BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 37(2), 217-239

Injectivity of differentiable maps $\mathbb{R}^2 \to \mathbb{R}^2$ at infinity

Carlos Gutierrez and Roland Rabanal

Abstract. The main result given in Theorem 1.1 is a condition for a map X, defined on the complement of a disk D in $\mathbb{R}^2$ with values in $\mathbb{R}^2$, to be extended to a topological embedding of $\mathbb{R}^2$, not necessarily surjective. The map X is supposed to be just differentiable with the condition that, for some $\epsilon > 0$, at each point the eigenvalues of the differential do not belong to the real interval $(-\epsilon, \infty)$. The extension is obtained by restricting X to the complement of some larger disc. The result has important connections with the property of asymptotic stability at infinity for differentiable vector fields.

Keywords: injectivity, reeb component, continuous vector fields.

Mathematical subject classification: Primary: 26B99, 58C25; Secondary: 37E30, 37C10.

1 Introduction

Given an open subset U of $\mathbb{R}^2$ and a differentiable (not necessarily of class C^1) map $X : U \to \mathbb{R}^2$, we shall denote by $\text{Spec}(X)$ the set of all eigenvalues of the derivative DX_z, when z varies in U.

Our main result is the following

Theorem 1.1. *Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable (but not necessarily C^1) map, where $\sigma > 0$ and $\overline{D}_\sigma = \{z \in \mathbb{R}^2 : ||z|| \leq \sigma\}$. If for some $\epsilon > 0$, $\text{Spec}(X) \cap (-\epsilon, +\infty) = \emptyset$, then there exists $s \geq \sigma$ such that $X|_{\mathbb{R}^2 \setminus \overline{D}_s}$ can be extended to a globally injective local homeomorphism $\widetilde{X} = (\widetilde{f}, \widetilde{g}) : \mathbb{R}^2 \to \mathbb{R}^2$.*

This theorem generalizes Gutierrez and Sarmiento injectivity work [15] who proved the corresponding C^1 version.

Received 11 May 2005.

The map $\widetilde{X}$ of Theorem 1.1 is not necessarily a homeomorphism of $\mathbb{R}^2$; it is a differentiable embedding, the image of which may be properly contained in $\mathbb{R}^2$.

Theorem 1.1 is valid for vector fields X such that $\mathrm{Spec}(X) \cap (-\infty, \epsilon) = \emptyset$. In fact, if in Theorem 1.1 we change the pair $(X, (\epsilon, \infty))$ by $(-X, (-\infty, \epsilon))$, we may see that its conclusions remain valid. Also, if $A \colon \mathbb{R}^2 \to \mathbb{R}^2$ is an arbitrary invertible linear map, then Theorem 1.1 applies to the map $A \circ X \circ A^{-1}$.

Throughout this article, given a topological circle $C \subset \mathbb{R}^2$, the compact disc (resp. open disc) bounded by C, will be denoted by $\overline{D}(C)$ (resp. $D(C)$). The boundary of any set A will be denoted by ∂A.

Let us proceed to give an idea of the proof of this result. First it will be observed that the assumptions imply that the Local Inverse Function Theorem is true. As a consequence, the level curves $\{f = \text{constant}\}$ (resp. $\{g = \text{constant}\}$) make up a C^0-foliation $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) on the plane, without singularities, such that every leaf L of $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) is a differentiable curve and $g|_L$ (resp. $f|_L$) is strictly monotone; in particular $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are (topologically) transversal to each other.

We will need:

Theorem 2.1. *Let* $Y \colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a local homeomorphism such that, for some* $s > 0$, $Y|_{\mathbb{R}^2 \setminus D_s}$ *is differentiable. If there exists* $\epsilon > 0$ *such that, for all* $p \in \mathbb{R}^2 \setminus D_s$, *no eigenvalue of* DY_p *belongs to* $(-\epsilon, \epsilon)$, *then* Y *is injective.*

To prove Theorem 2.1, it will be seen that the foliation $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) is topologically equivalent to the foliation, on the (x, y)-plane, induced by the form dx (this foliation is made up by all the vertical straight lines). The injectivity of Y will follow from the fact that $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are topologically transversal everywhere.

Sections 3 and 4 are devoted to prove

Proposition 4.7. *Let* $X = (f, g) \colon \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ *be a differentiable map as in Theorem* 1.1. *There exists a topological circle* C *such that* $\mathcal{F}(f)$, *restricted to* $\mathbb{R}^2 \setminus D(C)$, *is topologically equivalent to the foliation, on* $\mathbb{R}^2 \setminus D_1$, *induced by* dx.

Observe that the foliation, on $\mathbb{R}^2 \setminus D_1$, induced by dx has exactly two tangencies with $\partial \overline{D}_1$ (at $(1, 0)$ and $(-1, 0)$) which are "generic" and "external". Let us say a little more about what is proved in Section 3 and 4:

We show, in Section 3, that given a topological cirle $C_1 \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ surrounding the origin, and having "generic" contact with $\mathcal{F}(f)$, the number of "external" tangencies of $\mathcal{F}(f)$ with C_1 is equal to 2 plus the number of "internal" tangencies of $\mathcal{F}(f)$ with C_1. We show, in Section 4, that the circle C_1 can be deformed to a new topological circle C_2 so that the referred "external" and "internal" tangencies cancel in pairs yielding exactly 2 tangencies which are "external".

The proof of Theorem 1.1 is finished in Section 5. First, it will be seen that, under conditions of Proposition 4.7, the circle C can be deformed so that, for the resulting new circle, still denoted by C:

(i) $\mathcal{F}(f)|_{\mathbb{R}^2\setminus D(C)}$, is topologically equivalent to the foliation, on $\mathbb{R}^2 \setminus D_1$, induced by dx;

(ii) X takes C homeomorphically to a circle; and

(iii) $X|_{\mathbb{R}^2\setminus D(C)}$ can be extended to a local homeomorphism $\widetilde{X}\colon \mathbb{R}^2 \to \mathbb{R}^2$.

Under these conditions, the proof of Theorem 1.1 is obtained by using Theorem 2.1

Concerning injectivity of maps $\mathbb{R}^n \mapsto \mathbb{R}^n$ (globally defined) we wish to mention the following results:

(1) Fernandes–Gutierrez–Rabanal [11] proved that if $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable (but not necessarily C^1) map and, for some $\epsilon > 0$, $\mathrm{Spec}(X) \cap [0, \epsilon) = \emptyset$, then X is injective. See also [8], [13], [14], [16]. Under additional assumptions, there is an extension of this result for maps from $\mathbb{R}^n$ to itself (See [12, Theorem 1]).

(2) Pinchuck [23] proved that there are non-injective polynomial maps X : $\mathbb{R}^2 \to \mathbb{R}^2$ such that $0 \notin \mathrm{Spec}(X)$. Also Smith and Xavier ([28], Theorem 4) proved that there exist integers $n > 2$ and non-injective polynomial maps $P\colon \mathbb{R}^n \to \mathbb{R}^n$ with $\mathrm{Spec}(P) \cap [0, +\infty) = \emptyset$.

(3) C. Olech [20] proved the existence of a strong connection between the injectivity of C^1 maps, from $\mathbb{R}^2$ into itself, and global asymptotic stability of C^1 planar vector fields (see also [21]) . In a similar way, if V is a neighborhood of ∞ in the Riemann Sphere $\mathbb{R}^2 \cup \{\infty\}$, the results of this work is used to prove the existence of a sufficient condition that imply that a vector field $X\colon (V, \infty) \to (\mathbb{R}^2, 0)$, which is differentiable in $V \setminus \{\infty\}$ but not necessarily continuous at ∞, has ∞ as an attracting or a repelling singularity (see [15] and [17]). Moreover, the methods used in this work are related to those used in the study of planar vector fields (see [7], [9], [26], [22], [10], [6]).

The structure of the proof of our main result is similar to that of [15]. Nevertheless, most of the arguments had to be reconstructed. The basic difficulty was that, in our case, the eigenvalues of DX_p do not depend continuously on p. In this respect, we mention below some of the facts that were used in a very important way for the C^1-case and were not available for the differentiable case

1. the assumption $\mathrm{Spec}(Y) \cap (-\epsilon, \epsilon) = \emptyset$ in Theorem 2.1 is open in the C^1-Whitney topology; this allowed the map Y to be C^1 approximated by a smooth map $\widetilde{Y} = (\widetilde{f}, \widetilde{g})$ such that $\mathrm{Spec}(\widetilde{Y}) \cap (-\epsilon/2, \epsilon/2) = \emptyset$ and $(\widetilde{f}_x, \widetilde{f}_y)$ had generic contact with the vertical foliation;

2. the Hamiltonian vector field $X_f = (-f_y, f_x)$, of a C^1 vector field $X = (f, g)$, was continuous and so its index along a circle was well defined.

