrak{D}$, we put $Z_{\mathcal{U}} := \{(x_d) \in \ell^\infty((E_d)) \mid \lim_{d,\mathcal{U}} \|x_d\| = 0\}$. Given $(x_d) \in \ell^\infty((E_d))$, its class in the quotient set $\ell^\infty((E_d))/Z_{\mathcal{U}}$ is denoted by $(x_d)_{\mathcal{U}}$. Then the ultraproduct space $(E_d)_{\mathcal{U}}$ is the quotient Banach space $\ell^\infty((E_d))/Z_{\mathcal{U}}$ whose canonical quotient norm equals the norm $\|(x_d)_{\mathcal{U}}\| := \lim_{\mathcal{U}} \|x_d\|$. To avoid trivialities *all the used ultrafilters $\mathcal{U}$ are assumed to be countably incomplete*, i. e. there is a sequence $\{U_n\}_{n=1}^\infty \subset \mathcal{U}$ such that $\cap_{n=1}^\infty U_n = \emptyset$. Given a family $\{T_d : E_d \longrightarrow F_d \mid d \in \mathfrak{D}\}$ of continuous linear maps between Banach spaces E_d and F_d such that $\sup_{d\in\mathfrak{D}} \|T_d\| < \infty$ we can define the canonical ultraproduct map $(T_d)_{\mathcal{U}} : (E_d)_{\mathcal{U}} \longrightarrow (F_d)_{\mathcal{U}}$ by

$$\forall\ (x_d)_{\mathcal{U}} \in (E_d)_{\mathcal{U}} \quad (T_d)_{\mathcal{U}}((x_d)_{\mathcal{U}}) = (T_d(x_d))_{\mathcal{U}}. \tag{2}$$

More generally, suppose we have a family $\{T_d : E_d \longrightarrow F_d \mid d \in \mathfrak{D}\}$ of (non necessarily linear) continuous maps which are *locally $\mathcal{U}$-globally uniformly continuous*, i.e. for every $R > 0$ and $\varepsilon > 0$ there is $\delta > 0$ and $D \in \mathcal{U}$ such that

$$\sup\{\|T_d(x_d) - T_d(y_d)\| \mid \|x_d - y_d\| \leq \delta, \|x_d\| < R,\ \|y_d\| < R,\ d \in D\} \leq \varepsilon.$$

Then definition (2) is meaningful still and produces a well defined (non linear) continuous map $(T_d)_{\mathcal{U}} : (E_d)_{\mathcal{U}} \longrightarrow (F_d)_{\mathcal{U}}$ as it is easily seen. We shall use this type of mappings in Section 2.

If every $E_d, d \in \mathfrak{D}$ is a Banach lattice, $(E_d)_{\mathcal{U}}$ also is a Banach lattice with the canonical order given by $(x_d)_{\mathcal{U}} \leq (y_d)_{\mathcal{U}}$ if and only if there is $(\overline{x}_d) \in (x_i)_{\mathcal{U}}$ and $(\overline{y}_d) \in (y_i)_{\mathcal{U}}$ such that $\overline{x}_d \leq \overline{y}_d$ for every $d \in \mathfrak{D}$. Then $(x_d)_{\mathcal{U}} \wedge (y_d)_{\mathcal{U}} = (x_d \wedge y_d)_{\mathcal{U}}$.

Remark 1. If $(x_d)_{\mathcal{U}} \wedge (y_d)_{\mathcal{U}} = 0$, we can choose representants $(\overline{x}_d) \in (x_d)_{\mathcal{U}}$ and $(\overline{y}_d) \in (y_d)_{\mathcal{U}}$ such that $\overline{x}_d \wedge \overline{y}_d = 0$ for all $d \in \mathfrak{D}$.

Proof. Our assumption implies $\lim_{\mathcal{U}} \|x_d \wedge y_d\| = 0$ and hence

$$(x_d - (x_d \wedge y_d))_{\mathcal{U}} = (x_d)_{\mathcal{U}},\ (y_d - (x_d \wedge y_d))_{\mathcal{U}} = (y_d)_{\mathcal{U}}\,,$$

and $(x_d - (x_d \wedge y_d)) \wedge (y_d - (x_d \wedge y_d)) = 0$ for every $d \in \mathfrak{D}$ (see [7], chapter 1, theorem 1). □

Remark 2. Let $x = (x_d)_{\mathcal{U}} \geq 0$ in $\in \mathcal{U}$. Let $y = (y_d)_{\mathcal{U}}$ be a component of x. There are representants $(\overline{x}_d) \in (x_d)_{\mathcal{U}}$ and $(\overline{y}_d) \in (y_d)_{\mathcal{U}}$ such that $\overline{y}_d$ is a component of $\overline{x}_d$ for every $d \in \mathfrak{D}$.

Proof. Since $(x - y) \wedge y = 0$ we can suppose $y_d \geq 0$ and $x_d \geq y_d$ for every $d \in \mathfrak{D}$. By remark 1, there are representations $(z_d) \in (x_d - y_d)_{\mathcal{U}}$ and $(\overline{y}_d) \in (y_d)_{\mathcal{U}}$ such that $z_d \wedge \overline{y}_d = 0$ for every $d \in \mathfrak{D}$. Then

$$\lim_{d,\mathcal{U}} \|(z_d - x_d + y_d)\| = 0$$

Put $\overline{x}_d := z_d + \overline{y}_d$ for every $d \in \mathfrak{D}$. Then

$$\lim_{d,\mathcal{U}} \|(\overline{x}_d - x_d)\| = \lim_{d,\mathcal{U}} \|(z_d + \overline{y}_d - x_d)\| = \lim_{d,\mathcal{U}} \|(z_d - x_d + (\overline{y}_d - y_d) + y_d)\| = 0$$

and hence $(\overline{x}_d)_{\mathcal{U}} = (x_d)_{\mathcal{U}}$. Moreover $(\overline{x}_d - \overline{y}_d) \wedge \overline{y}_d = z_d \wedge \overline{y}_d = 0$ for all $d \in \mathfrak{D}$. Hence every $\overline{y}_d$ is a component of $\overline{x}_d,\ d \in \mathfrak{D}$ and the proof is complete. □

We fix now some notation concerning the specific ultraproducts we shall use in this paper. Given fixed real numbers $1 \leq p_0, p_1, q < \infty$ and $0 < \theta < 1$, for every $d \in \mathfrak{D}$ we consider the spaces $L^{p_j}(\Omega_d, \mathcal{M}_d, \mu_d)$, $(L^{p_j}(\Omega_d, \mu_d)$ or

$L^{p_j}(\Omega_d)$ for short if there is no risk of confusion), $j = 0, 1$ and the interpolation space $\lambda_d := (L^{p_0}(\Omega_d, \mu_d), L^{p_1}(\Omega_d, \mu_d))_{\theta,q}$. By a known result of Krée (see [11]), every space $\lambda_d, d \in \mathfrak{D}$ is isomorphic to a Lorentz space $L^{\overline{p},q}(\Omega_d, \mu_d)$ (where $1/\overline{p} = (1-\theta)/p_0 + \theta/p_1$) with inequality constants independent on Ω_d. Hence, if $q < p_0 < p_1$, defining r by the equality $1/\overline{p} = (1-\theta)/q + \theta/r$ we obtain that λ_d is isomorphic to $(L^q(\Omega_d), L^r(\Omega_d))_{\theta,q}$. A corresponding result can be obtained if $p_0 < p_1 < q$. As a consequence, since in this paper we shall be concerned with ultraproducts of spaces λ_d, $d \in \mathfrak{D}$, *we shall suppose always* $p_0 \le q \le p_1$ *and* $p_0 < p_1$. For every $d \in \mathfrak{D}$ we have the canonical inclusion maps

$$I^d_{\cap\lambda} \colon L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d) \longrightarrow (L^{p_0}(\Omega_d), L^{p_1}(\Omega_d))_{\theta,q},$$

$$I^d_{\lambda\cup} \colon (L^{p_0}(\Omega_d), L^{p_1}(\Omega_d))_{\theta,q} \longrightarrow L^{p_0}(\Omega_d) + L^{p_1}(\Omega_d),$$

$$I^d_{\cap p_i} \colon L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d) \longrightarrow L^{p_i}(\Omega_d), \quad i = 0, 1$$

and

$$I_{p_i\cup} \longrightarrow L^{p_0}(\Omega_d) + L^{p_1}(\Omega_d), \quad i = 0, 1.$$

All these maps have norm not greater than 1 and, moreover,

$$I^d_{p_0\cup} I^d_{\cap p_0} = I^d_{p_1\cup} I^d_{\cap p_1} = I^d_{\lambda\cup} I^d_{\cap\lambda}.$$

We form now the ultraproducts

$$\mathcal{U}_{p_j} := (L^{p_0}(\Omega_d))_{\mathcal{U}}, \quad j = 0, 1,$$

$$\mathcal{U}_\Lambda := ((L^{p_0}(\Omega_d), L^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}},$$

$$\mathcal{U}_I := (L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d))_{\mathcal{U}}$$

and

$$\mathcal{U}_S := (L^{p_0}(\Omega_d) + L^{p_1}(\Omega_d))_{\mathcal{U}}$$

and the canonical ultraproduct maps

$$I_{\cap\Lambda} := (I^d_{\cap\lambda})_{\mathcal{U}} : \mathcal{U}_I \longrightarrow \mathcal{U}_\Lambda,$$

$$I_{\Lambda\cup} := (I^d_{\lambda\cup})_{\mathcal{U}} : \mathcal{U}_\Lambda \longrightarrow \mathcal{U}_S,$$

$$I_{\cap p_j} := (I^d_{\cap p_j})_{\mathcal{U}} : \mathcal{U}_I \longrightarrow \mathcal{U}_{p_j}, \quad j = 0, 1$$

and

$$I_{p_j\cup} := (I^d_{p_j\cup})_{\mathcal{U}} : \mathcal{U}_{p_j} \longrightarrow \mathcal{U}_S, \quad j = 0, 1$$

are well defined.

