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Bivariate gamma distributions, sums and ratios

Saralees Nadarajah and Samuel Kotz

Abstract. Exact distributions of $R = X + Y$ and $W = X/(X + Y)$ and the corresponding moment properties are derived when X and Y follow five flexible bivariate gamma distributions. The expressions turn out to involve several special functions.

Keywords: bivariate gamma distributions, ratios of random variables, sums of random variables.

Mathematical subject classification: 33C90, 62E99.

1 Introduction

Since the 1930s, the statistics literature has seen many developments in the theory and applications of linear combinations and ratios of random variables. Some of these include:

- Ratios of normal random variables appear as sampling distributions in single equation models, in simultaneous equations models, as posterior distributions for parameters of regression models and as modeling distributions, especially in economics when demand models involve the indirect utility function (details in Yatchew, 1986).

- Weighted sums of uniform random variables – in addition to the well known application to the generation of random variables – have applications in stochastic processes which in many cases can be modeled by these weighted sums. In computer vision algorithms these weighted sums play a pivotal role (Kamgar-Parsi et al., 1995). An earlier application of the linear combinations of uniform random variables is given in connection with the distribution of errors in nth tabular differences Δ^n (Lowan and Laderman, 1939).

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- Ratio of linear combinations of chi-squared random variables are part of von Neumann's (1941) test statistics (mean square successive difference divided by the variance). These ratios appear in various two-stage tests (Toyoda and Ohtani, 1986). They are also used in tests on structural coefficients of a multivariate linear functional relationship model (details in Chaubey and Nur Enayet Talukder (1983) and Provost and Rudiuk (1994)).

- Sums of independent gamma random variables have applications in queuing theory problems such as determination of the total waiting time and in civil engineering problems such as determination of the total excess water flow into a dam. They also appear in test statistics used to determine the confidence limits for the coefficient of variation of fiber diameters (Linhart (1965) and Jackson (1969)) and in connection with the inference about the mean of the two-parameter gamma distribution (Grice and Bain, 1980).

- Linear combinations of inverted gamma random variables are used for testing hypotheses and interval estimation based on generalized p-values, specifically for the Behrens-Fisher problem and variance components in balanced mixed linear models (Witkovský, 2001).

- As to the Beta distributions their linear combinations occur in calculations of the power of a number of tests in ANOVA (Monti and Sen, 1976) among other applications. More generally, the linear combinations are used for detecting changes in the location of the distribution of a sequence of observations in quality control problems (Lai, 1974). Pham-Gia and Turkkan (1993, 1994, 1998, 2002) and Pham-Gia (2000) provided applications of sums and ratios to availability, Bayesian quality control and reliability.

- Linear combinations of the form $T = a_1 t_{f_1} + a_2 t_{f_2}$, where t_f denotes the Student t random variable based on f degrees of freedom, represents the Behrens-Fisher statistic and – as early as the middle of the twentieth century – Stein (1945) and Chapman (1950) developed a two-stage sampling procedure involving the T to test whether the ratio of two normal random variables is equal to a specified constant.

- Weighted sums of the Poisson parameters are used in medical applications for directly standardized mortality rates (Dobson et al., 1991).

In this paper, we consider the distributions of $R = X + Y$ and $W = X/(X + Y)$ when X and Y are correlated gamma random variables arising from the following distributions:

1. McKay's bivariate gamma distribution (McKay, 1934) given by the joint pdf

$$f(x, y) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} x^{p-1}(y-x)^{q-1}\exp(-ay) \tag{1}$$

   for $y > x > 0$, $a > 0$, $p > 0$ and $q > 0$. This distribution has received applications in several areas. For example, Clarke (1979, 1980) has used McKay's distribution with X = annual stream flow, and Y = areal precipitation. The justification is apparently that $Y \geq X$ is reasonable on physical grounds (for water height basins with little over year storage).

2. Cherian's bivariate gamma distribution (Cherian, 1941) given by the joint pdf

$$f(x, y) = \frac{\exp(-x-y)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \int_0^{\min(x,y)} (x-z)^{\theta_1-1}(y-z)^{\theta_2-1} z^{\theta_3-1}\exp(z)\,dz \tag{2}$$

   for $x > 0$, $y > 0$, $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_3 > 0$. This distribution has received applications in several areas. For example, Prekopa and Szantai (1978) used this distribution to study the streamflows of a river in six months of the year.

3. Kibble's bivariate gamma distribution (Kibble, 1941) given by the joint pdf

$$f(x, y) = \frac{(xy)^{(\alpha-1)/2}}{\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}} \exp\left(-\frac{x+y}{1-\rho}\right) I_{\alpha-1}\left(\frac{2\sqrt{xy\rho}}{1-\rho}\right) \tag{3}$$

   for $x > 0$, $y > 0$, $\alpha > 0$ and $0 \leq \rho < 1$, where $I_\nu(\cdot)$ denotes the modified Bessel function of the first kind of order ν defined by

$$I_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k k!} \left(\frac{x^2}{4}\right)^k .$$

   This distribution has received applications in several areas. Some of them are:

(a) Electric counter system: Lampard (1968) built up Kibble's distribution in the conditional manner, $h = f(x)g(y \mid x)$; his context was a system of two reversible counters (i.e. an input can either increase or decrease the cumulative count), with two Poisson inputs (an increase process and a decrease process). Output events occur when either of the cumulative counts decreases to zero. The sequence of time intervals between output events forms a Markov chain, and the joint distribution of successive intervals is of Kibble's form. Lampard also gave an interpretation of the same process in terms of a queueing system.