Before continuing, the authors wish to express their gratitude to the referee whose comments have been appreciated and incorporated into this work.

2 Global injectivity result

This section is devoted to prove the following:

Theorem 2.1. *Let $Y = (f, g) \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism such that, for some $s > 0$, $Y|_{\mathbb{R}^2 \setminus D_s}$ is differentiable. If there exists $\epsilon > 0$ such that, for all $p \in \mathbb{R}^2 \setminus D_s$, no eigenvalue of DY_p belongs to $(-\epsilon, \epsilon)$, then Y is injective.*

Theorem 2.1 improves the main injectivity result of [8]. The proof of this theorem will be completed throughout this section; to this end we shall use the following Černavskii's Theorem [4], [5] (see also [29] and [25]).

Theorem 2.2. *Let U be an open subset of $\mathbb{R}^2$ and $X = (f, g) \colon U \to \mathbb{R}^2$ be a differentiable map such that, for all $p \in U$, DX_p is non-singular. Then, for all $p \in U$, there exists a neighborhood $V = V(p)$ and $\varepsilon = \varepsilon(p) > 0$ such that $X|_V \colon V \to (f(p) - \varepsilon, f(p) + \varepsilon) \times (g(p) - \varepsilon, g(p) + \varepsilon)$ is a differentiable homeomorphism whose inverse $(X|_V)^{-1}$ is also differentiable.*

As a consequence of this *Inverse Mapping Theorem* we obtain:

Corollary 2.3. *Let $X = (f, g) \colon U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map such that, for all $p \in U$, DX_p is non-singular. Then the level curves $\{f = \text{constant}\}$ (resp. $\{g = \text{constant}\}$) make up a C^0-foliation $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) on $U \subset \mathbb{R}^2$, without singularities, such that every leaf L of $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) is a differentiable curve and $g|_L$ (resp. $f|_L$) is strictly monotone; in particular $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are (topologically) transversal to each other.*

Orient $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) so that if L is an oriented leaf of $\mathcal{F}(f)$ (resp.$\mathcal{F}(g)$) then $g|_L$ (resp. $f|_L$) is an increasing function in conformity with the orientation of L.

Now, we introduce the notion of *half-Reeb component* for $\mathcal{F}(f)$. Let $h_0(x, y) = xy$ and consider the set

$$B = \Big\{(x, y) \in [0, 2] \times [0, 2] \colon 0 < x + y \leq 2\Big\}.$$

Definition 2.4. *Let $X = (f, g) \colon U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism. Given $h \in \{f, g\}$, we will say that $\mathcal{A} \subset U$ is a half-Reeb component for $\mathcal{F}(h)$ (or simply a hRc for $\mathcal{F}(h)$) if there is a homeomorphism $H \colon B \to \mathcal{A}$ which is a topological equivalence between $\mathcal{F}(h)|_{\mathcal{A}}$ and $\mathcal{F}(h_0)|_B$ such that:*

(1) *The segment $\{(x, y) \in B \colon x + y = 2\}$ is sent by H onto a transversal section for the foliation $\mathcal{F}(h)$ in the complement of the point $H(1, 1)$; this section is called the compact edge of $\mathcal{A}$.*

(2) *Both segments $\{(x, y) \in B \colon x = 0\}$ and $\{(x, y) \in B \colon y = 0\}$ are sent by H onto full half-trajectories of $\mathcal{F}(h)$. These two semi-trajectories of $\mathcal{F}(h)$ are called the non–compact edges of $\mathcal{A}$.*

Observe that $\mathcal{A}$ may not be a closed subset of $\mathbb{R}^2$.

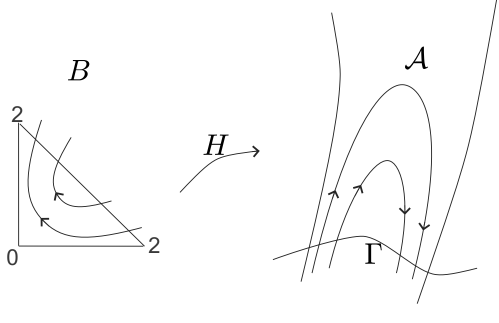

Figure 1: A half-Reeb component.

For each $\theta \in \mathbb{R}$ let R_θ denote the linear rotation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The following proposition will be needed. For the proof we refer the reader to [11, Proposition 1.5].

Proposition 2.5. *Let $Y = (f, g) \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism such that $\mathcal{F}(f)$ has a hRc $\mathcal{A}$. Let $(f_\theta, g_\theta) = R_\theta \circ Y \circ R_{-\theta}$, $\theta \in \mathbb{R}$. If $\Pi(\mathcal{A})$ is* *bounded, where* $\Pi\colon \mathbb{R}^2 \to \mathbb{R}$ *is given by* $\Pi(x, y) = x$*, then there is an* $\epsilon > 0$ *such that, for all* $\theta \in (-\epsilon, 0) \cup (0, \epsilon)$*,* $\mathcal{F}(f_\theta)$ *has a hRc* $\mathcal{A}_\theta$ *for which* $\Pi(\mathcal{A}_\theta)$ *is an interval of infinite length.*

The proof of the following lemma can be found in [3] (see also [18] and [27]).

Lemma 2.6. *Let* I *be a bounded interval of* $\mathbb{R}$ *and* $H\colon I \to \mathbb{R}$ *be a bounded measurable function. If* A *denote the set of* $x \in I$ *such that*

$$\lim_{h\to 0} \frac{|H(x+h) - H(x)|}{|h|} = +\infty.$$

Then A *is a (Lebesgue) measure–zero set.*

We will need the following proposition.

Proposition 2.7. *Let* $\sigma > 0$ *and* $X = (f, g)\colon \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ *be a differentiable map such that for some* $\epsilon > 0$*,* $\mathrm{Spec}(X) \cap (-\epsilon, \epsilon) = \emptyset$*. Then,*

(i) *any half-Reeb component of either* $\mathcal{F}(f)$ *or* $\mathcal{F}(g)$ *is a bounded subset of* $\mathbb{R}^2$*;*

(ii) *when* X *extends to a local homeomorphism* $\tilde{X} = (\tilde{f}, \tilde{g}) : \mathbb{R}^2 \to \mathbb{R}^2$*,* $\mathcal{F}(\tilde{f})$ *and* $\mathcal{F}(\tilde{g})$ *have no hRc's.*

Proof. Consider only the case (i). Suppose by contradiction that $\mathcal{F}(f)$ has an unbounded half-Reeb component $\mathcal{A}$. By composing with a linear rotation if necessary (see Proposition 2.5 and its notation) we may assume that $\Pi(\mathcal{A})$ is an unbounded interval. To simplify matters, let us suppose that $[b, +\infty) \subset \Pi(\mathcal{A})$. Then, if $a > b$ is enough large,

(a) for any $x \geq a$, the vertical line $\Pi^{-1}(x)$ intersects exactly one trajectory $\alpha_x \subset \mathcal{A}$ of $\mathcal{F}(f)|_{\mathcal{A}}$ such that $\Pi(\alpha_x) \cap (x, +\infty) = \emptyset$. In other words, x is the maximum for the restriction $\Pi|_{\alpha_x}$.

As α_x is a continuous curve, it follows that; if $x \geq a$, $\alpha_x \cap \Pi^{-1}(x)$ is a compact subset of $\mathcal{A}$.

Let $H\colon (a, +\infty) \to \mathbb{R}$ be defined by

$$H(x) = \sup\left\{y\colon (x, y) \in \alpha_x \cap \Pi^{-1}(x)\right\}.$$

When $x \geq a$ is kept fixed, every $q \in \Pi^{-1}(x) \cap \alpha_x$ is a local extremal of the differentiable function $(x, y) \mapsto f(x, y)$. Thus

(b) if $x \geq a,\ f_y(x, H(x)) = 0$.