It is known that $\mathcal{U}_{p_j}$, $j = 0, 1$ are order continuous (and hence order complete) Banach lattices (they are actually abstract L^{p_j}-spaces and hence order isometric to suitable L^{p_j} spaces). We also have

Lemma 3. *$\mathcal{U}_\Lambda$ and $\mathcal{U}_\mathcal{I}$ are order continuous Banach lattices.*

Proof. By the previously quoted result of Krée (see [11]), every space λ_d, $d \in \mathfrak{D}$, is isomorphic to the Lorentz space $L^{\overline{p},q}(\Omega_d, \mu_d)$ (where $\overline{p}$ is defined as above) with inequality constants independent on Ω_d. Then λ_d has a $\max\{\overline{p}, q\}$-lower estimate. Hence $\mathcal{U}_\Lambda$ also has a $\max\{\overline{p}, q\}$-lower estimate as a consequence of the definition of its norm. By a result of Maurey (see [13]), $\mathcal{U}_\Lambda$ is h-concave for every $h > \max\{\overline{p}, q\}$. Then $\mathcal{U}_\Lambda$ cannot contain c_0 as sublattice and hence its norm is order continuous (see theorems 2.4.12 and 2.4.2 in [14]). The proof for $\mathcal{U}_\mathcal{I}$ is similar. □

2 Preliminary technical results

Lemma 4. *The map $H_{p_0 p_1} : \mathcal{U}_{p_0} \longrightarrow \mathcal{U}_{p_1}$ defined by*

$$\forall\ f = (f_d)_{\mathcal{U}_{p_0}} \quad H_{p_0 p_1}(f) := \left(|f_d|^{\frac{p_0}{p_1}} \right)_{\mathcal{U}_{p_1}}$$

is a (non linear) locally uniform homeomorphism which preserves disjointness and sends components of positive elements $x \in \mathcal{U}_{p_0}$ into components of $H_{p_0 p_1}(x)$ and induces an isomorphism $\Phi_{p_0 p_1}$ between the respective boolean algebras of band projections given by

$$\Phi_{p_0 p_1}(P_B) = P_{H_{p_0 p_1}(B)}.$$

Proof. It is clear that $H_{p_0 p_1}$ is a bijective map. We use the elementary inequalities

$$\forall\ p \geq 1,\ a,\ b \in \mathbb{R} \quad \left| |a|^{\frac{1}{p}} - |b|^{\frac{1}{p}} \right| \leq |a - b|^{\frac{1}{p}}$$

and

$$\forall\ p \geq 1,\ |a| \geq |b| \geq 0 \quad |a|^p - |b|^p \leq p\ |a - b|\ |a|^{p-1}.$$

Let $f = (f_d)_{\mathcal{U}_{p_0}}, g = (g_d)_{\mathcal{U}_{p_0}}$. For every $d \in \mathfrak{D}$ we have

$$\begin{aligned} \left\| |f_d|^{\frac{p_0}{p_1}} - |g_d|^{\frac{p_0}{p_1}} \right\|_{L^{p_1}(\Omega_d)} &\leq \left\| |f_d|^{p_0} - |g_d|^{p_0} \right\|_{L^1(\Omega_d)}^{\frac{1}{p_1}} \\ &\leq p_0^{\frac{1}{p_1}} \left\| (f_d - g_d)(|f_d| \vee |g_d|)^{p_0 - 1} \right\|_{L^1(\Omega_d)}^{\frac{1}{p_1}} \end{aligned}$$

and by Hölder's inequality

$$\leq p_0^{\frac{1}{p_1}} \| f_d - g_d \|_{L^{p_0}(\Omega_d)}^{\frac{1}{p_1}} \| |f_d| \vee |g_d| \|_{L^{p_0}(\Omega_d)}^{\frac{p_0 - 1}{p_1}}. \tag{3}$$

Hence we obtain easily that the family $h^d_{p_0 p_1} : L^{p_0}(\Omega_d) \longrightarrow L^{p_1}(\Omega_d)$ of (non linear) maps defined by $h^d_{p_0 p_1}(f) = sg(f)|f|^{\frac{p_0}{p_1}}$ is a set of locally $\mathcal{U}$-globally uniformly continuous homeomorphisms ($sg(f)$ denotes the sign function of f defined by $sg(f)(t) = 1$ if $f(t) > 0$, $sg(f)(t) = 0$ if $f(t) = 0$ and $sg(f)(t) = -1$ if $f(t) < 0$). Every $h^d_{p_0 p_1}$, $d \in \mathfrak{D}$ sends open balls of center 0 and radius r in $L^{p_0}(\Omega_d)$ onto open balls of center 0 and radius $r^{\frac{p_0}{p_1}}$ in $L^{p_1}(\Omega_d)$. Then the ultraproduct map $H_{p_0 p_1} := (h^d_{p_0 p_1})_{\mathcal{U}}$, is well defined. Taking limits along $\mathcal{U}$ in (3), we see that $H_{p_0 p_1}$ is an homeomorphism which is uniformly continuous on bounded sets and has the same property about open balls in $\mathcal{U}_{p_0}$ and $\mathcal{U}_{p_1}$ that its component mappings.

It is now clear that $H_{p_0 p_1}$ preserves disjointness and, by Remark 2, sends components of positive elements in $\mathcal{U}_{p_0}$ onto components of its image because the components of an element in a function lattice (which is endowed with its canonical puntual order) are the product of such element by characteristic functions of measurable sets. With the same proof of proposition 1.1.1 of Raynaud in [20] it can be proved that the map defined by $\Phi_{p_0 p_1}(P_B) = P_{H_{p_0 p_1}(B)}$ is an isomorphism between the corresponding boolean algebras of band projections. □

For every $d \in \mathfrak{D}$ let

$$S^d_{p_i} = \{x \in L^{p_i}(\Omega_d) \mid \|x\| = 1\}, \quad i = 0, 1,$$
$$S^d_{\lambda} = \{x \in \lambda_d \mid \|x\| = 1\}, \quad S^d_{\cap} = \{x \in L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d) \mid \|x\| = 1\}$$

be the unit spheres in the respective spaces. Next lemma is essentially due to Raynaud [20] but we give a complete account of the more relevant details of the proof for better understanding of the paper.

Lemma 5. *There are (non linear) locally uniform isometries*

$$H_{\cap p_i} : \mathcal{U}_{\mathcal{I}} \longrightarrow \mathcal{U}_{p_i}, \quad i = 0, 1$$

and $H_{\cap\Lambda} : \mathcal{U}_{\mathcal{I}} \longrightarrow \mathcal{U}_{\Lambda}$ *which preserves disjointness, induces isomorphisms* $\Phi_{\cap p_i}$, $i = 0, 1$ *and* $\Phi_{\cap\Lambda}$ *between the boolean algebras of band projections in the corresponding spaces and moreover* $H_{p_0 p_1} H_{\cap p_0} = H_{\cap p_1}$.

Proof. By the deep results of Chaatit given in propositions 2.8 and 2.9 of [3], for each $d \in \mathfrak{D}$ there is a uniform surjective homeomorphism between the corresponding unit spheres

$$h^d_{\cap p_0} : S^d_{\cap} \longrightarrow S^d_{p_0}$$

which preserves the supports of the elements and has a modulus of continuity $\delta(\varepsilon)$ which depends on the modulus of uniform convexity of $L^{p_0}(\Omega_d)$ and the modulus of uniform smoothness of $L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d)$ exclusively. Hence $\delta(\varepsilon)$ is indeed independent of $d \in \mathfrak{D}$ and it depends on p_0 and p_1 only. This isometry can be extended to another support preserving surjective isometry (again denoted by $h^d_{\cap p_0}$) defined in the whole space $L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d)$ setting $h^d_{\cap p_0}(0) = 0$ and

$$\forall \ f_d \neq 0, \quad f_d \in L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d)$$

$$h^d_{\cap p_0}(f_d) = \| f_d \|_{L^{p_0} \cap L^{p_1}} \ h^d_{\cap p_0}\left(\frac{f_d}{\| f_d \|_{L^{p_0} \cap L^{p_1}}} \right).$$

Let us see that $h^d_{\cap p_0}$ is uniformly continuous in bounded sets of $L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d)$. Let B_α denotes the closed ball $B_\alpha := \{ f_d \mid \| f_d \|_{L^{p_0} \cap L^{p_1}} \leq \alpha \}, \alpha > 0$. Fix $M > 0$. Given $\varepsilon > 0$, we have

$$\| f_d \|_{L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d)} \leq \frac{\varepsilon}{2} \Longrightarrow \| h^d_{\cap p_0}(f_d) \| \leq \frac{\varepsilon}{2}$$

since $h^d_{\cap p_0}$ is an isometry. On the other hand there is $1 \geq \rho > 0$ such that $0 < \eta \leq \rho$ implies $\delta(\eta) \leq \frac{\varepsilon}{4M}$. Let $f_d \in L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d), \ f_d \neq 0$ be such that $\frac{\varepsilon}{4} < \| f_d \| < M$. If $g_d \in L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d)$ verifies $\| f_d - g_d \| \leq \frac{\rho \varepsilon}{8}$, having in mind that necessarily $\| g_d \| > \frac{\varepsilon}{8} > 0$ and

$$\begin{aligned} \left\| \frac{f_d}{\| f_d \|} - \frac{g_d}{\| g_d \|} \right\| &\leq \frac{\| f_d - g_d \|}{\| f_d \|} + \frac{| \, \| g_d \| - \| f_d \| \, |}{\| f_d \|} \\ &\leq 8 \ \frac{\| f_d - g_d \|}{\varepsilon} \\ &\leq \rho, \end{aligned}$$

we have

$$\left\| h^d_{\cap p_0}(f_d) - h^d_{\cap p_0}(g_d) \right\|$$

$$= \left\| \| f_d \| \left(h^d_{\cap p_0}\left(\frac{f_d}{\| f_d \|} \right) - h^d_{\cap p_0}\left(\frac{g_d}{\| g_d \|} \right) \right) + (\| f_d \| + \| g_d \|) \ h^d_{\cap p_0}\left(\frac{g_d}{\| g_d \|} \right) \right\|$$

$$\leq \| f_d \| \ \delta \left(\left\| \frac{f_d}{\| f_d \|} - \frac{g_d}{\| g_d \|} \right\| \right) + \| f_d - g_d \| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Hence

$$f_d, g_d \in B_M, \quad f_d - g_d \in B_{\frac{\rho}{8}} \Longrightarrow \|h^d_{\cap p_0}(f_d) - h^d_{\cap p_0}(g_d)\| \leq \varepsilon \tag{4}$$

as we claimed.

Noting that, *by the independence of $d \in D$ on the modulus of continuity of every $h^d_{\cap p_0}$*, (4) holds simultaneously for all $d \in \mathfrak{D}$ (for fixed $M > 0$ and $\varepsilon > 0$ given in advance), we see that the family $\{h^d_{\cap p_0} \mid d \in \mathfrak{D}\}$ is locally $\mathcal{U}$-globally uniformly continuous. Hence the ultraproduct map $H_{\cap p_0} := (h^d_{\cap p_0})_{\mathcal{U}} : \mathcal{U}_{\mathcal{I}} \longrightarrow \mathcal{U}_{p_0}$ is well defined, continuous and by remark 1 preserves disjointness of elements.