(b) Hydrology: Phatarford (1976) used Kibble's distribution as a model to describe summer and winter stream flows.

(c) Rain: As the gamma distribution is a popular univariate choice for the description of amount of rainfall, Izawa (1965) proposed Kibble's distribution to describe the joint distribution of rainfall at two nearby rain gauges.

(d) Wind gusts: Smith and Adelfang (1981) reported analyses of wind gust data using Kibble's distribution. The two variates were magnitude and length of the gust.

4. the beta Stacy distribution (Mihram and Hultquist, 1967) given by the joint pdf

$$f(x, y) = \frac{c}{a^{bc}\Gamma(b)B(p,q)} x^{p-1}(y-x)^{q-1} y^{bc-p-q} \exp\left\{-\left(\frac{y}{a}\right)^c\right\} \tag{4}$$

for $y > x > 0$, $a > 0$, $b > 0$, $c > 0$, $p > 0$ and $q > 0$.

5. Becker and Roux's bivariate gamma distribution (Becker and Roux, 1981) given by the joint pdf

$$f(x, y) = \begin{cases} \dfrac{\beta' \alpha^a}{\Gamma(a)\Gamma(b)} x^{a-1}\{\beta'(y-x)+\beta x\}^{b-1} \\ \quad \exp\{-\beta' y - (\alpha+\beta-\beta')x\}, & \text{if } y > x > 0, \\ \dfrac{\alpha' \beta^b}{\Gamma(a)\Gamma(b)} y^{b-1}\{\alpha'(x-y)+\alpha y\}^{a-1} \\ \quad \exp\{-\alpha' x - (\alpha+\beta-\alpha')y\}, & \text{if } x > y > 0 \end{cases} \tag{5}$$

for $x > 0$, $y > 0$, $a > 0$, $b > 0$, $\alpha > 0$, $\beta > 0$, $\alpha^{'} > 0$ and $\beta^{'} > 0$. As often with the gamma distribution, this distribution has applications in reliability studies. For example, Barlow and Proschan (1977) applied this distribution to data on failures of caterpillar tractors.

Explicit expressions for the pdfs and moments of $R = X + Y$ and $W = X/(X+Y)$ for these five distributions are derived in Sections 2–6. The calculations involve several special functions, including the complementary error function defined by

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp\left(-t^2\right) dt,$$

the confluent hypergeometric function defined by

$${}_1F_1(a; b; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!},$$

the degenerate hypergeometric function of one variable defined by

$$\begin{aligned}\Psi(a, b; x) = & \frac{\Gamma(1-b)}{\Gamma(1+a-b)} {}_1F_1(a, b; x) \\ & + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} {}_1F_1(1+a-b, 2-b; x),\end{aligned}$$

the degenerate hypergeometric function of two variables defined by

$$\Phi_1(a, b, c; x, y) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(a)_{k+\ell} (b)_\ell}{(c)_{k+\ell}} \frac{x^k y^\ell}{k! \ell!}$$

the Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

the hypergeometric function of two variables defined by

$$F_1(a, b, c, d; x, y) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(a)_{k+\ell} (b)_k (c)_\ell}{(d)_{k+\ell}} \frac{x^k y^\ell}{k! \ell!},$$

the incomplete beta function defined by

$$B_x(a, b) \;=\; \int_0^x t^{a-1}(1-t)^{b-1} dt,$$

the Jacobi polynomial defined by

$$P_n^{(\alpha,\beta)}(x) \;=\; \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\right\},$$

the modified Bessel function of the first kind defined by

$$I_\nu(x) \;=\; \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k k!}\left(\frac{x^2}{4}\right)^k,$$

the modified Bessel function of the third kind of order ν defined by

$$K_\nu(x) \;=\; \frac{\pi\left\{I_{-\nu}(x) - I_\nu(x)\right\}}{2\sin(\nu\pi)}$$

with $K_0(\cdot)$ interpreted as the limit

$$K_0(x) \;=\; \lim_{\nu \to 0} K_\nu(x),$$

the modified Laguerre polynomial defined by

$$L_n^\nu(x) \;=\; \frac{x^{-\nu}\exp(x)}{n!}\frac{d^n}{dx^n}\left\{x^{n+\nu}\exp(-x)\right\},$$

and the Whittaker function defined by

$$W_{\lambda,\mu}(a) \;=\; \frac{a^{\mu+1/2}\exp(-a/2)}{\Gamma(\mu-\lambda+1/2)}\int_0^\infty t^{\mu-\lambda-1/2}(1+t)^{\mu-\lambda-1/2}\exp(-at)dt,$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. We also need the following important lemmas.