As $\mathcal{F}(f)$ is a C^0-foliation, we may obtain that the function

(c) $\varphi(x) = f(x, H(x))$ is strictly monotone and continuous which, when restricted to any interval $(a, b]$, is bounded; in particular, φ is differentiable a.e.

We claim that

(d) H is upper semicontinuous; thus, H is a measurable function.

In fact, suppose by contradiction that H is not upper semicontinuous at $x_0 > a$. As H restricted to (a, x_0+1) is bounded there exist $c \in \mathbb{R}$ and a sequence $x_n \to x_0$ such that $H(x_0) < c$ and $H(x_n) \to c$. However, as φ is continuous,

$$f(x_0, c) = \lim_{n\to\infty} f(x_n, H(x_n)) = \lim_{n\to\infty} \varphi(x_n) = \varphi(x_0) = f(x_0, H(x_0)).$$

This contradiction with the definition of H proves (d).

By (d) above, Lemma 2.6 and by the fact that φ is differentiable a.e., we obtain that if $a > 0$ is large enough, there exists a full measure subset M of $(a, +\infty)$ such that

(e) if $x \in M$, then φ is differentiable at x and

$$\liminf_{h\to 0} \frac{|H(x+h) - H(x)|}{|h|} < +\infty.$$

To proceed we shall only consider the case in which φ is strictly increasing. We claim that

(f) if $x \in M$, then $\varphi'(x) = f_x(x, H(x)) \geq \varepsilon$.

In fact, if $x \in M$, there is a sequence $h_n \to 0$ such that $\lim_{n\to\infty} \frac{k_n}{h_n} \in \mathbb{R}$ where $k_n = H(x+h_n) - H(x)$. Also, by (b), $f_y(x, H(x)) = 0$. Hence, as f is differentiable at $(x, H(x))$, there are real valued functions $\varepsilon_1, \varepsilon_2$ defined in a neighborhood of $(0, 0)$ such that

$$\begin{aligned} f(x+h_n, H(x)+k_n) \;=\; & f(x, H(x)) + f_x(x, H(x))h_n \\ & + \varepsilon_1(h_n, k_n)h_n + \varepsilon_2(h_n, k_n)k_n \end{aligned}$$

and $\lim_{n\to\infty} \varepsilon_1(h_n, k_n) = \lim_{n\to\infty} \varepsilon_2(h_n, k_n) = 0$. Therefore, when n is large enough,

$$\frac{\varphi(x+h_n)-\varphi(x)}{h_n} = f_x(x, H(x)) + \varepsilon_1(h_n, k_n) + \varepsilon_2(h_n, k_n)\frac{k_n}{h_n}$$

which implies that

$$\varphi'(x) = \lim_{n\to\infty} \frac{\varphi(x+h_n)-\varphi(x)}{h_n} = f_x(x, H(x)).$$

Therefore,

$$DX(x, H(x)) = \begin{pmatrix} \varphi'(x) & 0 \\ g_x(x, H(x)) & g_y(x, H(x)) \end{pmatrix}$$

i.e. $\varphi'(x)$ is an eigenvalue of $DX(x, H(x))$. By the assumption of the proposition and the fact that φ is a strictly increasing function, (f) is proved.

As $f|_{\mathcal{A}}$ is bounded, φ is bounded too. Hence, there is a constant $K > 0$ such that for all $x > a$, $0 \le \varphi(x) - \varphi(a) \le K$. Take $c > a$ so that $(c-a)\varepsilon > K$. Then we have that

$$K > \varphi(c) - \varphi(a) \ge \int_a^c \varphi'(x)dx \ge \int_a^c \varepsilon dx = (c-a)\varepsilon > K.$$

This contradiction proves the proposition. □

Let $a > 0$ and $\sigma, \gamma \colon [-a, a) \to \mathbb{R}^2$ be injective C^0-curves such that $\sigma(0) = \gamma(0) = 0$. We say that γ is *transversal* (resp. *tangent*) to σ at $\gamma(0) = \sigma(0)$, if there exist $\varepsilon > 0$, neighborhoods V of $\gamma(0)$ and U of $(0, 0)$, in $\mathbb{R}^2$, and a homeomorphism $H \colon V \to U$ such that for all $|t| < \varepsilon$, $H \circ \sigma(t) = (t, 0)$ and $H \circ \gamma(t) = (t, t)$ (resp. $H \circ \gamma(t) = (t, \phi(t))$, where $\phi(t) \ge 0$ and $\phi(0) = 0$). If γ is tangent to σ at $\gamma(0) = \sigma(0)$, we say that the tangency is *generic* if H and ϕ (as right above) can be taken so that $\phi(t) = |t|$. In particular, when $\sigma([-a, a)) = C$ is a topological circle in $\mathbb{R}^2 \setminus \overline{D}_\sigma$, we will say that the generic tangency in $p = \sigma(0) = \gamma(0)$ is *external* (resp. internal) if in the definition of generic we have that $\gamma(t) \in \mathbb{R}^2 \setminus D(C)$ (resp. $\gamma(t) \in \overline{D}(C)$) for all $0 < |t| < \epsilon$.

Now we prove the main result of this section.

Proof of Theorem 2.1. Suppose by contradiction that the map Y is not injective. Let $p_1 \neq p_2$ be points in $\mathbb{R}^2$, such that $Y(p_1) = Y(p_2)$. For $i = 1, 2$, let α_i denote the leaf of $\mathcal{F}(f)$ passing through p_i. As $g|_{\alpha_i}$ is strictly monotone and $g(p_1) = g(p_2)$, we obtain $\alpha_1 \cap \alpha_2 = \emptyset$. Let $\Omega(p_1, p_2)$ be the set of all the compact arcs Γ_1 embedded in the plane such that: (1) for $i = 1, 2$, Γ_1 meets α_i transversally at p_i; (2) all the tangencies of $\mathcal{F}(f)$ with Γ_1 are *generic*.

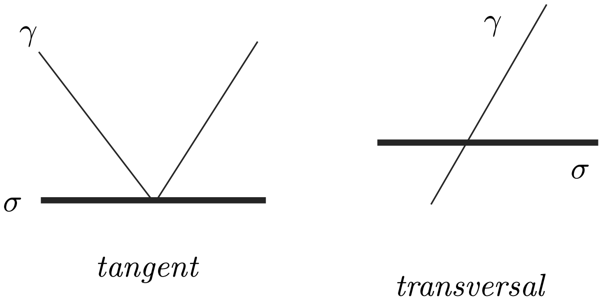

Figure 2

(a) Among all elements of $\Omega(p_1, p_2)$ take $\Gamma \in \Omega(p_1, p_2)$ which minimizes the number of (generic) tangencies with $\mathcal{F}(f)$.

We claim that:

(b) $\alpha_i \cap \Gamma = \{p_i\}, \quad$ for $i = 1, 2$.

If we assume, by contradiction, that $\alpha_1 \cap \Gamma$ contains properly $\{p_1\}$, we may find $q \in \Gamma \setminus \{p_1, p_2\}$ and a closed subinterval α of α_1, with endpoints p_1, q, such that $\alpha \cap \Gamma = \{p_1, q\}$. Let γ denote the connected component of $\Gamma \setminus \{q\}$ containing $\{p_2\}$. We can see that $\alpha \cup \gamma$ is an arc connecting p_1 and p_2 and also that $\mathcal{F}(f)$ is tangent to Γ at some point of $\Gamma \setminus (\gamma \cup \{p_1\} \cup \{q\})$. Under these conditions, we may approximate $\alpha \cup \gamma$ by an arc of $\Omega(p_1, p_2)$ which has less number of generic tangencies with $\mathcal{F}(f)$ than Γ. This contradiction with (a) proves (b).

As $f(p_1) = f(p_2)$, $\mathcal{F}(f)$ is tangent to Γ at some point $q \notin \{p_1, p_2\}$. All tangencies of $\mathcal{F}(f)$ with Γ are generic. Therefore, by looking at the trajectories of $\mathcal{F}(f)$ around q, we may see that there exist closed subintervals $[p, q]$, $[q, Tp]$ of Γ with $[p, q] \cap [q, Tp] = \{q\}$, and a homeomorphism $T : [p, q] \to [q, Tp]$ such that,

(c.1) $Tq = q$ and for every $x \in (p, q)$, there is an arc $[x, Tx]_f$ of $\mathcal{F}(f)$, starting at x, ending at Tx and meeting Γ exactly and transversally at $\{x, Tx\}$,

(c.2) the family $\{[x, Tx]_f : x \in (p, q)\}$ depends continuously on x and tends to the one point $\{q\}$ as $x \to q$.