Then, since $\mathcal{U}_{\mathcal{I}}$ and $\mathcal{U}_{p_0}$ are order complete Banach lattices, for each band B in $\mathcal{U}_{\mathcal{I}}$ we have

$$H_{\cap p_0}(B)^{\perp\perp} = (H_{\cap p_0}(B)^{\perp})^{\perp} = H_{\cap p_0}(B^{\perp})^{\perp} = H_{\cap p_0}(B^{\perp\perp}) = H_{\cap p_0}(B),$$

that is, $H_{\cap p_0}(B)$ is also a band in $\mathcal{U}_{p_0}$. In particular

$$\forall\, x \in \mathcal{U}_{\mathcal{I}} \quad H_{\cap p_0}(B_x) = H_{\cap p_0}(x^{\perp\perp}) = (H_{\cap p_0}(x^{\perp}))^{\perp} = H_{\cap p_0}(x)^{\perp\perp}$$

and $H_{\cap p_0}(B_x) = B_{H_{\cap p_0}(x)}$, the band generated by the image $H_{\cap p_0}(x)$. With the same proof of proposition 1.1.1 of Raynaud in [20] it can be proved now that the map defined by $\Phi_{\cap p_0}(P_B) = P_{H_{\cap p_0}(B)}$ is an isomorphism between the corresponding boolean algebras of band projections.

Finally, defining $H_{\cap p_1} := H_{p_0 p_1} H_{\cap p_0}$ and $\Phi_{\cap p_1}(B) = P_{H_{\cap p_1}(B)}$ for every band B in $\mathcal{U}_{\mathcal{I}}$, with a similar argumentation we obtain the second desired homeomorphism. Concerning $H_{\cap\Lambda}$ and $\Phi_{\cap\Lambda}$ the proof is analogous using moreover proposition 2.4 in [3] and starting with Chaatit's homeomorphisms $h^d_{\cap\lambda}$ between the unit spheres of the spaces $L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d)$ and λ_d. □

Set $I_{\cap\cup} := I_{\Lambda\cup} I_{\cap\Lambda}$. Let $\mathfrak{D}_1 := \{e^v := (e^v_d)_{\mathcal{U}_{\mathcal{I}}}, v \in \mathcal{V}_1\}$ be a maximal system of pairwise disjoint elements in the band $(Ker(I_{\cap\cup}))^{\perp}$ in $\mathcal{U}_{\mathcal{I}}$ and such that $\|e^v\|_{\mathcal{U}_{\mathcal{I}}} = 1$ for every $v \in \mathcal{V}_1$. As a consequence, for every $v \in \mathcal{V}_1$ and every component x of e^v we have necessarily $I_{\cap\cup}(x) \neq 0$. Let $\mathfrak{D}_2 := \{e^v := (e^v_d)_{\mathcal{U}_{\mathcal{I}}}, v \in \mathcal{V}_2\}$ be a maximal system of *atoms* in the band $Ker(I_{\cap\cup})$ such that $\|e^v\|_{\mathcal{U}_{\mathcal{I}}} = 1$ for each $v \in \mathcal{V}_2$. Remark that $\mathfrak{D}_2$ can be void. Finally, let $\mathfrak{D}_3 := \{e^v := (e^v_d)_{\mathcal{U}_{\mathcal{I}}}, v \in \mathcal{V}_3\}$ be a maximal system of pairwise disjoint elements in the band $Ker(I_{\cap\cup}) \cap \mathfrak{D}_2^{\perp}$ such that, moreover, $\|e^v\|_{\mathcal{U}_{\mathcal{I}}} = 1$ for each $v \in \mathcal{V}_3$. By maximality of $\mathfrak{D}_2$ no e^v, $v \in \mathcal{V}_3$ has atomic components. As above, perhaps $\mathfrak{D}_3 = \emptyset$ but we have always $\mathfrak{D}_2 \cup \mathfrak{D}_3 \neq \emptyset$ because $I_{\cap\cup}$ is injective if and only if every $I^d_{\lambda\cup} I^d_{\cap\lambda}$, $d \in \mathfrak{D}$ is an isomorphism onto the image and the set of norms of $I^d_{\lambda\cup} I^d_{\cap\lambda}$ and its inverse mappings is uniformly bounded with respect to $d \in \mathfrak{D}$.

Put $\mathcal{V}_0 := \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$. Clearly $\mathfrak{D}_0 := \{e^v \mid v \in \mathcal{V}_0\}$ is a maximal system of pairwise disjoint elements in $\mathcal{U}_1$ since $\mathcal{U}_1 = B \oplus B^\perp$ for every band B in $\mathcal{U}_1$.

With help of $\mathfrak{D}_0$ we select now some special sets of pairwise disjoint elements in $\mathcal{U}_{p_i}$, $i = 0, 1$ and $\mathcal{U}_\Lambda$. We define

$$u^v := \begin{cases} I_{\cap p_0}(e^v) & \text{if } v \in \mathcal{V}_1 \\ H_{\cap p_0}(e^v) & \text{if } v \in \mathcal{V}_2 \cup \mathcal{V}_3, \end{cases} \qquad w^v := \begin{cases} I_{\cap p_1}(e^v) & \text{if } v \in \mathcal{V}_1 \\ H_{\cap p_1}(e^v) & \text{if } v \in \mathcal{V}_2 \cup \mathcal{V}_3 \end{cases}$$

and

$$z^v := \begin{cases} I_{\cap \Lambda}(e^v) & \text{if } v \in \mathcal{V}_1 \\ H_{\cap \Lambda}(e^v) & \text{if } v \in \mathcal{V}_2 \cup \mathcal{V}_3. \end{cases}$$

Remark that

$$\forall\, v \in \mathcal{V}_2 \cup \mathcal{V}_3 \quad \|u^v\|_{\mathcal{U}_{p_0}} = \|w^v\|_{\mathcal{U}_{p_1}} = \|z^v\|_{\mathcal{U}_\Lambda} = 1 \tag{5}$$

and

$$\forall\, v \in \mathcal{V}_1 \;\; \|w^v\|_{\mathcal{U}_{p_1}} \le \|z^v\|_{\mathcal{U}_\Lambda} \le \|u^v\|_{\mathcal{U}_{p_0}} \le 1. \tag{6}$$

Fix, for future work, the notation $u^v = (u^v_d)_{\mathcal{U}_{p_0}}$, $z^v = (z^v_d)_{\mathcal{U}_\Lambda}$ and $w^v = (w^v_d)_{\mathcal{U}_{p_1}}$ for every $v \in \mathcal{V}_0$.

Lemma 6. *$\{u^v \mid v \in \mathcal{V}_0\}$ and $\{w^v \mid v \in \mathcal{V}_0\}$ are maximal systems of pairwise disjoint elements in $\mathcal{U}_{p_0}$ and $\mathcal{U}_{p_1}$ respectively.*

Proof. Let $x = (x_d)_{\mathcal{U}_{p_0}} \in \mathcal{U}_{p_0}$ verifies $|x| \wedge u^v = 0$ for every $v \in \mathcal{V}_0$. Set $y := (y_d)_{\mathcal{U}} := (H_{\cap p_0})^{-1}(|x|)$. Fix $v \in \mathcal{V}_0$. As $|x| \wedge u^v = 0$ we can suppose that $|x_d| \wedge u^v_d = 0$ for every $d \in \mathfrak{D}$ (Remark 1). Then, by Lemma 5 $|y_d| \wedge (h^d_{\cap p_0})^{-1}(u^v_d) = 0$ and consequently $|y_d| \wedge e^v_d = 0$ because $h^d_{\cap p_0}$ preserves the support of functions. In this way we get $|y| \wedge e^v = 0$. As $v \in \mathcal{V}_0$ is arbitrary, necessarily $y = 0$ and hence $|x| = H_{\cap p_0}(y) = 0$. The proof for $\mathcal{U}_{p_1}$ is similar. □

Lemma 7. *For every $v \in \mathcal{V}_2$, u^v, z^v and w^v are atoms.*

Proof. Assume $x \wedge y = 0$, $0 < x \le u^v$ and $0 < y \le u^v$. Then $x \wedge u^s = 0$ and $y \wedge u^s = 0$ for each $s \in \mathcal{V}_0$, $s \neq v$. As $H_{\cap p_0}$ preserves disjointness,

$$|H^{-1}_{\cap p_0}(x)| \wedge |H^{-1}_{\cap p_0}(y)| = 0, \quad |H^{-1}_{\cap p_0}(x)| \wedge e^s = 0 \quad \text{and} \quad |H^{-1}_{\cap p_0}(y)| \wedge e^s = 0$$

for such $s \in \mathcal{V}_0$. By the maximal property of $\mathfrak{D}_0$ necessarily

$$0 \neq e^v \wedge |H^{-1}_{\cap p_0}(x)| \leq e^v \quad \text{and} \quad 0 \neq e^v \wedge |H^{-1}_{\cap p_0}(y)| \leq e^v$$

since $|H^{-1}_{\cap p_0}(x)| \neq 0$ and $|H^{-1}_{\cap p_0}(y)| \neq 0$, $H_{\cap p_0}$ being an isometry, a contradiction with the atomic character of e^v. The proof of the other statements is similar. $\square$

For simplicity of notation, put $h^d_{p_0\lambda} := h^d_{\cap\lambda}(h^d_{\cap p_0})^{-1}$ for every $d \in \mathfrak{D}$ and $H_{p_0\Lambda} := (h^d_{p_0\lambda})_{\mathcal{U}} = H_{\cap\Lambda} H^{-1}_{\cap p_0}$ which still is a homeomorphism. For every $d \in \mathfrak{D}$ and every $x_d \in C(u^v_d)$, we put

$$g^d_{p_0\lambda}(x_d) = \begin{cases} I^d_{\cap\lambda}(I^d_{\cap p_0})^{-1}(x_d) & \text{if } \ v \in \mathcal{V}_1 \\ P_{h^d_{p_0\lambda}(x_d)}(z^v_d) & \text{if } \ v \in \mathcal{V}_2 \cup \mathcal{V}_3. \end{cases}$$

Clearly $\sup_{d\in\mathfrak{D}} \|g^d_{p_0\lambda}(x_d)\| = \sup_{d\in\mathfrak{D}} \|x_d\|_{\mathcal{U}_1} < \infty$ if $v \in \mathcal{V}_1$. In other case we have

$$\sup_{d\in\mathfrak{D}} \|g^d_{p_0\lambda}(x_d)\| = \sup_{d\in\mathfrak{D}} \| \sup_{n\in\mathbb{N}} \ z^v_d \wedge n \ h^d_{p_0\lambda}(x_d)\| \leq \sup_{d\in\mathfrak{D}} \|z^v_d\| < \infty. \tag{7}$$

Then the map

$$\forall \ v \in \mathcal{V}_0, \ \ \forall \ x \in C(u^v) \quad G_{p_0\Lambda}(x) := (g^d_{p_0\lambda}(x_d))_{\mathcal{U}_\Lambda}$$

is well defined on $\bigcup\{C(u^v) \mid v \in \mathcal{V}_0\}$.