Lemma 1 [Equation (2.3.6.1), Prudnikov et al., 1986, volume 1]. *For* $\alpha > 0$ *and* $\beta > 0$,

$$\int_0^a x^{\alpha-1}(a-x)^{\beta-1}\exp(-px)\,x \;=\; B(\alpha,\beta)\,a^{\alpha+\beta-1}\,{}_1F_1(\alpha;\alpha+\beta;-ap).$$

Lemma 2 [Equation (2.2.6.1), Prudnikov et al., 1986, volume 1]. *For $\alpha > 0$ and $\beta > 0$,*

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma} dx \; = \; (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma} B(\alpha,\beta) \; {}_2F_1\left(\alpha, -\gamma; \alpha+\beta; \frac{c(a-b)}{ac+d}\right).$$

Lemma 3 [Equation (2.2.8.5), Prudnikov et al., 1986, volume 1]. *For $a > 0$, $\alpha > 0$, $\beta > 0$, $a \mid u \mid < 1$ and $a \mid v \mid < 1$,*

$$\int_0^a x^{\alpha-1}(a-x)^{\beta-1}(1-ux)^{-\rho}(1-vx)^{-\lambda} dx \; = \; a^{\alpha+\beta-1} B(\alpha,\beta)\, F_1(\alpha,\rho,\lambda,\alpha+\beta; ua, va).$$

Lemma 4 [Equation (2.15.3.2), Prudnikov et al., 1986, volume 2]. *For $\alpha+\nu > 0$ and $p > c$,*

$$\int_0^\infty x^{\alpha-1} \exp(-px)\, I_\nu(cx) dx \; = \; p^{-(\alpha+\nu)} \left(\frac{c}{2}\right)^{\nu} \frac{\Gamma(\nu+\alpha)}{\Gamma(\nu+1)} \, {}_2F_1\left(\frac{\alpha+\nu}{2}, \frac{\alpha+\nu+1}{2}; \nu+1; \frac{c^2}{p^2}\right).$$

Lemma 5 [Equation (2.15.2.6), Prudnikov et al., 1986, volume 2]. *For $a > 0$, $\beta > 0$ and $\nu > -1$,*

$$\int_0^a x^{\nu+1}\left(a^2-x^2\right)^{\beta-1} I_\nu(cx) dx \; = \; 2^{\beta-1} a^{\nu+\beta} c^{-\beta} \Gamma(\beta) I_{\nu+\beta}(ac).$$

Lemma 6 [Equation (2.15.5.4), Prudnikov et al., 1986, volume 2]. *For $p > 0$ and $\nu > -n-1$,*

$$\int_0^\infty x^{\nu+2n+1} \exp\left(-px^2\right) I_\nu(cx) dx \; = \; \frac{n!c^{\nu}}{2^{\nu+1} p^{n+\nu+1}} \exp\left(\frac{c^2}{4p}\right) L_n^{\nu}\left(-\frac{c^2}{4p}\right).$$

Lemma 7 [Equation (2.19.3.2), Prudnikov et al., 1986, volume 2]. *For $p > 0$ and $\alpha > 0$,*

$$\int_0^\infty x^{\alpha-1} \exp(-px)\, L_n^{\lambda}(cx) dx \; = \; \frac{\Gamma(\alpha)}{p^{\alpha}} P_n^{(\lambda,\alpha-\lambda-n-1)}\left(1-\frac{2c}{p}\right).$$

Lemma 8 [Equation (2.263.1), Gradshteyn and Ryzhik, 2000]. *For* $m < 2n$*,*

$$\int \frac{x^m}{\left(a+bx+cx^2\right)^{n+1/2}} dx = \frac{x^{m-1}}{(m-2n)c\left(a+bx+cx^2\right)^{n-1/2}} - \frac{(2m-2n-1)b}{2(m-2n)c} \int \frac{x^{m-1}}{\left(a+bx+cx^2\right)^{n+1/2}} dx - \frac{(m-1)a}{(m-2n)c} \int \frac{x^{m-2}}{\left(a+bx+cx^2\right)^{n+1/2}} dx.$$

Lemma 9 [Equation (2.263.2), Gradshteyn and Ryzhik, 2000].

$$\int \frac{x^{2n}}{\left(a+bx+cx^2\right)^{n+1/2}} dx = -\frac{x^{2n-1}}{(2n-1)c\left(a+bx+cx^2\right)^{n-1/2}} - \frac{b}{2c} \int \frac{x^{2n-1}}{\left(a+bx+cx^2\right)^{n+1/2}} dx + \frac{1}{c} \int \frac{x^{2n-2}}{\left(a+bx+cx^2\right)^{n-1/2}} dx.$$

Lemma 10 [Equation (2.263.4), Gradshteyn and Ryzhik, 2000]. *For* $n \geq 1$*,*

$$\int \frac{dx}{\left(a+bx+cx^2\right)^{n+1/2}} = \frac{2(2cx+b)}{(2n-1)\left(4ac-b^2\right)\left(a+bx+cx^2\right)^{n-1/2}} \times \left\{1 + \sum_{k=1}^{n-1} \frac{8^k (n-1)(n-2)\cdots(n-k)c^k}{(2n-3)(2n-5)\cdots(2n-2k-1)\left(4ac-b^2\right)^k} \left(a+bx+cx^2\right)^k \right\}.$$

Lemma 11 [Equation (2.3.8.1), Prudnikov et al., 1986, volume 1]. *For* $a > 0$*,* $\alpha > 0$ *and* $\beta > 0$*,*

$$\int_0^a x^{\alpha-1}(a-x)^{\beta-1}(x+z)^{-\rho} \exp(-px)\, dx = B(\alpha,\beta)\, z^{-\rho} a^{\alpha+\beta-1} \Phi_1\left(\alpha, \rho, \alpha+\beta; -a/z, ap\right).$$

Lemma 12 [Equation (2.3.6.9), Prudnikov et al., 1986, volume 1]. *For* $\alpha > 0$ *and* $p > 0$*,*

$$\int_0^\infty \frac{x^{\alpha-1}\exp(-px)}{(x+z)^\rho} dx = \Gamma(\alpha) z^{\alpha-\rho} \Psi\left(\alpha, \alpha+1-\rho; pz\right).$$