From now on, suppose that

(d) $[p, q]$ is maximal with respect to properties (c.1) and (c.2) above.

Then, using (b) and the fact that $\mathcal{F}(f)$ has no half-Reeb components (see Proposition 2.7), we obtain $\{p, Tp\} \cap \{p_1, p_2\} = \emptyset$. We claim that

(e) there is no arc of trajectory $[p, Tp]_f$ of $\mathcal{F}(f)$ connecting p and Tp such that the family $\{[x, Tx]_f : x \in (p, q]\}$ approaches continuously to $[p, Tp]_f$ as x goes to p.

In fact, suppose that (e) is false. Then, by using (d) and the fact that $\mathcal{F}(f)$ has no half-Reeb components, we conclude $[p, Tp]_f$ is tangent to Γ at least at one of the points of $\{p, Tp\}$. Under these circumstances, it is not difficult to approximate the curve, which is the union of $[p, Tp]_f$ with $\Gamma \setminus ((p, q] \cup [q, Tp))$, by a curve $\Gamma_1 \in \Omega(p_1, p_2)$ which has less tangencies with $\mathcal{F}(f)$ than Γ. This contradiction with (a) proves (e). Therefore, the subinterval $[p, q] \cup [q, Tp]$ is the compact edge of a half-Reeb component of $\mathcal{F}(f)$ made up of two half trajectories of $\mathcal{F}(f)$ starting at p and Tp, respectively, together with the union of the arcs $[x, Tx]_f$, with $x \in (p, q]$. Thus we have found an unbounded half-Reeb component of $\mathcal{F}(f)$. This contradiction with Proposition 2.7 finishes the proof. $\square$

3 A local flow associated to $\mathcal{F}(f)$

Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable map such that for all $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, DX_p is non-singular (See Theorem 2.2). Let L_p be the connected component of the level curve $\{f = f(p)\}$ passing through p. Since $g|_{L_p}$ is strictly monotone, given $q \in L_p$ and $t = g(q) - g(p)$ we define $\varphi(t, p)$ as the unique point which is the intersection of L_p with the level curve $\{g = g(q)\}$. For each $p \in \mathbb{R}^2$, let $a_m(p) = \inf\{g(q) : q \in L_p\}$ and $a_M(p) = \sup\{g(q) : q \in L_p\}$. If $p \in \mathbb{R}^2$ and $t \in (a_m(p) - g(p), a_M(p) - g(p))$ then $\varphi(t, p)$ is well defined and determines a continuous local flow around any point of $\mathbb{R}^2$. This map φ will be called *the local flow associated to $\mathcal{F}(f)$.*

Proposition 3.1. *Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable map with* $\mathrm{Spec}(X) \cap [0, +\infty) = \emptyset$*. If $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ is a topological circle surrounding the origin , there exists $\varepsilon_0 > 0$ such that:*

(a) *the local flow φ associated to $\mathcal{F}(f)$ is defined in $(-\varepsilon_0, \varepsilon_0) \times C$.*

(b) *Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. If $u \in (-\varepsilon_0, 0) \cup (0, \varepsilon_0)$ and $Z_u = (A_u, B_u) : C \to \mathbb{S}^1$ is defined as*

$$Z_u(p) = \frac{\varphi(u, p) - p}{\| \varphi(u, p) - p \|}.$$

Then $A_u(p_0) = 0$, for some $p_0 \in C$, implies that $B_u(p_0) < 0$. In particular, the degree of Z_u is zero.

To prove this proposition we shall need the following lemmas.

Lemma 3.2. *Let $Z\colon \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be given by*

$$Z(p) = \frac{(-f_y(p),\, f_x(p))}{||(-f_y(p),\, f_x(p))||}.$$

(a) *If $p \in C$, $Z_u(p) \to Z(p)$ as $u \to 0^+$.*

(b) *The curve $t \mapsto \varphi(t, p)$ is differentiable and*

$$\frac{\partial \varphi}{\partial t}(s, p) = \frac{1}{\det(DX_q)}(-f_y(q),\, f_x(q)),$$

where $q = \varphi(s, p)$.

Proof. If W denotes the local inverse of $X = (f, g)$ at $X(p) = (c, d)$, by using the fact that this inverse is differentiable, (see Theorem 2.2) we have

$$\begin{aligned} \frac{\partial W}{\partial y}(c, d) &= \lim_{u\to 0} \frac{W(c, d+u) - W(c, d)}{u} \\ &= \lim_{u\to 0} \frac{\varphi(u, p) - p}{u} = \frac{\partial \varphi}{\partial u}(0, p), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial W}{\partial y}(c, d) &= (DX_p)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{\det(DX_p)}(-f_y(p),\, f_x(p)). \end{aligned}$$

As $\mathrm{Spec}(X) \cap [0, +\infty) = \emptyset$ we obtain that $\det(DX_p) > 0$. This finishes the proof of (a). The proof of (b) follows from the last computations. $\square$

Lemma 3.3. *Let φ be the local flow associated to $\mathcal{F}(f)$. There does not exist a compact disc D whose boundary is made up of the union of a vertical segment $A = \{(a, y)\colon c \le y \le d\}$ and an arc of trajectory $B = \{(\varphi(t, p))\colon 0 \le t \le t_0\}$ such that $c < d$, $p = (a, c)$, $\varphi(t_0, p) = (a, d)$ and, for all $0 < t < t_0$, $\Pi(\varphi(t, p)) \neq a$, where $\Pi\colon \mathbb{R}^2 \to \mathbb{R}$ is given by $\Pi(x, y) = x$.*

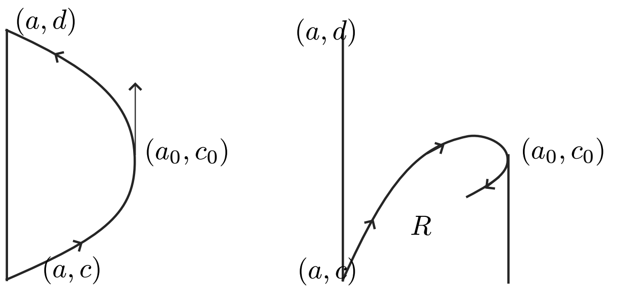

Figure 3a Figure 3b

Figure 3

Proof. Suppose by contradiction that such a disc D exists (see Figure 3a). We shall only consider the case in which, for all $0 < t < t_0$, $\Pi(\varphi(t, p)) > a$. Let $[a, a_0]$ be the interval $\Pi(D)$. Let $s_0 \in [0, t_0]$ be the smallest value such that $\Pi(\varphi(s_0, p)) = a_0$. Let $(a_0, c_0) = \varphi(s_0, p)$ and let R be the closed region bounded by the union of $\{(a, y) \colon y \leq c\}$, $\{(a_0, y) \colon y \leq c_0\}$ and $\{\varphi(t, p) \colon 0 \leq t \leq s_0\}$. It follows from (b) of Lemma 3.2 that $f_y(a_0, c_0) = 0$. This implies that $f_x(a_0, c_0) \in \operatorname{Spec}(X)$. By the assumptions of Proposition 3.1 about $\operatorname{Spec}(X)$, $f_x(a_0, c_0) < 0$ which in turn implies that the arc $\{\varphi(t, p) \colon s_0 \leq t \leq t_0\}$ must enter into R and cannot cross the boundary of R (see Figure 3b). This contradicts the fact that $\varphi(t_0, p) = (a, d) \notin R$. $\square$

Proof of Proposition 3.1. The proof of (a) is immediate. Let us proceed to prove (b). Orient C and $\mathbb{S}^1$ with the usual positive orientation. Suppose by contradiction that, for some $(p, u) \in C \times [(-\varepsilon_0, 0) \cup (0, \varepsilon_0)]$ we have that $Z_u(p) = (0, 1)$. Hence $\varphi(u, p)$ is of the form $\varphi(u, p) = (a, d)$, with $c < d$.