We extend $G_{p_0\Lambda}$ to the set

$$\mathcal{F}^v_{p_0} := \left\{ \sum_{i=1}^n \alpha_i x_i \,/\, \alpha_i \in \mathbb{R}, x_i \in C(u^v), x_i \wedge x_j = 0 \right. \\ \left. \text{if } \ i \neq j, i, j = 1, 2, ..., n; \quad n \in \mathbb{N} \right\}$$

putting

$$\forall \ \sum_{i=1}^n \alpha_i x_i \in \mathcal{F}^v_{p_0} \quad G_{p_0\Lambda}\left(\sum_{i=1}^n \alpha_i x_i\right) = \sum_{i=1}^n \alpha_i G_{p_0\Lambda}(x_i).$$

Finally, we extend the definition of $G_{p_0\Lambda}$ by linearity to the linear span $\mathcal{F}^{p_0}_{\mathcal{V}_0}$ of $\cup_{v\in\mathcal{V}_0}\mathcal{F}^v_{p_0}$. We have

Lemma 8.

1) *$G_{p_0\Lambda}$ is well defined on $\mathcal{F}^{p_0}_{\mathcal{V}_0}$ and $G_{p_0\Lambda}(x) \in C(z^v)$ for every $v \in \mathcal{V}_0$ and $x \in C(u^v)$.*

2) *$G_{p_0\Lambda}$ is continuos on the linear span $\mathcal{F}^{p_0}_{\mathcal{V}_1\cup\mathcal{V}_2}$ of $\bigcup\{C(u^v) \mid v \in \mathcal{V}_1\cup\mathcal{V}_2\}$ and hence it can be extended by continuity to the band $\{u^v \mid v \in \mathcal{V}_1 \cup \mathcal{V}_2\}^{\perp\perp}$ in $\mathcal{U}_{p_0}$.*

Proof. 1) If $v \in \mathcal{V}_1$, $G_{p_0\Lambda}$ is well defined on $\mathcal{F}^v_{p_0}$ trivially. Let $v \in \mathcal{V}_2 \cup \mathcal{V}_3$. Suppose

$$\sum_{i=1}^{n} \alpha_i x^i = \sum_{j=1}^{m} \beta_j y^j \tag{8}$$

where $\{x^i\}_{i=1}^n$ and $\{y^j\}_{j=1}^m$ are sets of pairwise disjoint components of u^v. By well known properties of vector lattices, (see for instance proposition 1.2.17 in [14]), there is another set $\{\overline{z}^k\}_{k=1}^h := \{(\overline{z}^k_d)_{\mathcal{U}_{p_0}}\}_{k=1}^h$ of pairwise disjoint components of u^v such that

$$\begin{aligned} &\forall\, i = 1, 2, ..., n \quad x^i = \sum_{k=1}^{h} \gamma_{ik}\overline{z}^k \quad \text{and} \\ &\forall\, j = 1, 2, ..., m \quad y^j = \sum_{k=1}^{h} \rho_{jk}\overline{z}^k \end{aligned} \tag{9}$$

and, moreover $\gamma_{ik} = 0$ or $\gamma_{ik} = 1$ and $\rho_{jk} = 0$ or $\rho_{jk} = 1$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$. By remark 1 we can also suppose $\overline{z}^k_d \wedge \overline{z}^s_d = 0$ for $d \in \mathfrak{D}$ and $1 \leq k \neq s \leq h$. Then

$$\sum_{i=1}^{n} \alpha_i x^i = \sum_{k=1}^{h} \left(\sum_{i=1}^{n} \alpha_i \gamma_{ik} \right) \overline{z}^k$$

and

$$\sum_{j=1}^{m} \beta_j y^j = \sum_{k=1}^{h} \left(\sum_{j=1}^{m} \beta_j \rho_{jk} \right) \overline{z}^k$$

and from (8) we obtain

$$\forall\, k = 1, 2, ..., h \qquad \sum_{i=1}^{n} \alpha_i \gamma_{ik} = \sum_{j=1}^{m} \beta_j \rho_{jk}. \tag{10}$$

According to (9), for each $d \in \mathfrak{D}$ let

$$\begin{aligned} &\forall\ i = 1, 2, ..., n \ \ x_d^i = \sum_{k=1}^{h} \gamma_{ik} \overline{z}_d^k \ \text{ and} \\ &\forall\ j = 1, 2, ..., m \ \ y_d^j = \sum_{k=1}^{h} \rho_{jk} \overline{z}_d^k. \end{aligned} \tag{11}$$

Clearly, by (9) we have $x^i = (x_d^i)_{\mathcal{U}_{p_0}}, i = 1, 2, ..., n$ and $y^j = (y_d^j)_{\mathcal{U}_{p_0}}$, $j = 1, 2, ..., m$. Furthermore, (11) implies

$$\forall\, d \in \mathfrak{D}, \ \ \forall\ i = 1, 2, ..., n \quad Supp(x_d^i) = \bigcup \left\{ Supp(\overline{z}_d^k) \mid \gamma_{ik} = 1 \right\}$$

and

$$\forall\, d \in \mathfrak{D}, \ \ \forall\ j = 1, 2, ..., m \quad Supp(y_d^j) = \bigcup \left\{ Supp(\overline{z}_d^k) \mid \rho_{jk} = 1 \right\}.$$

Then we can suppose that for every $i = 1, 2, ..., n$ and every $t \in \Omega_d$ with $x_d^i(t) \neq 0$ there is $1 \leq k \leq h$ such that $\overline{z}_d^k(t) \neq 0$ and since $h_{p_0\lambda}^d$ preserves the supports, $h_{p_0\lambda}^d(x_d^i)(t) \neq 0$ and $h_{p_0\lambda}^d(\overline{z}_d^k)(t) \neq 0$. As a consequence we get

$$\begin{aligned} P_{h_{p_0\lambda}^d(x_d^i)}(z_d^v) &= \sup_{r\in\mathbb{N}} z_d^v \wedge r\ h_{p_0\lambda}^d(x_d^i) = \sup_{r\in\mathbb{N}} z_d^v \wedge r\ h_{p_0\lambda}^d\left(\sum_{k=1}^{h} \gamma_{ik}\overline{z}_d^k\right) \\ &= \sup_{r\in\mathbb{N}} z_d^v \wedge r\ \left(\sum_{k=1}^{h} \gamma_{ik} h_{p_0\lambda}^d(\overline{z}_d^k)\right) = \sum_{k=1}^{h} \gamma_{ik} \left(\sup_{r\in\mathbb{N}} z_d^v \wedge r\ h_{p_0\lambda}^d(\overline{z}_d^k)\right) \\ &= \sum_{k=1}^{h} \gamma_{ik} P_{h_{p_0\lambda}^d(\overline{z}_d^k)}(z_d^v) \end{aligned}$$

having in mind that the supports of $\overline{z}_d^k$'s are pairwise disjoint.

Then after a similar computation with the elements $\{y_d^j\}_{j=1}^m$, by (10) we get

$$\begin{aligned} \forall\ d \in \mathfrak{D} \quad \sum_{i=1}^{n} \alpha_i P_{h_{p_0\lambda}^d(x_d^i)}(z_d^v) &= \sum_{k=1}^{h} \left(\sum_{i=1}^{n} \alpha_i \gamma_{ik}\right) P_{h_{p_0\lambda}^d(\overline{z}_d^k)}(z_d^v) \\ &= \sum_{k=1}^{h} \left(\sum_{j=1}^{m} \beta_j \rho_{jk}\right) P_{h_{p_0\lambda}^d(\overline{z}_d^k)}(z_d^v) \\ &= \sum_{j=1}^{m} \beta_j P_{h_{p_0\lambda}^d(y_d^j)}(z_d^v). \end{aligned}$$

That means that $G_{p_0\Lambda}$ is well defined on $\mathcal{F}^{v}_{p_0}$ if $v \in \mathcal{V}_2 \cup \mathcal{V}_3$.

Since every element of $\mathcal{F}^{p_0}_{\mathcal{V}_0}$ can be written in a unique way as finite sum $\sum_{i=1}^{s} x_i$ with every $x_i \in B_{u^{v_i}}$, $v_i \in \mathcal{V}_0$, $i = 1, 2, ..., s$, it is now clear that $G_{p_0\Lambda}$ is well defined on $\mathcal{F}^{p_0}_{\mathcal{V}_0}$.

On the other hand, for every projection band B in a Banach lattice E and every $x \in E$, $x \geq 0$, $P_B(x)$ is a component of x. Then, by Remark 2 $g^d_{p_0\lambda}(x_d) \in C(z^v_d)$ for every $v \in \mathcal{V}_0$, $x := (x_d)_{\mathcal{U}_{p_0}} \in C(u^v)$ and $d \in \mathfrak{D}$. Then $G_{p_0\Lambda}(x) \in C(z^v)$ for every $x \in C(u^v)$.