Lemma 13 [Equation (2.19.3.6), Prudnikov et al., 1986, volume 3]. *For* $\sigma - 1/2 < \alpha < -\rho$,

$$\int_0^\infty x^{\alpha-1} \exp(cx/2) W_{\rho,\sigma}(cx) dx =$$

$$\frac{\Gamma(1/2+\alpha+\sigma)\,\Gamma(1/2+\alpha-\sigma)\,\Gamma(-\alpha-\sigma)}{c^\alpha \Gamma(1/2-\alpha-\sigma)\,\Gamma(1/2-\alpha+\sigma)}.$$

The properties of the above special functions can be found in Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

2 McKay's Bivariate Gamma

Theorems 1 and 2 derive the pdfs of $R = X + Y$ and $W = X/(X+Y)$ when X and Y are distributed according to (1).

Theorem 1. *If X and Y are jointly distributed according to* (1) *then*

$$f_R(r) = \frac{a^{p+q} r^{p+q-1} \exp(-ar)}{2^p \Gamma(p)\Gamma(q)}\, {}_1F_1\left(p;\, p+q;\, \frac{ar}{2}\right) \tag{6}$$

for $0 < r < \infty$.

Proof. From (1), the joint pdf of $(R, W) = (X+Y, X/R)$ becomes

$$f(r, w) = \frac{a^{p+q} r^{p+q-1}}{\Gamma(p)\Gamma(q)} w^{p-1} (1-2w)^{q-1} \exp\{-ar(1-w)\}. \tag{7}$$

Thus, the pdf of R can be written as

$$f_R(r) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} r^{p+q-1} \exp(-ar)\, J(r), \tag{8}$$

where

$$J(r) = \int_0^{1/2} w^{p-1} (1-2w)^{q-1} \exp(arw)\, dw.$$

Substituting $u = 2w$, the integral $J(r)$ can be rewritten as

$$J(r) = 2^{-p} \int_0^1 u^{p-1} (1-u)^{q-1} \exp(aru/2)\, du. \tag{9}$$

Direct application of Lemma 1 shows that (9) can be calculated as

$$J(r) = 2^{-p} B(p, q)\, {}_1F_1\left(p;\, p+q;\, \frac{ar}{2}\right). \tag{10}$$

The result of the theorem follows by combining (8) and (10). □

Theorem 2. *If X and Y are jointly distributed according to* (1) *then*

$$f_W(w) = \frac{\Gamma(p+q) w^{p-1} (1-2w)^{q-1}}{\Gamma(p)\Gamma(q)(1-w)^{p+q}} \tag{11}$$

for $0 < w < 1/2$.

Proof. Using (7), one can write

$$f_W(w) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} w^{p-1} (1-2w)^{q-1} \int_0^{\infty} r^{p+q-1} \exp\left\{-ar(1-w)\right\}. \tag{12}$$

The result of the theorem follows by calculating the gamma integral on the right hand side of (12). □

Using special properties of the hypergeometric functions, one can derive elementary forms for the pdf in (6). This is illustrated in the corollaries below.

Corollary 1. *If X and Y are jointly distributed according to* (1) *and if $p = q$ then*

$$f_R(r) = \frac{\sqrt{\pi} a^{p+1/2} (-r)^{p-1/2}}{\sqrt{2}\Gamma(p)} \exp\left(\frac{ar}{4}\right) I_{p-1/2}\left(-\frac{ar}{4}\right)$$

for $0 < r < \infty$.

Corollary 2. *If X and Y are jointly distributed according to* (1) *and if $p = 3/2$ and $q = 1/2$ then*

$$f_R(r) = 2^{-3/2} a^2 r \exp\left(-\frac{3ar}{4}\right) \left\{ I_0\left(-\frac{ar}{4}\right) - I_1\left(-\frac{ar}{4}\right) \right\}$$

for $0 < r < \infty$.

Now, we derive the moments of $R = X + Y$ and $W = X/(X+Y)$ when X and Y are distributed according to (1). We need the following lemma.

Lemma 14. *If X and Y are jointly distributed according to* (1) *then*

$$E\left(X^m Y^n\right) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} \sum_{k=0}^{n} \binom{n}{k} \Gamma(p+m+n-k)\, \Gamma(q+k)$$

for $m \geq 1$ and $n \geq 1$.

Proof. The result following by writing

$$
\begin{aligned}
E\left(X^m Y^n\right) &= \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} \int_0^\infty \int_x^\infty x^{p+m-1} y^n (y-x)^{q-1} \exp(-ay)\, dy dx \\
&= \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} \int_0^\infty \int_0^\infty x^{p+m-1} (w+x)^n w^{q-1} \exp(-aw-ax)\, dw dx \\
&= \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} \sum_{k=0}^{n} \binom{n}{k} \left\{ \int_0^\infty x^{p+m+n-k-1} \exp(-ax)\, dx \right\} \\
&\qquad \left\{ \int_0^\infty w^{q+k-1} \exp(-aw)\, dw \right\} \\
&= \frac{1}{a^{m+n}\Gamma(p)\Gamma(q)} \sum_{k=0}^{n} \binom{n}{k} \Gamma(p+m+n-k)\Gamma(q+k),
\end{aligned}
$$

where we have set $w = y - x$ and used the binomial expansion for the term $(w + x)^n$. □

The moments of $R = X + Y$ are now simple consequences of this lemma as illustrated in Theorem 3. The moments of $W = X/(X + Y)$ require a separate treatment as shown by Theorem 4.