By applying Lemma 3.3 we conclude that the segment connecting (a, c) with (a, d) must be an arc of trajectory. However, this would imply that $f_y(p) = 0$ and $f_x(p) > 0$ which would be a contradiction with the assumptions of this proposition because $f_x(p) \in \operatorname{Spec}(X)$. $\square$

4 Avoiding internal tangencies

We say that a topological circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ is in *general position with* $\mathcal{F}(f)$ if there exist a set $T \subset C$, at most finite such that: (i) $\mathcal{F}(f)$ is transversal to $C \setminus T$, (ii) $\mathcal{F}(f)$ has a generic tangency with C at every point of T and, (iii) a leaf of $\mathcal{F}(f)$ can meet tangentially C at most at one point.

Denote by $\mathcal{GP}(f) = \mathcal{GP}(f, \sigma)$ the set of all topological circles $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ in general position with $\mathcal{F}(f)$ and surrounding the origin.

Remark 4.1. Let suppose that $C \in \mathcal{GP}(f)$. If $q \in C$ is an (internal) tangency of $\mathcal{F}(f)$ with C, we have that:

(i) For some closed subintervals $[p, q]$, $[q, r]$ in C there exist an orientation reversing homeomorphism $\varphi\colon [p, q] \to [q, r]$ such that, for all $z \in (p, q)$, $f(z) = f((\varphi(z)))$ and there is an oriented arc T_z of a leaf of $\mathcal{F}(f)$, connecting z with $\varphi(z)$ and meeting C exactly and transversally at its endpoints.

(ii) The family $\{T_z : z \in (p, q)\}$ depends continuously on z and tends to the one point set $\{q\}$ as $z \to q$.

The following definition was introduced in [1] (see also [24]).

Definition 4.2. *Let $C \in \mathcal{GP}(f)$. The Index of $\mathcal{F}(f)$ along C is the integer number*

$$I_{\mathcal{F}(f)}(C) := \frac{2 - n^e(f, C) + n^i(f, C)}{2}$$

where $n^e(f, C)$ (resp. $n^i(f, C)$) is the number of generic tangencies of $\mathcal{F}(f)$ with C, which are external (resp. internal).

Let $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ be a topological circle surrounding the origin. Let $Z_u : C \to \mathbb{S}^1$ be as in Proposition 3.1. We say that Z_u has a *generic contact* (resp. generic tangency with; resp. transversal to; etc) with C at $p \in C$ if every small local integral curve of Z_u at p has such property.

It is well known that if Z_u is in general position with C,

$$\deg(Z_u) = \frac{2 - n^e(Z_u, C) + n^i(Z_u, C)}{2}$$

where $n^i(Z_u, C)$ (resp. $n^e(Z_u, C)$) is the number of internal tangency (resp. external tangency) of Z_u with C (see [19, Theorems 9.1 and 9.2, p. 166-174]).

By using a standard homotopy argument we may conclude that

Lemma 4.3. *If $Z_u : C \to \mathbb{S}^1$ is as in Proposition* 3.1,

$$\deg(Z_u) = I_{\mathcal{F}(f)}(C).$$

As a consequence

Lemma 4.4. *Let $C \in \mathcal{GP}(f)$ be such that $n^i(f, C) = 0$. If $n^e(f, C)$ is greater than two, the degree of Z_u is different from zero.*

Next proposition will shows us that we can always select $C \in \mathcal{GP}(f)$ such that, $\mathcal{F}(f)$ has no internal tangencies with C and exactly two external ones. The definitions of external, internal and generic tangencies are given right after the proof of Proposition 2.7.

We shall need the following two lemmas, the first of which is proved in [15, Lemma 2].

Lemma 4.5. *Let $C \in \mathcal{GP}(f)$. Suppose that a leaf γ of $\mathcal{F}(f)$ meets C transversally somewhere and with an external tangency at a point $p \in C$. Then, γ contains a closed subinterval $[p, r]_f$ which meets C exactly at $\{p, r\}$ (doing it transversally at r) and the following is satisfied:*

(i) *If $[p, r]$ denotes the closed subinterval of C such that $\Gamma = [p, r] \cup [p, r]_f$ bounds a compact disc $\overline{D}(\Gamma)$ contained in $\mathbb{R}^2 \setminus D(C)$, then points of $\gamma \setminus [p, r]_f$ nearby p do not belong to $\overline{D}(\Gamma)$.*

(ii) *Let $(\tilde{p}, \tilde{r})$ and $[\tilde{p}, \tilde{r}]$ be subintervals of C satisfying $[p, r] \subset (\tilde{p}, \tilde{r}) \subset [\tilde{p}, \tilde{r}]$. If $\tilde{p}$ and $\tilde{r}$ are close enough to p and r, respectively, then we may deform C into $C_1 \in \mathcal{GP}(f)$ in such a way that the deformation fixes $C \setminus (\tilde{p}, \tilde{r})$ and takes $[\tilde{p}, \tilde{r}] \subset C$ to a closed subinterval $[\tilde{p}, \tilde{r}]_1 \subset C_1$ which is close to $[p, r]_f$. Furthermore, the number of generic tangencies of $\mathcal{F}(f)$ with C_1 is smaller than that of $\mathcal{F}(f)$ with C.*

Lemma 4.6. *Let $X = (f, g)\colon \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable map as in Theorem* 1.1. *If $C \in \mathcal{GP}(f)$ minimizes the number of tangencies with $\mathcal{F}(f)$, then every tangency is external.*

Proof. Suppose, by contradiction, that $q \in C$ is an internal tangency of $\mathcal{F}(f)$ with C, we shall proceed using Remark 4.1 and its notation, so we may select the maximal interval $[p, q]$ with properties (i) and (ii) of this remark. Assume that

(a) the family $\{T_z\}$, with $z \in (p, q)$, can be extend continuously to $z = p$ in such a way that T_p is a compact arc.

In this case, by our selection of $[p, q]$, the arc T_p has to meet C at a generic tangency. By Lemma 4.5 we may select $C_1 \in \mathcal{GP}(f)$ having smaller number of tangencies with $\mathcal{F}(f)$ than that of C. This contradiction proves that (a) is not possible. Therefore, the level curve $\{f = f(p)\}$ has two connected components:

$p \in C$ belongs to one connected component and $r \in C$ belongs to the other. By Remark 4.1, $[p, r]_C \subset C$ is the compact edge of an unbounded *hRc*. This contradiction with Proposition 2.7 finishes the proof of the lemma. □

Proposition 4.7. *Let $X = (f, g) \colon \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable map as in Theorem* 1.1. *There exists a topological circle $C \in \mathcal{GP}(f)$ and there are two points $a, b \in C$, with $f(a) < f(b)$, such that $\mathcal{F}(f)$ is tangent to C exactly at a and b; moreover, these tangencies are generic and external.*

Proof. Take $C \in \mathcal{GP}(f)$ as Lemma 4.6, so $n^i(f, C) = 0$. If $a, b \in C$ are such that $f(C) = [f(a), f(b)]$, the circle C has two external tangencies: one at a and the other at b.

Suppose by contradiction that a and b are not the only tangencies; so $n^e(f, C)$ is greater than two. This implies, by Lemma 4.4, that the degree of Z_u is different from zero, contradicting Proposition 3.1. □

5 Proof of Theorem 1.1

We shall say that a collar neighborhood U of a topological circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ is *interior* (resp. exterior), if U is contained in $\overline{D}(C)$ (resp. $\mathbb{R}^2 \setminus D(C)$).

Proposition 5.1. *Let $X = (f, g) \colon \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable map as in Theorem 1.1. There exists a topological circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ surrounding the origin such that:* (i) *$X(C)$ is a topological circle;* (ii) *for some exterior collar neighborhood U of C, its image $X(U)$ is an exterior collar neighborhood of $X(C)$ and* (iii) *$X|_U \colon U \to X(U)$ is a homeomorphism.*

The proof of this proposition needs some preparatory lemmas.