2) Again it is enough to do the proof for the linear span $\mathcal{F}^{p_0}_{\mathcal{V}_2}$ of $\bigcup\{C(u^v) \mid v \in \mathcal{V}_2\}$. Let $v \in \mathcal{V}_2$. By Lemma 7 every u^v is an atom and hence the unique component of u^v is the same u^v. Then

$$\begin{aligned} G_{p_0\Lambda}(u^v) &= G_{p_0\Lambda}((u^v_d)_{\mathcal{U}_{p_0}}) = (P_{h^d_{p_0\lambda}(u^v_d)}(z^d_v))_{\mathcal{U}_\Lambda} \\ &= (P_{z^d_v}(z^d_v))_{\mathcal{U}_\Lambda} = (z^d_v)_{\mathcal{U}_\Lambda} = z^v. \end{aligned} \tag{12}$$

$\mathcal{U}_\Lambda$ having an r upper estimate, where $r = \min\{\overline{p}, q\}$, (notation of Lemma 3), $\mathcal{U}_{p_0}$ being an abstract L^{p_0}-space and noticing that $p_0 \leq r$, there is $C > 0$ such that, for every finite set $\{u^{v_n}\}_{n=1}^t \subset \mathcal{D}_2$ and $\{\alpha_n\}_{n=1}^t \subset \mathbb{R}$, by (12) and (6) we have

$$\begin{aligned} \left\| \sum_{n=1}^{t} \alpha_n G_{p_0\Lambda}(u^{v_n}) \right\|_{\mathcal{U}_\Lambda} &\leq C \left(\sum_{n=1}^{t} |\alpha_n|^r \|G_{p_0\Lambda}(u^{v_n})\|^r_{\mathcal{U}_\Lambda} \right)^{\frac{1}{r}} \\ &\leq C \left(\sum_{n=1}^{t} |\alpha_n|^{p_0} \|z^{v_n}\|^{p_0}_{\mathcal{U}_\Lambda} \right)^{\frac{1}{p_0}} \\ &\leq C \left(\sum_{n=1}^{t} |\alpha_n|^{p_0} \|u^{v_n}\|^{p_0}_{\mathcal{U}_{p_0}} \right)^{\frac{1}{p_0}} = C \left\| \sum_{n=1}^{t} \alpha_n u^{v_n} \right\|_{\mathcal{U}_{p_0}} \end{aligned}$$

which shows the continuity of $G_{p_0\Lambda}$ in $\mathcal{F}^{p_0}_{\mathcal{V}_2}$. By order continuity of $\mathcal{U}_{p_0}$ and Freudenthal´s spectral theorem, $G_{p_0\Lambda}$ can be continuously extended to the band $\{u^v \mid v \in \mathcal{V}_2\}^{\perp\perp}$. $\square$

Now we do a similar work with the ultraproducts $\mathcal{U}_\Lambda$ and $\mathcal{U}_{p_1}$. For every $d \in \mathfrak{D}$ we define $h^d_{\lambda p_1} := h^d_{\cap p_1}(h^d_{\cap\lambda})^{-1}$,

$$\forall v \in \mathcal{V}_0, \quad \forall x_d \in C(z^v_d) \quad g^d_{\lambda p_1}(x_d) := \begin{cases} I^d_{\cap p_1}(I^d_{\cap\lambda})^{-1}(x_d) & \text{if } v \in \mathcal{V}_1 \\ P_{h^d_{\lambda p_1}(x_d)}(w^v_d) & \text{if } v \in \mathcal{V}_2 \cup \mathcal{V}_3 \end{cases}$$

and, after similar computations to that (7), we define

$$\forall v \in \mathcal{V}_0, \quad \forall x = (x_d)_{\mathcal{U}_\Lambda} \in C(z^v) \quad G_{\Lambda p_1}(x) = (g^d_{\lambda p_1}(x_d))_{\mathcal{U}_{p_1}}.$$

We extend $G_{\Lambda p_1}$ by linearity to

$$\mathcal{F}^v_\Lambda := \left\{ \sum_{n=1}^{k} \alpha_n y_n \;\; y^n \in C(z^v), n = 1, 2, ..., k, \; k \in \mathbb{N}, y^n \wedge y^m = 0 \text{ if } n \neq m \right\}$$

and in a further step to the linear span $\mathcal{F}^\Lambda_{\mathcal{V}_0}$ of $\bigcup \{C(z^v) \mid v \in \mathcal{V}_0\}$. Note that

$$\forall\, v \in \mathcal{V}_0 \quad w^v = G_{\Lambda p_1}(z^v). \tag{13}$$

We have

Lemma 9.

1) *$G_{\Lambda p_1}$ is well defined on $\mathcal{F}^\Lambda_{\mathcal{V}_0}$ and $G_{\Lambda p_1}(x) \in C(w^v)$ for every $v \in \mathcal{V}_0$ and $x \in C(z^v)$.*

2) *$G_{\Lambda p_1}$ is continuos on the linear span $\mathcal{F}^\Lambda_{\mathcal{V}_1 \cup \mathcal{V}_2}$ of $\bigcup \{C(z^v) \mid v \in \mathcal{V}_1 \cup \mathcal{V}_2\}$ and hence it can be extended by continuity to the band $\{z^v \mid v \in \mathcal{V}_1 \cup \mathcal{V}_2\}^{\perp\perp}$ in $\mathcal{U}_\Lambda$ generated by them.*

Proof. The proof is analogous to the given one in Lemma 2 but using the fact that $\mathcal{U}_\Lambda$ verifies a s-lower estimate, (where $s = \max\{\overline{p}, q\}$ and $s \leq p_1$). There is now $M > 0$ such that given $\{v_n\}_{n=1}^k \subset \mathcal{V}_2$ and $\{\alpha_n\}_{n=1}^k \subset \mathbb{R}$, by (13) and (5), $\mathcal{U}_{p_1}$ being an abstract L^{p_1}-space, we have

$$\begin{aligned}
\left\| \sum_{n=1}^{k} \alpha_n G_{\Lambda p_1}(z^{v_n}) \right\|_{\mathcal{U}_{p_1}} &= \left(\sum_{n=1}^{k} |\alpha_n|^{p_1} \| G_{\Lambda p_1}(z^{v_n}) \|^{p_1}_{\mathcal{U}_{p_1}} \right)^{\frac{1}{p_1}} \\
&\leq \left(\sum_{n=1}^{k} |\alpha_n|^{s} \| G_{\Lambda p_1}(z^{v_n}) \|^{s}_{\mathcal{U}_{p_1}} \right)^{\frac{1}{s}} \\
&\leq \left(\sum_{n=1}^{h} |\alpha_n|^{s} \| z^{v_n} \|^{s}_{\mathcal{U}_{p_1}} \right)^{\frac{1}{s}} \leq M \left\| \sum_{n=1}^{k} \alpha_n z^{v_n} \right\|_{\mathcal{U}_\Lambda}.
\end{aligned}$$

□

Consider now the linear map $G_{p_0 p_1} \colon \mathcal{U}_{p_0} \longrightarrow \mathcal{U}_{p_1}$ defined by $G_{p_0 p_1} = G_{\Lambda p_1} G_{p_0 \Lambda}$. We have

Lemma 10. *For every $v \in \mathcal{V}_0$ and every $x \in C(u^v)$ the equality*

$$G_{p_0 p_1}(x) = H_{p_0 p_1}(x)$$

holds.

Proof. There is nothing to prove if $v \in \mathcal{V}_1$. Let $v \in \mathcal{V}_2 \cup \mathcal{V}_3$. Let $x = (x_d)_{\mathcal{U}_{p_0}} \in C(u^v)$. Since every x_d and u_d^v are functions on the measure space $\Omega_d, d \in \mathfrak{D}$, by Remark 2 we can suppose there are measurable sets A_d in Ω_d such that $x_d = \chi_{A_d} u_d^v$ for every $d \in \mathfrak{D}$. For every $d \in \mathfrak{D}$, let $y_d := g^d_{p_0\lambda}(x_d)$ and $s_d := (h^d_{p_0\lambda})^{-1}(y_d)$. By known properties of band projections and the definition of the homeomorphism $H_{p_0 p_1}$, $\mathcal{U}_{p_0}$ and $\mathcal{U}_{p_1}$ being order continuous lattices we obtain

$$\begin{aligned} g^d_{\lambda p_1}(y_d) = P_{h^d_{\lambda p_1}(y_d)}(w_d) &= \sup_{n\in\mathbb{N}} h^d_{p_0 p_1}(u^v_d) \wedge n\, h^d_{\lambda p_1}(y_d) \\ &= \sup_{n\in\mathbb{N}} h^d_{p_0 p_1}(u^v_d) \wedge n\, h^d_{p_0 p_1}(h^d_{p_0\lambda})^{-1}(y_d) \\ &= \sup_{n\in\mathbb{N}} (u^v_d)^{\frac{p_0}{p_1}} \wedge n\, s_d^{\frac{p_0}{p_1}}. \end{aligned} \tag{14}$$

By properties of $h^d_{p_0\lambda}$ the equalities

$$Supp\left(x_d^{\frac{p_0}{p_1}}\right) = Supp(x_d) = Supp(y_d) = Supp(s_d) = Supp\left(s_d^{\frac{p_0}{p_1}}\right),$$

hold. As $x_d = \chi_{A_d} u^v_d$, we get as continuation of (14)

$$g^d_{\lambda p_1}(y_d) = \sup_{n\in\mathbb{N}} (\chi_{A_d} u^v_d)^{\frac{p_0}{p_1}} \wedge n\, s_d^{\frac{p_0}{p_1}} = (\chi_{A_d} u^v_d)^{\frac{p_0}{p_1}} = x_d^{\frac{p_0}{p_1}} = h^d_{p_0 p_1}(x_d).$$

As a consequence

$$\begin{aligned} G_{\Lambda p_1} G_{p_0 \Lambda}(x) &= G_{\Lambda p_1}((g^d_{p_0\lambda}(x_d))_{\mathcal{U}_\Lambda}) = (g^d_{\lambda p_1}(y_d))_{\mathcal{U}_{p_1}} \\ &= (h^d_{p_0 p_1}(x_d))_{\mathcal{U}_{p_1}} = H_{p_0 p_1}(x). \end{aligned}$$

□

3 Main results

Theorem 11. *There is a measure space $(\Omega_\mathcal{U}, \mathcal{M}_\mathcal{U}, \mu_\mathcal{U})$ and isometric order isomorphisms*

$$\Psi_{p_j} : \mathcal{U}_{p_j} \longrightarrow L^{p_j}((\Omega_\mathcal{U}, \mathcal{M}_\mathcal{U}, \mu_\mathcal{U}), \quad j = 0, 1$$

such that

$$\forall x \in \mathcal{U}_{p_0} \quad \Psi_{p_0}(x) = \Psi_{p_1}(G_{p_0 p_1}(x)). \tag{15}$$