Theorem 3. *If X and Y are jointly distributed according to* (1) *then*

$$
E\left(R^n\right) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{j} \binom{j}{k} \Gamma(p+n-k)\Gamma(q+k)
$$

for $n \geq 1$.

Proof. The result follows by writing

$$
E\left((X+Y)^n\right) = \sum_{j=0}^{n} \binom{n}{j} E\left(X^{n-j} Y^j\right)
$$

and applying Lemma 14 to each expectation in the sum. □

Theorem 4. *If X and Y are jointly distributed according to* (1) *then*

$$
E\left(W^n\right) = \frac{B(p+n, q)}{2^{p+n} B(p, q)} \, {}_2F_1\left(p+n, p+q; p+q+n; \frac{1}{2}\right) \tag{13}
$$

for $n \geq 1$.

Proof. The result following by writing

$$\begin{aligned} E\left(W^n\right) &= \frac{1}{B(p,q)} \int_0^{1/2} \frac{w^{p+n-1}(1-2w)^{q-1}}{(1-w)^{p+q}} dw \\ &= \frac{1}{2^{p+n} B(p,q)} \int_0^1 \frac{u^{p+n-1}(1-u)^{q-1}}{(1-u/2)^{p+q}} du \\ &= \frac{B(p+n,q)}{2^{p+n} B(p,q)} {}_2F_1\left(p+n, p+q; p+q+n; \frac{1}{2}\right), \end{aligned}$$

where we have set $u = 2w$ and applied Lemma 2 in the last step. □

Using special properties of the Gauss hypergeometric function, one can derive elementary forms of (13) when p and q are integers. This is shown in the corollary below.

Corollary 3. *If X and Y are jointly distributed according to* (1) *and if both $p \geq 1$ and $q \geq 1$ are integers then*

$$E\left(W^n\right) =$$

$$\frac{1}{B(p,q)} \sum_{i=0}^{p+k-1} \sum_{j=0}^{q-1} \binom{p+k-1}{i} \binom{q-1}{j} (-1)^{i+j} 2^{q-1-j} \delta(i-j-p-1)$$

for $n \geq 1$, where $\delta(m) = (1-2^{-m-1})/(m+1)$ if $m \neq -1$ and $\delta(-1) = \log 2$.

3 Cherian's Bivariate Gamma

Theorem 5 derives the pdf of $W = X/(X+Y)$ when X and Y are distributed according to (2).

Theorem 5. *If X and Y are jointly distributed according to* (2) *then*

$$\begin{aligned} f_W(w) &= \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_2)\Gamma(\theta_1+\theta_3)} w^{\theta_1+\theta_3-1}(1-w)^{\theta_2-1} \\ &\quad \times F_1\left(\theta_1, 1-\theta_2, \theta_1+\theta_2+\theta_3, \theta_1+\theta_3; \frac{w}{1-w}, w\right) \end{aligned} \tag{14}$$

if $0 < w \leq 1/2$, *and*

$$f_W(w) = \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2+\theta_3)} w^{\theta_1-1}(1-w)^{\theta_2+\theta_3-1} \times F_1\left(\theta_3, 1-\theta_1, \theta_1+\theta_2+\theta_3, \theta_2+\theta_3; \frac{1-w}{w}, 1-w\right) \tag{15}$$

if $1/2 < w \leq 1$.

Proof. From (2), the joint pdf of $(R, W) = (X+Y, X/R)$ becomes

$$f(r,w) = \frac{r\exp(-r)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \int_0^{r\min(w,1-w)} (rw-z)^{\theta_1-1}(r(1-w)-z)^{\theta_2-1} z^{\theta_3-1}\exp(z)dz. \tag{16}$$

After setting $v = z/r$, (16) can be written as

$$f(r,w) = \frac{r^{\theta_1+\theta_2+\theta_3-1}\exp(-r)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \int_0^{\min(w,1-w)} (w-v)^{\theta_1-1}(1-w-v)^{\theta_2-1} v^{\theta_3-1}\exp(rv)dv.$$

Consider the cases $w \leq 1/2$ and $w > 1/2$ separately. If $w \leq 1/2$ then the pdf of W can be written as

$$\begin{aligned} f_W(w) &= \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \int_0^{\min(w,1-w)} \frac{(w-v)^{\theta_1-1}(1-w-v)^{\theta_2-1}v^{\theta_3-1}}{(1-v)^{\theta_1+\theta_2+\theta_3}} dv \\ &= \frac{\Gamma(\theta_1+\theta_2+\theta_3)\, w^{\theta_1+\theta_3-1}(1-w)^{\theta_2-1}}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \\ &\quad \times \int_0^1 (1-u)^{\theta_1-1}\left(1-\frac{wu}{1-w}\right)^{\theta_2-1} u^{\theta_3-1}(1-wu)^{-(\theta_1+\theta_2+\theta_3)}du \\ &= \frac{\Gamma(\theta_1+\theta_2+\theta_3)\, w^{\theta_1+\theta_3-1}(1-w)^{\theta_2-1}}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \\ &\quad \times B(\theta_1,\theta_3)\, F_1\left(\theta_1, 1-\theta_2, \theta_1+\theta_2+\theta_3, \theta_1+\theta_3; \frac{w}{1-w}, w\right), \end{aligned} \tag{17}$$

which follows by setting $u = v/w$ and applying Lemma 3. This establishes the result in (14). The result in (15) can be established similarly. □

Using special properties of the hypergeometric function, one can derive elementary forms for the pdfs in (14) and (15). This is illustrated in the corollary below.