We say that a topological circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ is of ETT (*i.e external tangency type*) for $\mathcal{F}(f)$, if the following is satisfied: C surrounds the origin, there are two points $a, b \in C$, with $f(a) < f(b)$, $f(C) = [f(a), f(b)]$, and there are points $a_1, a_2, \ldots, a_n \in C^-$ and $b_1, b_2, \ldots, b_n \in C^+$, where C^- and C^+ are the connected components of $C \setminus \{a, b\}$, such that:

(a.1) $\mathcal{F}(f)$ is tangent to C exactly at a and b; also, these tangencies are generic and external;

(a.2) $f(a) = \inf\{f(z) \colon z \in C\} < \sup\{f(z) \colon z \in C\} = f(b)$;

(a.3) f takes homeomorphically each C^i, with $i \in \{-, +\}$, onto the open interval $(f(a), f(b))$ (i.e., $X(C^i)$ is the graph of a map $(f(a), f(b)) \mapsto \mathbb{R}$);

(a.4) X restricted to $C \setminus \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ is a topological embedding and also, $X(C^-)$ and $X(C^+)$ meet transversally to each other;

(a.5) $(X(a_1), X(a_2), \ldots, X(a_n)) = (X(b_1), X(b_2), \ldots, X(b_n))$ and $f(a) < f(a_1) = f(b_1) < \cdots < f(a_n) = f(b_n) < f(b)$;

(a.6) there are sequences $x_n \to a$ and $y_n \to b$ of points x_n and y_n in $\mathbb{R}^2 \setminus \overline{D}(C)$ such that, for all n, $f(x_n) < f(a) < f(b) < f(y_n)$. This means that the local exterior of C around a (resp. around b) is taken to the unbounded connected component of $\mathbb{R}^2 \setminus X(C)$. In particular, $n \geq 0$ is an even number;

(a.7) if $x \in \mathbb{R}^2 \setminus \overline{D}(C)$ is close enough to $y \in C^+$ (resp. $y \in C^-$) and $f(x) = f(y)$, then $g(y) < g(x)$ (resp. $g(y) > g(x)$).

(a.8) if $\overline{a}_1, \overline{a}_n \in C^-$ and $\overline{b}_1, \overline{b}_n \in C^+$ close enough to a_1, a_n and b_1, b_n respectively, and $[a_1, a_n] \subset (\overline{a}_1, \overline{a}_n)$, $[b_1, b_n] \subset (\overline{b}_1, \overline{b}_n)$ then, $X([\overline{a}_1, a_1) \cup (a_n, \overline{a}_n])$ is below $X([\overline{b}_1, b_1) \cup (b_n, \overline{b}_n])$ (i.e. if $a' \in [\overline{a}_1, a_1) \cup (a_n, \overline{a}_n]$ and $b' \in [\overline{b}_1, b_1) \cup (b_n, \overline{b}_n]$ are such that $f(a') = f(b')$ then $g(a') < g(b')$).

Lemma 5.2. *There exists a topological circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ of* ETT *for $\mathcal{F}(f)$.*

Proof. By Proposition 4.7 we may take a topological circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$, surrounding the origin, such that there are two points $a, b \in C$ with $f(a) < f(b)$, $f(C) = [f(a), f(b)]$, and so that (a.1) above is satisfied. This implies that (a.2) and (a.3) of the definition above are also satisfied.

By deforming C around a small open neighborhood $V_a \subset C$ of a (resp. $V_b \subset C$ of b) we may also assume that $g|_{V_a}$ (resp. $g|_{V_b}$) is a topological embedding. In this way, if V_a and V_b are small enough $X|_{V_a \cup V_b}$ is a topological embedding. This implies that

$$X(C^+) \cap X(C^-) = X(C^+ \setminus (V_a \cup V_b)) \cap X(C^- \setminus (V_a \cup V_b))$$

is a compact set. As $C^+ \setminus (V_a \cup V_b)$ and $C^- \setminus (V_a \cup V_b)$ are disjoints sets we may deform C so that

(i) $X(C^+)$ and $X(C^-)$ meet transversally doing this along a set which is at most finite.

Thus, (i) implies that (a.4) and (a.5) of the definition above are satisfied too. Item (a.6) follows directly from the preceeding properties. As $X(C)$ is tangent to the vertical foliation at the points $X(a)$ and $X(b)$ and by using (a.6), the connected components C^- and C^+ can be named to satisfy (a.7). Item (a.7) implies (a.8). $\square$

In the following of this section, C will be a topological circle of ETT for $\mathcal{F}(f)$ and we shall use all corresponding introduced notation.

Given $\alpha, \beta \in C^-$ (resp. $\alpha, \beta \in C^+$), $[\alpha, \beta]$,(α, β),$[\alpha, \beta)$ will denote subintervals of C^- (resp. C^+) with endpoints α, β. Let L denote the straight line which passes through the points $X(a_1)$ and $X(a_n)$. Let $\mathcal{L}$ be the foliation of $\mathbb{R}^2$ made up by all the straight lines parallel the line L. By a small deformation of C with support in $[a_1, a_n] \cup [b_1, b_n]$, we may assume that

(b) every point of tangency of $X([a_1,\ a_n] \cup [b_1,\ b_n])$ with $\mathcal{L}$ is generic, $X([a_1,\ a_n])$ and $X([b_1,\ b_n])$ are transversal to L.

From (a.8), by taking $\overline{a}_1, \overline{a}_n \in C^-$ and $\overline{b}_1, \overline{b}_n \in C^+$ close to a_1, a_n and b_1, b_n respectively, such that $[a_1, a_n] \subset (\overline{a}_1, \overline{a}_n)$ and $[b_1, b_n] \subset (\overline{b}_1, \overline{b}_n)$, we may suppose as well that

(c) $X([\overline{a}_1, a_1) \cup (a_n, \overline{a}_n])$ and $X([\overline{b}_1, b_1) \cup (b_n, \overline{b}_n])$ are disjoint of L.

Let $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ be such that $R_\theta(\mathcal{L})$ is made up of vertical lines, where R_θ is the linear rotation of angle θ. Recall that $X_\theta = (f_\theta, g_\theta) = R_\theta \circ X \circ R_{-\theta}$. By means of a small deformation of C, we may also assume that

(d) $R_\theta(C)$ is in general position with $\mathcal{F}(f_\theta)$, i.e. $R_\theta(C) \in \mathcal{GP}(f_\theta)$.

Remark 5.3. X_θ takes any leaf of $\mathcal{F}(f_\theta)$ into a leaf of $R_\theta(\mathcal{L})$, where the foliation $R_\theta(\mathcal{L})$ is made up by vertical lines.

Lemma 5.4. *Let denote* $\mathbf{a_j} = R_\theta(a_j)$, $\overline{\mathbf{a}}_\mathbf{j} = R_\theta(\overline{a}_j)$, $\mathbf{b_j} = R_\theta(b_j)$ *and* $\overline{\mathbf{b}}_\mathbf{j} = R_\theta(\overline{b}_j)$. *If* $X_\theta([\overline{\mathbf{a}}_\mathbf{1}, \mathbf{a_1}) \cup (\mathbf{a_n}, \overline{\mathbf{a}}_\mathbf{n}])$ *is on the left to the vertical line* $R_\theta(L)$ *and* $X_\theta([\overline{\mathbf{b}}_\mathbf{1}, \mathbf{b_1}) \cup (\mathbf{b_n}, \overline{\mathbf{b}}_\mathbf{n}])$ *is on the right to* $R_\theta(L)$ *(see fig.* 4*); then there is a circle* $C_1 \subset \mathbb{R}^2 \setminus D(R_\theta(C))$, *surrounding the origin, obtained from* $R_\theta(C)$ *by a deformation which fixes* $R_\theta(C) \setminus ((\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}) \cup (\overline{\mathbf{b}}_\mathbf{1}, \overline{\mathbf{b}}_\mathbf{n}))$ *and takes* $[\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}] \subset R_\theta(C)$ *and* $[\overline{\mathbf{b}}_\mathbf{1}, \overline{\mathbf{b}}_\mathbf{n}] \subset R_\theta(C)$ *to the closed subintervals* $[\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]_{C_1} \subset C_1$ *and* $[\overline{\mathbf{b}}_\mathbf{1}, \overline{\mathbf{b}}_\mathbf{n}]_{C_1} \subset C_1$ *respectively, which satisfy* $X_\theta([\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]_{C_1})$ *is on the left to* $R_\theta(L)$ *and* $X_\theta([\overline{\mathbf{b}}_\mathbf{1}, \overline{\mathbf{b}}_\mathbf{n}]_{C_1})$ *is on the right to* $R_\theta(L)$. *In particular,* $R_{-\theta}(C_1)$ *is as requested to prove Proposition* 5.1.

Proof. Item (d) mentioned in this proof, refers to the one considered right before the statement of this lemma.