Proof. Let $v \in \mathcal{V}_0$ and let $\mathcal{P}_{p_0}(u^v)$ and $\mathcal{P}_{p_1}(w^v)$ be the boolean algebras of principal band projections generated by the components of u^v and w^v respectively. By Lemma 10 we have

$$\forall\, x \in C(u^v) \quad \forall\, f \in \mathcal{U}_{p_1} \quad P_{H_{p_0 p_1}(x)}(f) = P_{G_{p_0 p_1}(x)}(f)$$

and by lemmata 8 and 9 $P_x(u^v) = x$ and $P_{G_{p_0 p_1}(x)}(w^v) = G_{p_0 p_1}(x)$. Then by lemmata 4 and 10, the restriction to the set of principal band projections of $\Phi_{p_0 p_1}$ given by

$$\forall x \in C(u^v) \quad \Phi_{p_0 p_1}(P_x) = P_{H_{p_0 p_1}(x)} = P_{G_{p_0 p_1}(x)}$$

is an isomorphism between the respective boolean algebras of principal band projections. By the Stone representation theorem there is an extremely disconnected compact space X_v and isomorphisms of boolean algebras

$$\mathcal{H}^v_{p_0} : C(u^v) \longrightarrow \mathcal{O}_v, \qquad \mathcal{H}^v_{p_1} : C(w^v) \longrightarrow \mathcal{O}_v$$

onto the boolean algebra $\mathcal{O}_v$ of clopen sets of X_v such that

$$\forall x \in C(u^v) \quad \mathcal{H}^v_{p_0}(x) = \mathcal{H}^v_{p_1}(H_{p_0 p_1}(x)) \in \mathcal{O}_v. \tag{16}$$

Since every $\mathcal{U}_{p_i}$, $i = 0, 1$ is an abstract L^{p_i}-space, if we define

$$\forall A \in \mathcal{O}_v \quad \mu^v_i(A) := \|(\mathcal{H}^v_{p_i})^{-1}(A)\|^{p_i}_{\mathcal{U}_{p_i}} \quad i = 0, 1$$

we obtain measures μ^v_i, $i = 0, 1$ in $\mathcal{O}_v$ such that, by (16) and the definition of $H_{p_0 p_1}$

$$\begin{gathered}\forall A \in \mathcal{O}_v \quad \mu^v_1(A) := \|(\mathcal{H}^v_{p_1})^{-1}(A)\|^{p_1}_{\mathcal{U}_{p_1}} \\ = \|H_{p_0 p_1}(\mathcal{H}^v_{p_0})^{-1}(A)\|^{p_1}_{\mathcal{U}_{p_1}} = \|(\mathcal{H}^v_{p_0})^{-1}(A)\|^{p_0}_{\mathcal{U}_{p_0}} = \mu^v_0(A),\end{gathered}$$

i.e. we have a unique measure μ_v defined on $\mathcal{O}_v$. By the standard Caratheodory procedure μ_v can be extended to a measure (again denoted by μ_v,) defined in the σ-algebra $\mathcal{M}_v$ of μ_v-measurable sets of X_v. The map

$$\Psi^v_{p_0} : \mathcal{F}^v_{p_0} \longrightarrow L^{p_0}(X_v, \mu_v)$$

given by

$$\Psi^v_{p_0}\left(\sum_{i=1}^n \alpha_i x_i\right) = \sum_{i=1}^n \alpha_i \chi_{\mathcal{H}^v_{p_0}(x_i)}$$

is well defined by the same argumentation used in Lemma 8. As $\mathcal{U}_{p_0}$ is an abstract L^{p_0}-space we have

$$\left\| \Psi_{p_0}^v \left(\sum_{i=1}^n \alpha_i x_i \right) \right\|_{L^{p_0}(\mu_v)}^{p_0} = \sum_{i=1}^n \int_{\mathcal{H}_{p_0}^v(x_i)} |\alpha_i|^{p_0} d\mu_v = \sum_{i=1}^n |\alpha_i|^{p_0} \mu_v(\mathcal{H}_{p_0}^v(x_i))$$
$$= \sum_{i=1}^n |\alpha_i|^{p_0} \|x_i\|_{\mathcal{U}_{p_0}}^{p_0} = \|x\|^{p_0}$$

and hence, $\Psi_{p_0}^v$ is an isometry. Analogously, if

$$\mathcal{F}_{p_1}^v = \left\{ y = \sum_{i=1}^n \alpha_i y_i \quad y_i \in C(w^v), \quad i = 1, 2, ..., n, \quad n \in \mathbb{N}, y_i \wedge y_j = 0 \text{ if } i \neq j \right\},$$

the map

$$\Psi_{p_1}^v : \mathcal{F}_{p_1}^v \longrightarrow L^{p_1}(X_v, \mu_v)$$

defined by

$$\Psi_{p_1}^v \left(\sum_{i=1}^n \alpha_i y_i \right) = \sum_{i=1}^n \alpha_i \chi_{\mathcal{H}_{p_1}^v(y_i)}$$

is an isometry. Since $H_{p_0 p_1}$ *sends components into components* (Lemma 4), by Lemma 10, for every finite set $\{x_i\}_{i=1}^n \subset C(u^v)$ and $\{\alpha_i\}_{i=1}^n \subset \mathbb{R}$ we have

$$\Psi_{p_0}^v \left(\sum_{i=1}^n \alpha_i x_i \right) = \sum_{i=1}^n \alpha_i \chi_{\mathcal{H}_{p_0}^v(P_{x_i})} = \sum_{i=1}^n \alpha_i \chi_{\mathcal{H}_{p_1}^v(P_{H_{p_0 p_1}(x_i)})}$$
$$= \Psi_{p_1}^v \left(\sum_{i=1}^n \alpha_i H_{p_0 p_1}(x_i) \right) = \Psi_{p_1}^v \left(\sum_{i=1}^n \alpha_i G_{p_0 p_1}(x_i) \right)$$
$$= \Psi_{p_1}^v \left(G_{p_0 p_1} \left(\sum_{i=1}^n \alpha_i x_i \right) \right).$$

To finish we only have to define the measure space $(\Omega_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}, \mu_{\mathcal{U}})$ where $\Omega_{\mathcal{U}} := \bigcup_{v \in \mathcal{V}_0} X_v$,

$$\mathcal{M}_{\mathcal{U}} := \left\{ A := \bigcup_{v \in \mathcal{V}_0} A_v \mid A \in \mathcal{M}_v \ \forall \ v \in \mathcal{V}_0 \right\}$$

and

$$\forall \ A := \bigcup_{v \in \mathcal{V}_0} A_v \in \mathcal{M}_{\mathcal{U}} \quad \mu_{\mathcal{U}}(A) = \sum_{v \in \mathcal{V}_0} \mu_v(A_v)$$

and to glue the isomorphisms $\Psi^v_{p_0}$, $i = 0, 1$ defining

$$\forall\ x \in \mathcal{U}_{p_0} \quad \Psi_{p_0}(x) = \sum_{v \in \mathcal{V}_0} \Psi^v_{p_0}(P_{u^v}(x))$$

and

$$\forall\ x \in \mathcal{U}_{p_1} \quad \Psi_{p_1}(x) = \sum_{v \in \mathcal{V}_0} \Psi^v_{p_1}(P_{w^v}(x)). \qquad \square$$

Theorem 12. *There are a measure space $(\Omega_1, \mathcal{M}_1, \mu_1)$, a discrete measure space $(\Omega_2, \mathcal{M}_2, \mu_2)$ (eventually empty) and an atomless measure space $(\Omega_3, \mathcal{M}_3, \mu_3)$, (eventually empty) such that $\Omega_2 \cup \Omega_3 \neq \emptyset$ and such that the ultraproduct of interpolation spaces $((L^{p_0}(\Omega_d, \mu_d), L^{p_1}(\Omega_d, \mu_d))_{\theta,q})_{\mathcal{U}}$ is isomorphic to the direct sum*

$$(L^{p_0}(\Omega_1, \mu_1), L^{p_1}(\Omega_1, \mu_1))_{\theta,q} \oplus K(\Omega_2) \oplus X(\Omega_3)$$

where $K(\Omega_2)$ is an intermediate space of the couple $(\ell^{p_0}(\Omega_2), \ell^{p_1}(\Omega_2))$ and $X(\Omega_3)$ is an order continuous Köthe function space over Ω_3.

Proof. For every $i = 1, 2, 3$ let $P^i_{\mathcal{U}_{p_0}} : \mathcal{U}_{p_0} \longrightarrow \mathcal{U}_{p_0}$ be the canonical projection onto the band generated in $\mathcal{U}_{p_0}$ by the set $\{u^v \mid v \in \mathcal{V}_i\}$. Analogously we define $P^i_{\mathcal{U}_{p_1}} : \mathcal{U}_{p_1} \longrightarrow \mathcal{U}_{p_1}$ as the canonical projection onto the band generated in $\mathcal{U}_{p_1}$ by the set $\{w^v \mid v \in \mathcal{V}_i\}$ and $P^i_{\mathcal{U}_\Lambda} : \mathcal{U}_\Lambda \longrightarrow \mathcal{U}_\Lambda$ as the canonical projection onto the band generated in $\mathcal{U}_\Lambda$ by the set $\{z^v \mid v \in \mathcal{V}_i\}$. Clearly we have

$$\mathcal{U}_\Lambda = P^1_{\mathcal{U}_\Lambda}(\mathcal{U}_\Lambda) \oplus P^2_{\mathcal{U}_\Lambda}(\mathcal{U}_\Lambda) \oplus P^3_{\mathcal{U}_\Lambda}(\mathcal{U}_\Lambda).$$

We consider the measure space $(\Omega_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}, \mu_{\mathcal{U}},)$ constructed in Theorem 11. Let $\Omega_i = \cup_{v \in \mathcal{V}_i} Supp(\Psi_{p_0}(u^v))$, $i = 1, 2, 3$. Let $\mathcal{F}^{p_0}_{\mathcal{V}_2 \cup \mathcal{V}_3}$ be the linear span of the set of components $\cup \{C(u^v) \mid v \in \mathcal{V}_2 \cup \mathcal{V}_3\}$. We define $\Psi^{23}_{\mathcal{U}} : \mathcal{F}^{p_0}_{\mathcal{V}_2 \cup \mathcal{V}_3} \longrightarrow L^0(\Omega_2 \cup \Omega_3, \mu_{\mathcal{U}})$ by

$$\forall\ \sum_{i=1}^{n} \alpha_i x_i \in \mathcal{F}^{p_0}_{\mathcal{V}_2 \cup \mathcal{V}_3} \quad \Psi^{23}_{\mathcal{U}}\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{i=1}^{n} \alpha_i \chi_{\Psi_{p_0}\left(G^{-1}_{p_0 \Lambda}(x_i)\right)}$$

and we define a norm on $\Psi^{23}_{\mathcal{U}}(\mathcal{F}^{p_0}_{\mathcal{V}_2 \cup \mathcal{V}_3})$ by

$$\forall\ x \in \mathcal{F}^{p_0}_{\mathcal{V}_2 \cup \mathcal{V}_3} \quad |||\Psi^{23}_{\mathcal{U}}(x)||| = \|x\|_{\mathcal{U}_\Lambda}.$$