Corollary 4. *If X and Y are jointly distributed according to* (2) *and if* $\theta_1 \geq 1$, $\theta_2 \geq 1$ *and* $\theta_3 \geq 1$ *are integers then*

$$f_W(w) =$$
$$\frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \sum_{i=0}^{\theta_1-1} \sum_{j=0}^{\theta_2-1} \sum_{k=0}^{\theta_3-1} \binom{\theta_1-1}{i} \binom{\theta_2-1}{j} \binom{\theta_3-1}{k} (-1)^{i+j+k+\theta_1+\theta_2-2}$$
$$\times (1-w)^{\theta_1-1-i} w^{\theta_2-1-j} \delta_1 (i+j+k-\theta_1-\theta_2-\theta_3)$$

if $0 < w \leq 1/2$, *and*

$$f_W(w) =$$
$$\frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} \sum_{i=0}^{\theta_1-1} \sum_{j=0}^{\theta_2-1} \sum_{k=0}^{\theta_3-1} \binom{\theta_1-1}{i} \binom{\theta_2-1}{j} \binom{\theta_3-1}{k} (-1)^{i+j+k+\theta_1+\theta_2-2}$$
$$\times (1-w)^{\theta_1-1-i} w^{\theta_2-1-j} \delta_2 (i+j+k-\theta_1-\theta_2-\theta_3)$$

if

$$\frac{1}{2} < w \leq 1, \quad \textit{where} \quad \delta_1(m) = \frac{1-(1-w)^{m+1}}{(m+1)}$$

if $m \neq -1$, $\delta_2(m) = \dfrac{1-w^{m+1}}{(m+1)}$ *if* $m \neq -1$, $\delta_1(m) = -\log(1-w)$ *and* $\delta_2(m) = -\log w$.

Now, we derive the moments of $W = X/(X+Y)$ when X and Y are distributed according to (2).

Theorem 6. *If X and Y are jointly distributed according to* (2) *then*

$$\begin{aligned} E\left(W^n\right) = & \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_2)\Gamma(\theta_1+\theta_3)} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(\theta_1)_{k+\ell}(1-\theta_2)_k(\theta_1+\theta_2+\theta_3)_\ell}{(\theta_1+\theta_3)_{k+\ell}} \\ & \times \frac{B_{1/2}(n+\theta_1+\theta_3+k+\ell, \theta_2-k)}{k!\ell!} \\ & + \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2+\theta_3)} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(\theta_3)_{k+\ell}(1-\theta_1)_k(\theta_1+\theta_2+\theta_3)_\ell}{(\theta_2+\theta_3)_{k+\ell}} \\ & \times \frac{1-B_{1/2}(n+\theta_1-k, \theta_2+\theta_3+k+\ell)}{k!\ell!} \end{aligned} \tag{18}$$

for $n \geq 1$.

Proof. From (14)–(15), using the definition of the hypergeometric function of two variables, one can write

$$E\left(W^n\right) = \frac{\Gamma\left(\theta_1+\theta_2+\theta_3\right)}{\Gamma\left(\theta_2\right)\Gamma\left(\theta_1+\theta_3\right)} \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{\left(\theta_1\right)_{k+\ell}\left(1-\theta_2\right)_k\left(\theta_1+\theta_2+\theta_3\right)_\ell I_1(k,\ell)}{\left(\theta_1+\theta_3\right)_{k+\ell}k!\ell!} + \frac{\Gamma\left(\theta_1+\theta_2+\theta_3\right)}{\Gamma\left(\theta_1\right)\Gamma\left(\theta_2+\theta_3\right)} \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{\left(\theta_3\right)_{k+\ell}\left(1-\theta_1\right)_k\left(\theta_1+\theta_2+\theta_3\right)_\ell I_2(k,\ell)}{\left(\theta_2+\theta_3\right)_{k+\ell}k!\ell!}, \tag{19}$$

where

$$I_1(k,\ell) = \int_0^{1/2} w^{k+\ell+\theta_1+\theta_3-1}(1-w)^{\theta_2-k-1}dw$$

and

$$I_2(k,\ell) = \int_{1/2}^1 w^{\theta_1-k-1}(1-w)^{k+\ell+\theta_2+\theta_3-1}dw.$$

Using the definition of the incomplete beta function, it follows that

$$I_1(k,\ell) = B_{1/2}\left(n+\theta_1+\theta_3+k+\ell, \theta_2-k\right) \tag{20}$$

and

$$I_2(k,\ell) = 1-B_{1/2}\left(n+\theta_1-k, \theta_2+\theta_3+k+\ell\right). \tag{21}$$

The result of the theorem follows by (19), (20) and (21). □

4 Kibble's Bivariate Gamma

Theorems 7 and 8 derive the pdfs of $R = X+Y$ and $W = X/(X+Y)$ when X and Y are distributed according to (3).

Theorem 7. *If X and Y are jointly distributed according to* (3) *then*

$$f_R(r) = \frac{\sqrt{\pi}2^{1/2-\alpha}r^{\alpha-1/2}\rho^{1/4-\alpha/2}}{\Gamma(\alpha)\sqrt{1-\rho}} I_{\alpha-1/2}\left(\frac{r\sqrt{\rho}}{1-\rho}\right) \tag{22}$$

for $0 < r < \infty$.