Suppose that $\mathcal{F}(f_\theta)$ has an internal tangency with $R_\theta(C)$ at $q \in (\mathbf{a_1}, \mathbf{a_n})$. By Item (d), we may proceed as in Remark 4.1 (applied to $\mathcal{F}(f_\theta)$ and considering the notation introduced there) to obtain subintervals $[p, q]$, $[q, r]$ of $R_\theta(C)$, (generated by $q \in R_\theta(C)$), determined by the condition that

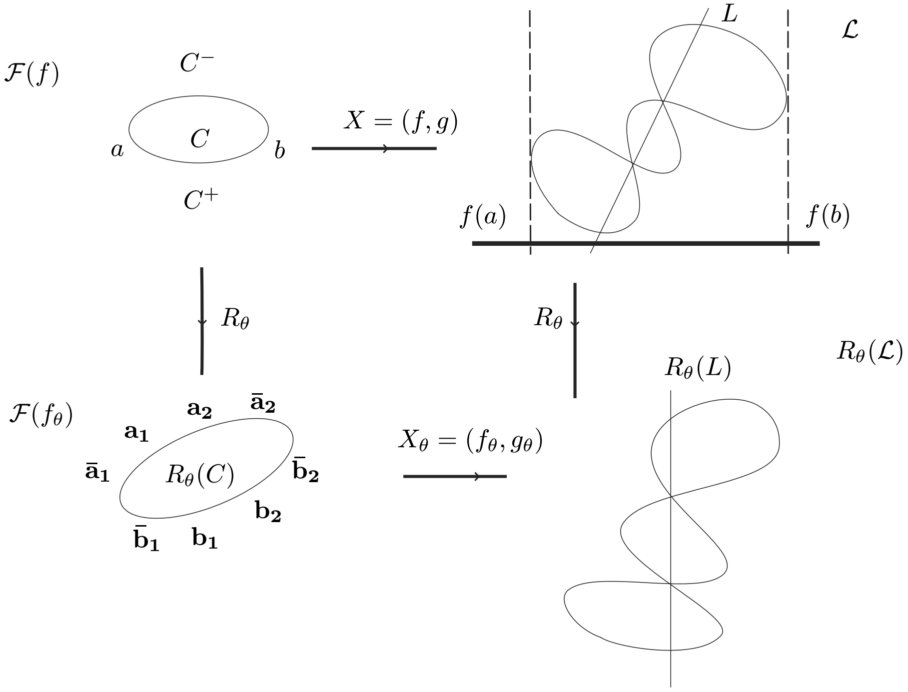

Figure 4: case $n = 2$ in Lemma 5.4.

(e.1) $(p, q]$ is the maximal interval satisfying properties (i) and (ii) of Remark 4.1, and also $(p, q]$ contained in $[\mathbf{a_1}, \mathbf{a_n}]$.

By Remark 4.1, every element of the family $\{T_z : z \in (p, q)\}$ is an arc of a leaf of $\mathcal{F}(f_\theta)$.

As $\mathbb{R}^2 \setminus D(R_\theta(C))$ is not bounded, $[p, q] \cup [q, r]$ is properly contained in $R_\theta(C)$. Therefore,

(e.2) the family $\{T_z\}$, with $z \in (p, q)$, extends continuously to $z = p$, in such a way that T_p is a compact arc connecting p and r.

In fact, if (e.2) is false the positive (resp. negative) half-leaf L_p^+ (resp. L_r^-) of the foliation $\mathcal{F}(f_\theta)$ starting at p (resp. at r) does not meet $R_\theta(C)$ and so accumulate at the point ∞ of the Riemann sphere $\mathbb{R}^2 \cup \infty$. Under these circumstances, Remark 4.1 implies that the subinterval $[p, q] \cup [q, r]$ is the compact edge of a unbounded *hRc* for $\mathcal{F}(f_\theta)$. This contradiction with Proposition 2.7 shows (e.2).

It follows from (e.1), (e.2) and (d) that (e.3) If $\{p, r\} \cap \{\mathbf{a_1}, \mathbf{a_n}\} = \emptyset$, then, between p and r, exactly one of them is an external tangency point of $\mathcal{F}(f)$ with $R_\theta(C)$. See Fig. 5.

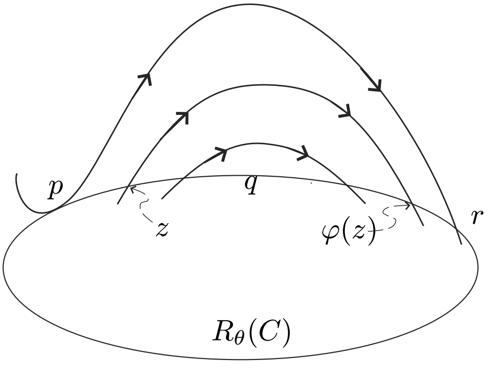

Figure 5

Let us to perform a sequence of adequate deformations of $R_\theta(C)$, in order to obtain the circle C_1 as requested. We meet two possible cases:

The *first* one is that $\{p, r\} \cap \{\mathbf{a_1}, \mathbf{a_n}\} \neq \emptyset$. Consider only the case in which $p = \mathbf{a_1}$ and $r \neq \mathbf{a_n}$. From (ii) of Remark 4.1 and (e.2), we may deform $R_\theta(C)$ into a new circle C_1 in such a way that: the deformation fixes $R_\theta(C) \setminus (\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n})$ and takes $[\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]$ to a closed subinterval $[\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]_{C_1} \subset C_1$ such that

(f) the cardinality of $R_\theta(L) \cap X_\theta([\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]_{C_1})$ is less than that of (the finite set) $R_\theta(L) \cap X_\theta([\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}])$; and, concerning the number of tangencies with the vertical foliation, that are on the right to $R_\theta(L)$, the curve $X_\theta([\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]_{C_1})$ has less number than that of $X_\theta([\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}])$. See fig. 6.

In this deformation the arc $[p, r] \subset R_\theta(C)$ has been taken to an interval whose image under X_θ is on the left of the vertical line $R_\theta(L)$. This deformation takes place inside a small neighborhood of $\bigcup\{T_z : z \in [p, q]\}$ and so C_1 is surrounding the origin and satisfies (f). Also as $[\overline{a}_1, \overline{a}_n] \subset C$ is transversal to $\mathcal{F}(f)$, $[\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]_{C_1} \subset C_1$ can be obtained to be transversal to the foliation $R_\theta(\mathcal{F}(f))$. We do not care if $X_\theta(C_1)$ has more self-intersections than $X_\theta \circ R_\theta(C)$.

The *second* case happens when $\{p, r\} \cap \{\mathbf{a_1}, \mathbf{a_n}\} = \emptyset$. We shall only consider the case in which p is the external tangency (see e.3). This and Remark 4.1 imply that

(g.1) if $\Gamma = [p, r] \cup T_p$, then, $\overline{D}(\Gamma)$ is contained in $\mathbb{R}^2 \setminus D(R_\theta(C))$ and the points of $L_p \setminus T_p$ (here L_p is the leaf of $\mathcal{F}(f_\theta)$ passing through p) near p do not belong to $\overline{D}(\Gamma)$.

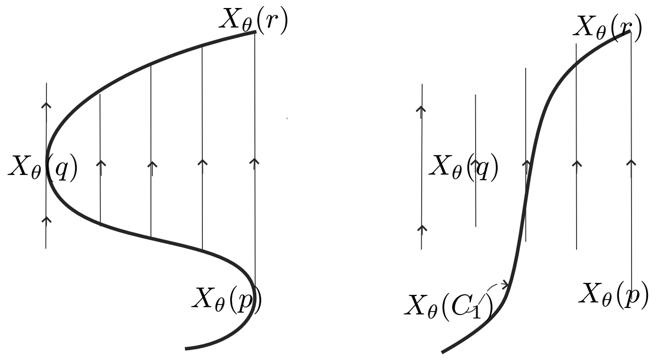

Figure 6

The arguments of Lemma 4.5 imply that we may deform of $R_\theta(C)$ into a new circle C_1 according to the following conditions. The deformation fixes $R_\theta(C) \setminus (\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n})$ and takes $[\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]$ to a closed subinterval $[\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]_{C_1} \subset C_1$ such that

(g.2) $R_\theta(L) \cap X_\theta([\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]_{C_1})$ has the same number of elements than $R_\theta(L) \cap X_\theta([\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}])$; and, concerning the number of tangencies with the vertical foliation, that are on the right to $R_\theta(L)$, $X_\theta([\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]_{C_1})$ has one less than that of $X_\theta([\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}])$.