By the construction and the isomorphic properties of Ψ_{p_0}, $\Psi^{23}_{\mathcal{U}}$ is an isometric order isomorphism from $\mathcal{F}^{p_0}_{\mathcal{V}_2\cup\mathcal{V}_3}$ onto $\Psi^{23}_{\mathcal{U}}(\mathcal{F}^{p_0}_{\mathcal{V}_2\cup\mathcal{V}_3})$ and hence it can be extended by continuity to another isometric order isomorphism (again denoted by $\Psi^{23}_{\mathcal{U}}$) between the respective completions. But the completion of $\mathcal{F}^{p_0}_{\mathcal{V}_2\cup\mathcal{V}_3}$ is $P^2_{\mathcal{U}_\Lambda}(\mathcal{U}_\Lambda)\oplus P^3_{\mathcal{U}_\Lambda}(\mathcal{U}_\Lambda)$ by Freudenthal's spectral theorem, $\mathcal{U}_\Lambda$ being order continuous (Lemma 3). In this way, $P^2_{\mathcal{U}_\Lambda}(\mathcal{U}_\Lambda)$ and $P^3_{\mathcal{U}_\Lambda}(\mathcal{U}_\Lambda)$ are isometric to certain atomic Köthe space $K(\Omega_2)$ and certain Köthe function space $X(\Omega_3)$ defined on Ω_2 and Ω_3 respectively (the completion of $\Psi^{23}_{\mathcal{U}}(\mathcal{F}^{p_0}_{\mathcal{V}_i})$, $i=2,3$ (where $\mathcal{F}^{p_0}_{\mathcal{V}_i}$ is the linear span of $\bigcup\{C(u^v)\mid v\in\mathcal{V}_i\}$, $i=2,3$) which can be identified with a subspace of $L^0(\Omega_2\cup\Omega_3,\mathcal{M}_{\mathcal{U}},\mu_{\mathcal{U}})$).

As Ψ_{p_0} is an order isomorphism, every set $Supp(\Psi_{p_0}(u^v))$, $v\in\mathcal{V}_2$ is an atom in $(\Omega_{\mathcal{U}},\mathcal{M}_{\mathcal{U}},\mu_{\mathcal{U}})$ and $\mu_{\mathcal{U}}\ (Supp(\Psi_{p_0}(u^v)))=\|u^v\|^{p_0}_{\mathcal{U}_{p_0}}=1$ (remember lemmata 7 and 5). Then $(\Omega_2,\mathcal{M}_{\mathcal{U}},\mu_{\mathcal{U}})$ is a purely atomic space. That $K(\Omega_2)$ is an intermediate space of the couple $(\ell^{p_0}(\Omega_2),\ell^{p_1}(\Omega_2))$ follows from Propositions 8 and 9 and Theorem 11.

Finally we prove that $P^1_{\mathcal{U}_\Lambda}(\mathcal{U}_\Lambda)$ is isomorphic to the interpolation space $(L^{p_0}(\Omega_1,\mu_{\mathcal{U}}),L^{p_1}(\Omega_1,\mu_{\mathcal{U}}))_{\theta,q}$. Take a family $(\varepsilon_d)_{\mathcal{U}}$ such that $\lim_{d,\mathcal{U}}\varepsilon_d=0$ and $\varepsilon_d<1$ for each $d\in\mathfrak{D}$ (remember that $\mathcal{U}$ is countably incomplete). Let $(f_d)_{\mathcal{U}_\Lambda}\in\mathcal{U}_\Lambda$. For every $d\in\mathfrak{D}$ and every $n\in\mathbb{Z}$ there is a representation

$$f_d=f_0^{nd}+f_1^{nd}\quad\forall\, n\in\mathbb{Z}\tag{17}$$

such that

$$\max_{j=0,1}\left(\sum_{n\in\mathbb{Z}}e^{(j-\theta)nq}\|(f_j^{nd})\|_{L^{p_j}(\Omega_d)}\right)^{\frac{1}{q}}\le\|f_d\|_{\lambda_d}+\varepsilon_d.\tag{18}$$

Since $\sup_{d\in\mathfrak{D}}\|f_d\|_{\lambda_d}<\infty$, using (18) we obtain for every $n\in\mathbb{Z}$ and $j=0,1$

$$\begin{aligned}\sup_{d\in\mathfrak{D}}\|(f_j^{nd})\|_{L^{p_j}(\Omega_d)}&\le e^{-(j-\theta)n}\left(\sum_{k\in\mathbb{Z}}e^{(j-\theta)kq}\|(f_j^{nd})\|_{L^{p_j}(\Omega_d)}\right)^{\frac{1}{q}}\\&\le e^{-(j-\theta)n}\left(\sup_{d\in\mathfrak{D}}\|f_j^{nd}\|_{\lambda_d}+1\right)<\infty\end{aligned}$$

and as a consequence $(f_j^{nd})_{\mathcal{U}}\in\mathcal{U}_{p_j}$.

Then given $(f_d)_{\mathcal{U}_\Lambda}\in\mathcal{U}_\Lambda$ we can choose $(f_j^d)_{\mathcal{U}_{p_j}}\in\mathcal{U}_{p_j}$, $j=0,1$ such that $(f_d)_{\mathcal{U}_\Lambda}=(f_0^d+f_1^d)_{\mathcal{U}_\Lambda}$. We define

$$\begin{aligned}\forall\,(f_d)_{\mathcal{U}_\Lambda}\in\mathcal{U}_\Lambda\quad\Psi^1_{\mathcal{U}}((f_d)_{\mathcal{U}_\Lambda})&=\Psi^1_{\mathcal{U}}((f_0^d+f_1^d)_{\mathcal{U}_\Lambda})\\&=\Psi_{p_0}(P^1_{\mathcal{U}_{p_0}}((f_0^d)_{\mathcal{U}_{p_0}}))+\Psi_{p_1}(P^1_{\mathcal{U}_{p_1}}((f_1^d)_{\mathcal{U}_{p_1}})).\end{aligned}\tag{19}$$

Claim 1. *The definition of* $\Psi^1_{\mathcal{U}}((f^d)_{\mathcal{U}_\Lambda})$ *does not depend on the selected decomposition for* $(f^d)_{\mathcal{U}_\Lambda}$.

Proof. Suppose

$$0 = \left(f_0^d + f_1^d\right)_{\mathcal{U}_\Lambda} \tag{20}$$

with $(f_j^d)_{\mathcal{U}_{p_j}} \in \mathcal{U}_{p_j}$, $j = 0, 1$. Having in mind the order continuity of every $\mathcal{U}_{p_j}$, $j = 0, 1$ to finish the proof of the claim it is enough to see that for every $v \in \mathcal{V}_1$ the equality

$$0 = \Psi_{p_0}(P_{u^v}((f_0^d)_{\mathcal{U}_{p_0}})) + \Psi_{p_1}(P_{w^v}((f_1^d)_{\mathcal{U}_{p_1}}))$$

holds. To see that, define

$$A_0 := \left\{t \in \Omega_1 \mid \Psi_{p_0}(P_{u^v}((f_0^d)_{\mathcal{U}_{p_0}}))(t) + \Psi_{p_1}(P_{w^v}((f_1^d)_{\mathcal{U}_{p_1}}))(t) > 0\right\}.$$

Assume $\mu_{\mathcal{U}}(A_0) > 0$. Then there would be a measurable set $A \in \mathcal{O}_v$ and a number $\delta > 0$ such that $A \subset A_0$, $0 < \mu_{\mathcal{U}}(A) < \infty$ and

$$\delta\chi_A \leq (\Psi_{p_0}(P_{u^v}((f_0^d)_{\mathcal{U}_{p_0}})) + \Psi_{p_1}(P_{w^v}((f_1^d)_{\mathcal{U}_{p_1}})))\chi_A =$$

and by Theorem 11

$$\begin{aligned} &= \Psi_{p_1}(P_{w^v}((f_0^d)_{\mathcal{U}_{p_1}})) + \Psi_{p_1}(P_{w^v}((f_1^d))_{\mathcal{U}_{p_1}}))\chi_A \\ &= \Psi_{p_1}(P_{w^v}((f_0^d + f_1^d)_{\mathcal{U}_{p_1}}))\chi_A. \end{aligned} \tag{21}$$

We obtain $0 \neq \Psi_{p_1}^{-1}(\chi_A) \in \mathcal{U}_{p_1}$ since Ψ_{p_1} is an isomorphism. On the other hand (20) implies

$$0 = P_{z^v}((f_0^d + f_1^d)_{\mathcal{U}_\Lambda}),$$

and taking images by $I_{\Lambda p_1}$

$$0 = P_{w^v}((f_0^d + f_1^d)_{\mathcal{U}_{p_1}}).$$

If $\gamma := \Psi_{p_1}^{-1}(\chi_A) \in C(w^v) \subset \mathcal{U}_{p_1}$ we have

$$P_\gamma P_{w^v}((f_0^d + f_1^d)_{\mathcal{U}_{p_1}}) \wedge \Psi_{p_1}^{-1}(\chi_A) = 0 \wedge \Psi_{p_1}^{-1}(\chi_A) = 0.$$

But, Ψ_{p_1} being an order isomorphism and using (21) and (1)

$$\Psi_{p_1}\left(P_\gamma P_{w^v}((f_0^d + f_1^d))_{\mathcal{U}_{p_1}}) \wedge \Psi_{p_1}^{-1}(\chi_A)\right)$$

$$= \chi_A \Psi_{p_1}(P_{w^v}((f_0^d + f_1^d)_{\mathcal{U}_{p_1}})) \wedge \chi_A \geq \delta\chi_A \wedge \chi_A = \min\{1, \delta\}\chi_A > 0,$$

a contradiction. Analogously we can prove that, if

$$B_0 := \left\{t \in \Omega \mid \Psi_{p_0}(P_{u^v}((f_0^d)_{\mathcal{U}_{p_0}}))(t) + \Psi_{p_1}(P_{w^v}((f_1^d)_{\mathcal{U}_{p_1}}))(t) < 0\right\}$$

necessarily we have $\mu_{\mathcal{U}}(B_0) = 0$. Hence $\Psi^1_{\mathcal{U}}$ is well defined.