Proof. From (3), the joint pdf of $(R, W) = (X+Y, X/R)$ becomes

$$f(r,w) = \frac{r^\alpha\{w(1-w)\}^{(\alpha-1)/2}}{\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}} \exp\left(-\frac{r}{1-\rho}\right) I_{\alpha-1}\left(\frac{2r\sqrt{\rho}\sqrt{w(1-w)}}{1-\rho}\right). \tag{23}$$

Thus, the pdf of R can be written as

$$f_R(r) \;=\; \frac{r^\alpha}{\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}} \exp\left(-\frac{r}{1-\rho}\right) J(r), \tag{24}$$

where

$$J(r) \;=\; \int_0^1 \{w(1-w)\}^{(\alpha-1)/2} I_{\alpha-1}\left(\frac{2r\sqrt{\rho}\sqrt{w(1-w)}}{1-\rho}\right) dw.$$

Substituting $u = \sqrt{w(1-w)}$, the integral $J(r)$ can be rewritten as

$$J(r) \;=\; 2\int_0^{1/2} \frac{u^\alpha}{\sqrt{1/4-u^2}} I_{\alpha-1}\left(\frac{2r\sqrt{\rho}u}{1-\rho}\right) du. \tag{25}$$

Direct application of Lemma 5 shows that (25) can be calculated as

$$J(r) \;=\; 2^{3/2-\alpha}\sqrt{\pi}\left(\frac{2r\sqrt{\rho}}{1-\rho}\right)^{-1/2} I_{\alpha-1/2}\left(\frac{r\sqrt{\rho}}{1-\rho}\right). \tag{26}$$

The result of the theorem follows by combining (24) and (26). □

Theorem 8. *If X and Y are jointly distributed according to* (3) *then*

$$f_W(w) \;=\; \frac{\Gamma(2\alpha)(1-\rho)^\alpha}{\Gamma^2(\alpha)} \frac{w^{\alpha-1}(1-w)^{\alpha-1}}{\{1-4\rho w(1-w)\}^{\alpha+1/2}} \tag{27}$$

for $0 < w < 1$.

Proof. Using (23), one can write

$$f_W(w) \;=\; \frac{\{w(1-w)\}^{(\alpha-1)/2}}{\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}} J(w), \tag{28}$$

where

$$J(w) \;=\; \int_0^\infty r^\alpha \exp\left(-\frac{r}{1-\rho}\right) I_{\alpha-1}\left(\frac{2r\sqrt{\rho}\sqrt{w(1-w)}}{1-\rho}\right) dr.$$

Direct application of Lemma 4 shows that one can calculate $J(w)$ as

$$\begin{aligned} J(w) \;=&\; \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \rho^{(\alpha-1)/2}(1-\rho)^{\alpha+1}\{w(1-w)\}^{(\alpha-1)/2} \\ &{}_2F_1\left(\alpha, \alpha+\frac{1}{2}; \alpha; 4\rho w(1-w)\right). \end{aligned} \tag{29}$$

Upon using the property that

$$ {}_2F_1\left(a, b; a; x\right) = (1-x)^{-b}, $$

one can reduce (29) to

$$ \begin{aligned} J(w) = & \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \rho^{(\alpha-1)/2} (1-\rho)^{\alpha+1} \\ & \{w(1-w)\}^{(\alpha-1)/2} \{1 - 4\rho w(1-w)\}^{-(\alpha+1/2)} . \end{aligned} \tag{30} $$

The result of the theorem follows by combining (28) and (30). □

Using special properties of the Bessel function of the first kind, one can derive elementary forms for the pdfs in (22). This is illustrated in the corollary below.

Corollary 5. *If X and Y are jointly distributed according to* (3) *and if $\alpha \geq 1$ is an integer then*

$$ \begin{aligned} f_R(r) = & \frac{2^{1-\alpha} r^{\alpha-1/2} \rho^{1/4-\alpha/2}}{\Gamma(\alpha)\sqrt{1-\rho}\sqrt{z}} \\ & \left\{ a \sum_{j=0}^{[(2|\alpha-1/2|-3)/4]} \frac{(-1)^j (2j + \mid \alpha - 1/2 \mid + 1/2)! (2z)^{-2j-1}}{(2j+1)!(-2j + \mid \alpha - 1/2 \mid -3/2)!} \right. \\ & \left. - \ b \sum_{j=0}^{[(2|\alpha-1/2|-1)/4]} \frac{(-1)^j (2j + \mid \alpha - 1/2 \mid - 1/2)!}{(2j)!(-2j + \mid \alpha - 1/2 \mid -1/2)!(2z)^{2j}} \right\}, \end{aligned} $$

where

$$ z = r\sqrt{\rho}/(1-\rho), a = \cos(\pi(\alpha-1)/2 - z), \quad \text{and} \quad b = \sin(\pi(\alpha-1)/2 - z). $$

Now, we derive the moments of $R = X + Y$, $P = XY$ and $W = X/(X+Y)$ when X and Y are distributed according to (3). We need the following lemma.

Lemma 15. *If X and Y are jointly distributed according to* (3) *then*

$$ E\left(X^m Y^n\right) = \frac{n!\Gamma(m+\alpha)(1-\rho)^n}{\Gamma(\alpha)} P_n^{(\alpha-1, m-n)} \left(\frac{1+\rho}{1-\rho}\right) $$

for $m \geq 1$ and $n \geq 1$.