As above, this deformation takes place inside a small neighborhood of T_p and so C_1 surrounds the origin. Also $[\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]_{C_1} \subset C_1$ can be constructed to be transversal to the foliation $R_\theta(\mathcal{F}(f))$. Again, as in the case above, we do not care if $X_\theta(C_1)$ has more self-intersections than $X_\theta \circ R_\theta(C)$.

As these cases are the only possible ones, and thanks to (f) and (g.2), we only need to perform finitely many times the process (just described above) of obtaining new circles such that their image under $R_{-\theta}$ is of ETT for $\mathcal{F}(f)$, in order to finally obtain a circle, say C_2, such that $X_\theta([\overline{\mathbf{a}}_\mathbf{1}, \overline{\mathbf{a}}_\mathbf{n}]_{C_2})$ is on the left to $R_\theta(L)$. Similarly, by a deformation that fixes $C_2 \setminus [\overline{\mathbf{b}}_\mathbf{1}, \overline{\mathbf{b}}_\mathbf{n}]$ we will finally obtain one circle as requested in this lemma. □

Proof of Proposition 5.1. By Lemma 5.2, there exists a topological circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ of ETT for $\mathcal{F}(f)$. The property (a.8) of the definition of ETT tell us that $X([\overline{a}_1, a_1) \cup (a_n, \overline{a}_n])$ is below $X([\overline{b}_1, b_1) \cup (b_n, \overline{b}_n])$. Now, we select an adequate θ so that we may deform $R_\theta(C)$, locally around $\{R_\theta(a_1), R_\theta(b_1), R_\theta(a_n), R_\theta(b_n)\}$ in such a way that its image under $R_{-\theta}$ is of ETT for $\mathcal{F}(f)$, and $R_\theta(C)$ satisfies the conditions of Lemma 5.4. Therefore, $R_\theta(C)$ can be deformed into one as requested to prove this proposition. □

Proof of Theorem 1.1. Let C and U be as in Proposition 5.1.

By Schoenflies Theorem [2, Theorem III.6.B], the map $X|_C\colon C \to X(C)$, can be extended to a homeomorphism $Y_1\colon \overline{D}(C) \to \overline{D}(X(C))$. In this way, we extend $X\colon \mathbb{R}^2 \setminus D(C) \to \mathbb{R}^2$ to $\widetilde{X}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by defining $\widetilde{X}|_{\overline{D}(C)} = Y_1$. As $\widetilde{X}|_U\colon U \to X(U)$ is a homeomorphism and U and $X(U)$ are exterior collar neighborhoods of C and $X(C)$, respectively, $\widetilde{X}$ is a local homeomorphism everywhere. By Theorem 2.1 $\widetilde{X}$ is globally injective. $\square$

Acknowledgments. The first author was supported in part by FAPESP Grant # TEMÁTICO 03/03107-9, and by CNPq Grant # 306992/2003-5. The second author was supported in part by CNPq Grant # 141853/2001-8.

References

[1] S.Kh. Aranson, G.R. Belitsky and E.V. Zhuzhoma, *Introduction to the qualitative theory of dynamical system on surfaces.* Trans. of Mathematical Monographs, AMS, **143** (1996). xiii+325 pp.

[2] R.H. Bing, *The geometric topology of 3-manifolds*. American Mathematical Society Colloquium Publications, **40** (1983). x+238 pp.

[3] H. Blumberg, *The measurable boundaries of an arbitrary function.* Acta Math., **65** (1935), 263–282.

[4] A.V. Černavskii, *Finite–to–one open mappings of manifolds.* Mat. Sbornik **65** (1964), 357–369.

[5] A.V. Černavskii, *Addendum to the paper "Finite-to-one open mappings of manifolds"*. Mat. Sbornik **66** (1965), 471–472.

[6] M. Chamberland, J. Llibre and G. Świrszcz, *Weakened Markus-Yamabe conditions for 2-dimensional global asymptotic stability.* Nonlinear Anal. **59** 6 (2004), 951–958.

[7] C. Chicone, *Ordinary differential equations with applications.* Texts in Applied Mathematics, **34** (1999), Springer-Verlag, New York. xvi+561 pp.

[8] M. Cobo, C. Gutierrez and J. Llibre, *On the injectivity of C^1 maps of the real plane.* Canadian Journal of Mathematics **54** (6) (2002), 1187–1201.

[9] F. Dumortier, P. De Maesschalck, *Topics on singularities and bifurcations of vector fields. Normal forms, bifurcations and finiteness problems in differential equations,* 33–86, NATO Sci. Ser. II Math. Phys. Chem., **137** (2004), Kluwer Acad. Publ., Dordrecht.

[10] F. Dumortier, R. Roussarie, J. Sotomayor and H. Żoladek, *Bifurcations of planar vector fields. Nilpotent singularities and Abelian integrals.* Lecture Notes in Mathematics, **1480** (1991), Springer-Verlag, Berlin. viii+226 pp.

[11] A. Fernandes, C. Gutierrez and R. Rabanal, *Global asymptotic stability for differentiable vector fields of* $\mathbb{R}^2$. Journal of Differential Equations, **206** (2) (2004), 470–482.

[12] A. Fernandes, C. Gutierrez and R. Rabanal, *On local diffeomorphisms of* $\mathbb{R}^n$ *that are injective.* Qualitative Theory of Dynamical Systems, **4** (2004), 255–262.

[13] R. Fessler, *A proof of the two dimensional Markus–Yamabe Stability Conjecture and a generalization.* Annales Polonici Math.**LXII** (1995), 45–74.

[14] C. Gutierrez, *A solution to the bidimensional global asymptotic stability conjecture.* Ann. Inst. H. Poincaré. Analyse non linearaire, **12** (1995) 627–671.

[15] C. Gutierrez and A. Sarmiento, *Injectivity of* C^1 *maps* $\mathbb{R}^2 \to \mathbb{R}^2$ *at infinity and planar vector fields.* Asterisque, **287** (2003), 89–102.

[16] C. Gutierrez and Nguyen Van Chau, *A remark on an eigenvalue condition for the global injectivity of differentiable maps of* $\mathbb{R}^2$. To appear in Discrete and Continuous Dynamical Systems.

[17] C. Gutierrez, R. Rabanal and B.F. Pires, *Asymptotic stability at infinity for differentiable vector fields of the plane.* To appear in Journal of Differential Equations.

[18] E.H. Hanson, *A theorem of Denjoy, Young and Saks.* Bull. Amer. Math. Soc., **40** (1934), 691–694.

[19] P. Hartman, *Ordinary differential equations.* John Wiley and Sons, Inc. 1973. xx+612 pp.

[20] C. Olech, *On the global stability of an autonomous system on the plane.* Contributions to Differential Equations **1** (1963) 389–400.

[21] C. Olech, *Introduction. New directions in differential equations and dynamical systems*, viii–x, Royal Soc. Edinburgh, Edinburgh, 1991.

[22] C. Olech, *Global phase-portrait of plane autonomous system.* Ann. Inst. Fourier (Grenoble) **14** (fasc. 1) (1964), 87–97.

[23] S. Pinchuck, *A counterexample to the strong Jacobian conjecture*. Math. Z. **217** (1994), 1–4.

[24] H. Poincaré, *Sur les courbes définies par les équations différentielles*, Œuvres, vol. I, Gauthier–Villars, Paris, 1928, pp. 3–84, 90–161, 167–221.

[25] S. Radulescu and M. Radulescu, *Local Inversion theorems without assuming continuous differenciability.* J. Math. Anal. Appl. **138** (2) (1989), 581–590.

[26] R. Roussarie, *Bifurcation of planar vector fields and Hilbert's sixteenth problem.* Progress in Mathematics, **164** (1998), Birkhäuser Verlag, Basel. xviii+204 pp.

[27] S. Saks, *Sur les nombres dérivés des fonctions.* Fundamenta Math., **5** (1924), 98–104.

[28] B. Smyth and F. Xavier, *Injectivity of local diffeomorphisms from nearly spectral conditions.* J. Differential Equations, **130** (1996), 406–414.

[29] J. Väisälä, *Discrete open mappings on manifolds.* Annales Academiae Scientiarum Fennicae. Ser. A I No. **392** (1966), 10 pages.

Carlos Gutierrez and **Roland Rabanal**
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo
Caixa Postal 668, CEP 13560-970
São Carlos, SP
BRASIL

E-mail: gutp@icmc.usp.br / roland@icmc.usp.br