Claim 2. *$\Psi^1_{\mathcal{U}}$ is an isomorphism from $P^1_{\mathcal{U}_\Lambda}(\mathcal{U}_\Lambda)$ onto the interpolation space $(L^{p_0}(\Omega_1), L^{p_1}(\Omega_1))_{\theta,q}$.*

Proof. Choose representations of type (18). By claim 1

$$\forall h \in \mathbb{Z} \quad \Psi^1_{\mathcal{U}}((f_d)_{\mathcal{U}_\Lambda}) = \Psi_{p_0}(P^1_{\mathcal{U}_{p_0}}((f_0^{hd})_{\mathcal{U}_{p_0}})) + \Psi_{p_1}(P^1_{\mathcal{U}_{p_1}}((f_1^{hd})_{\mathcal{U}_{p_1}})).$$

Then

$$\|\Psi^1_{\mathcal{U}}((f_d)_{\mathcal{U}_\Lambda})\| \le \max_{j=0,1} \left(\sum_{h\in\mathbb{Z}} e^{(j-\theta)hq} \|\Psi_{p_j}(P^1_{\mathcal{U}_{p_j}}((f_j^{hd})_{\mathcal{U}_{p_j}}))\|^q_{L^{p_j}(\Omega_1)} \right)^{\frac{1}{q}} \le$$

and Ψ_{p_0}, Ψ_{p_1} being isometries

$$\begin{aligned}
&\le \max_{j=0,1} \left(\sum_{h\in\mathbb{Z}} e^{(j-\theta)hq} \|P^1_{\mathcal{U}_{p_j}}((f_j^{hd})_{\mathcal{U}_{p_j}})\|^q_{\mathcal{U}_{p_j}} \right)^{\frac{1}{q}} \\
&\le \max_{j=0,1} \|P^1_{\mathcal{U}_{p_j}}\| \left(\sum_{h\in\mathbb{Z}} e^{(j-\theta)hq} \lim_{d,\mathcal{U}} \|f_j^{hd}\|^q_{L^{p_j}(\Omega_d)} \right)^{\frac{1}{q}} \\
&= \max_{j=0,1} \lim_{k\to\infty} \left(\lim_{d,\mathcal{U}} \sum_{|h|\le k} e^{(j-\theta)hq} \|f_j^{hd}\|^q_{L^{p_j}(\Omega_d)} \right)^{\frac{1}{q}} \qquad (22)\\
&\le \max_{j=0,1} \left(\lim_{d,\mathcal{U}} \sum_{h\in\mathbb{Z}} e^{(j-\theta)hq} \|f_j^{hd}\|^q_{L^{p_j}(\Omega_d)} \right)^{\frac{1}{q}} \\
&\le \lim_{d,\mathcal{U}} (\|f_d\|_{\lambda_d} + \varepsilon_d) = \|(f_d)_{\mathcal{U}_\Lambda}\|_{\mathcal{U}_\Lambda}
\end{aligned}$$

and Ψ^1_Λ becomes continuous.

Conversely, given $\varepsilon > 0$ and $f \in (L^{p_0}(\Omega_1, \mu_{\mathcal{U}}), L^{p_1}(\Omega_1, \mu_{\mathcal{U}}))_{\theta,q}$ there is a sequence $\{f^h\}_{h\in\mathbb{Z}} \subset L^{p_0}(\Omega_1, \mu_{\mathcal{U}}) \cap L^{p_1}(\Omega_1, \mu_{\mathcal{U}})$, such that

$$f = \sum_{h\in\mathbb{Z}} f^h \qquad (23)$$

in $(L^{p_0}(\Omega_1, \mu_{\mathcal{U}}), L^{p_1}(\Omega_1, \mu_{\mathcal{U}}))_{\theta,q}$ (recall that $1 \le q < \infty$ and proposition 3, chapter II in [2]) and

$$\max_{j=0,1} \left(\sum_{h\in\mathbb{Z}} e^{(j-\theta)hq} \|f^h\|^q_{L^{p_j}(\Omega)} \right)^{\frac{1}{q}} \le \|f\| + \varepsilon. \qquad (24)$$

We can suppose that f and each f^h, $h \in \mathbb{Z}$ are defined in all $\Omega_1 \cup \Omega_2 \cup \Omega_3$ and vanishes in $\Omega_2 \cup \Omega_3$.

Let us see that the series

$$\sum_{h\in\mathbb{Z}} G_{p_0\Lambda} P^1_{\mathcal{U}_{p_0}} \Psi^{-1}_{p_0}(f^h) \tag{25}$$

is convergent in the Banach space $P^1_{\mathcal{U}_\Lambda}(\mathcal{U}_\Lambda)$. Remark that, by our assumptions on every $f^h, h \in \mathbb{Z}$ and by Theorem 11, if $(f^h_d)_{\mathcal{U}_{p_0}} := P^1_{\mathcal{U}_{p_0}} \Psi^{-1}_{p_0}(f^h) = \Psi^{-1}_{p_0}(f^h)$, we have $f^h_d \in L^{p_0}(\Omega_d) \cap L^{p_1}(\Omega_d)$ for every $d \in \mathfrak{D}$. Then, by the convergence of the series of (24), given $\delta > 0$ there is a finite set $H_0 \subset \mathbb{Z}$ such that for every *finite* set $H \subset \mathbb{Z}$ such that $H \cap H_0 = \emptyset$ we have

$$\begin{aligned}
\left\| \sum_{h\in H_0} G_{p_0\Lambda}((f^h_d)_{\mathcal{U}}) \right\|_{\mathcal{U}_\Lambda} &= \left\| \left(\sum_{h\in H_0} f^h_d \right)_{\mathcal{U}_\Lambda} \right\|_{\mathcal{U}_\Lambda} \\
&= \lim_{d,\mathcal{U}} \left\| \sum_{h\in H_0} f^d_h \right\|_{\lambda_d} \le \lim_{d,\mathcal{U}} \max_{j=0,1} \left(\sum_{h\in H_0} e^{(j-\theta)hq} \| f^h_d \|^q_{L^{p_j}(\Omega_d)} \right)^{\frac{1}{q}} \\
&\le \max_{j=0,1} \left(\sum_{h\in H_0} e^{(j-\theta)hq} \lim_{d,\mathcal{U}} \| f^h_d \|^q_{L^{p_j}(\Omega_d)} \right)^{\frac{1}{q}} \\
&= \max_{j=0,1} \left(\sum_{h\in H_0} e^{(j-\theta)hq} \| (f^h_d)_{\mathcal{U}_{p_j}} \|^q_{\mathcal{U}_{p_j}} \right)^{\frac{1}{q}} \\
&= \max_{j=0,1} \left(\sum_{h\in H_0} e^{(j-\theta)hq} \| \Psi_{p_j}((f^h_d)_{\mathcal{U}_{p_j}}) \|_{L^{p_j}(\Omega_1)})^q \right)^{\frac{1}{q}} \\
&= \max_{j=0,1} \left(\sum_{h\in H_0} e^{(j-\theta)hq} \| f^h \|^q_{L^{p_0}(\Omega_1)} \right)^{\frac{1}{q}} \le \delta.
\end{aligned}$$

Then, (25) is convergent in $\mathcal{U}_\Lambda$ and with similar computations we get

$$\left\| \sum_{h\in\mathbb{Z}} (f^h_d)_{\mathcal{U}_\Lambda} \right\| \le \| f \| + \varepsilon. \tag{26}$$

Now we see that the sum of series (25) is independent on the selected sequence $\{f^h\}_{h\in\mathbb{Z}}$ in (23) and (24). Let $\{\overline{f}^h\}_{h\in\mathbb{Z}}$ be another sequence in $L^{p_0}(\Omega_1)\cap L^{p_1}(\Omega_1)$ for which (23) holds. Assume

$$\varphi := \left(\sum_{h\in\mathbb{Z}} G_{p_0\Lambda} P^1_{\mathcal{U}_{p_0}} \Psi^{-1}_{p_0}(f^h - \overline{f}^h)\right)^+ > 0.$$

By Freudenthal's spectral theorem, there is $v \in \mathcal{V}_1$, $x \in C(z^v)$ and some $\delta > 0$ such that $0 < x \leq \delta\ \varphi$. Clearly $\varphi \in P^1_{\mathcal{U}_\Lambda}(\mathcal{U}_\Lambda)$ and hence $x \in P^1_{\mathcal{U}_\Lambda}(\mathcal{U}_\Lambda)$ and $G_{\Lambda p_1}(x) \leq G_{\Lambda p_1}(\varphi)$. As $\Psi^1_{\mathcal{U}}$ is continuous, by definition of $\mathcal{V}_1$ and by Theorem 11 we have

$$0 \neq \Psi_{p_1}(G_{\Lambda p_1}(x)) = \Psi^1_{\mathcal{U}}(x) \leq \delta\ \Psi^1_{\mathcal{U}}(\varphi)$$
$$= \delta \left(\sum_{h\in\mathbb{Z}} \Psi^1_{\mathcal{U}} \left(G_{p_0\Lambda} P^1_{\mathcal{U}_{p_0}} \Psi^{-1}_{p_0}(f^h - \overline{f}^h)\right)\right)^+ = \delta\ \left(\sum_{h\in\mathbb{Z}} \Psi^1_{\mathcal{U}} \Psi^{-1}_{p_1}(f^h - \overline{f}^h)\right)^+$$
$$= \delta\ \left(\sum_{h\in\mathbb{Z}} \Psi_{p_1}\Psi^{-1}_{p_1}(f^h - \overline{f}^h)\right)^+ = \delta \left(\sum_{h\in\mathbb{Z}}(f^h - \overline{f}^h)\right)^+ = 0$$

a contradiction. Then $\varphi = 0$ and an analogous computation gives us the negative part is also 0.

Once (25) is well defined, $\varepsilon > 0$ being arbitrary in (26), we get

$$\left\| \sum_{h\in\mathbb{Z}} (f^h_d)_{\mathcal{U}_\Lambda} \right\| \leq \| f \|. \tag{27}$$

As $\Psi^1_{\mathcal{U}}$ is continuous, by (19)

$$\Psi^1_{\mathcal{U}} \left(\sum_{h\in\mathbb{Z}} (f^h_d)_{\mathcal{U}_\Lambda}\right) = \sum_{h\in\mathbb{Z}} \Psi^1_{\mathcal{U}}((f^h_d)_{\mathcal{U}_\Lambda}) = \sum_{h\in\mathbb{Z}} \Psi_{p_0}\Psi^{-1}_{p_0}(f^h) = \sum_{h\in\mathbb{Z}} f^h = f.$$

Then by (27) and (22) we get the continuity of $(\Psi^1_{\mathcal{U}})^{-1}$ and the bijectivity of $\Psi^1_{\mathcal{U}}$. Finally, defining $\Psi_{\mathcal{U}} = \Psi^1_{\mathcal{U}} P^1_{\mathcal{U}_\Lambda} + \Psi^{23}_{\mathcal{U}}(P^2_{\mathcal{U}_\Lambda} + P^3_{\mathcal{U}_\Lambda})$ the proof is complete. □

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