Proof. One can express

$$
\begin{aligned}
E\left(X^m Y^n\right) &= \frac{1}{\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}} \int_0^\infty x^{m+(\alpha-1)/2} \exp\left(-\frac{x}{1-\rho}\right) \\
&\quad \times \int_0^\infty y^{n+(\alpha-1)/2} \exp\left(-\frac{y}{1-\rho}\right) I_{\alpha-1}\left(\frac{2\sqrt{xy\rho}}{1-\rho}\right) dy dx \\
&= \frac{2}{\Gamma(\alpha)(1-\rho)\rho^{(\alpha-1)/2}} \int_0^\infty x^{m+(\alpha-1)/2} \exp\left(-\frac{x}{1-\rho}\right) \qquad (31) \\
&\quad \times \int_0^\infty w^{2n+\alpha} \exp\left(-\frac{w^2}{1-\rho}\right) I_{\alpha-1}\left(\frac{2\sqrt{x\rho}w}{1-\rho}\right) dw dx \\
&= \frac{n!(1-\rho)^n}{\Gamma(\alpha)} \int_0^\infty x^{m+\alpha-1} \exp(-x)\, L_n^{\alpha-1}\left(-\frac{x\rho}{1-\rho}\right) dx,
\end{aligned}
$$

which follows after setting $w = \sqrt{y}$ and applying Lemma 6. The integral in (31) can be calculated by direct application of Lemma 7 to yield

$$
\begin{aligned}
& \int_0^\infty x^{m+\alpha-1} \exp(-x)\, L_n^{\alpha-1}\left(-\frac{x\rho}{1-\rho}\right) dx \\
& = \Gamma(m+\alpha) P_n^{(\alpha-1,m-n)}\left(\frac{1+\rho}{1-\rho}\right).
\end{aligned}
\qquad (32)
$$

The result of the lemma follows by combining (31) and (32). □

The moments of $R = X + Y$ are now simple consequences of this lemma as illustrated in Theorem 9. The moments of $W = X/(X+Y)$ require a separate treatment as shown by Theorem 10.

Theorem 9. *If X and Y are jointly distributed according to* (3) *then*

$$
E\left(R^n\right) = \sum_{k=0}^{n} \frac{n!\Gamma(n-k+\alpha)(1-\rho)^k}{(n-k)!\Gamma(\alpha)} P_n^{(\alpha-1,n-2k)}\left(\frac{1+\rho}{1-\rho}\right) \qquad (33)
$$

for $n \geq 1$.

Proof. The result in (33) follows by writing

$$
E\left((X+Y)^n\right) = \sum_{k=0}^{n} \binom{n}{k} E\left(X^{n-k} Y^k\right)
$$

and applying Lemma 15 to each expectation in the sum. □

Theorem 10. *If X and Y are jointly distributed according to* (3) *and if α is an integer then*

$$E\left(W^n\right) = \frac{\Gamma(2\alpha)(1-\rho)^\alpha}{\Gamma^2(\alpha)} \sum_{m=0}^{\alpha-1} (-1)^m \binom{\alpha-1}{m} J(n+m+\alpha-1), \qquad (34)$$

for $n \geq 1$, where $J(m)$ satisfies the recurrence relations

$$J(m) = \frac{1}{4\rho(m-2\alpha)} + \frac{2m-2\alpha-1}{2(m-2\alpha)} J(m-1) - \frac{m-1}{4\rho(m-2\alpha)} J(m-2) \qquad (35)$$

and

$$J(2\alpha) = \frac{1}{4\rho(1-2\alpha)} + \frac{1}{2} J(2\alpha-1) - \frac{1}{4\alpha} J(2\alpha-2) \qquad (36)$$

for $m \neq 2\alpha$, with the initial values

$$J(0) = \frac{1}{(2\alpha-1)(1-\rho)} \left\{1 + \sum_{k=1}^{\alpha-1} \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-k)(2\rho)^k}{(2\alpha-3)(2\alpha-5)\cdots(2\alpha-2k-1)\rho^k(1-\rho)^k}\right\} \qquad (37)$$

and $J(1) = (1/2)J(0)$.

Proof. It follows from (27) that

$$E\left(W^n\right) = \frac{\Gamma(2\alpha)(1-\rho)^\alpha}{\Gamma^2(\alpha)} \int_0^1 \frac{w^{n+\alpha-1}(1-w)^{\alpha-1}}{(4\rho w^2 - 4\rho w + 1)^{\alpha+1/2}} dw$$

$$= \frac{\Gamma(2\alpha)(1-\rho)^\alpha}{\Gamma^2(\alpha)} \int_0^1 \left\{\sum_{m=0}^{\alpha-1} (-1)^m \binom{\alpha-1}{m} w^m\right\} \frac{w^{n+\alpha-1}}{(4\rho w^2 - 4\rho w + 1)^{\alpha+1/2}} dw$$

$$= \frac{\Gamma(2\alpha)(1-\rho)^\alpha}{\Gamma^2(\alpha)} \sum_{m=0}^{\alpha-1} (-1)^m \binom{\alpha-1}{m} J(n+m+\alpha-1),$$

where

$$J(m) = \int_0^1 \frac{w^m}{\left(4\rho w^2 - 4\rho w + 1\right)^{\alpha+1/2}} dw.$$

This establishes the result in (34). The recurrence relations in (35) and (36) follow by applying Lemmas 8 and 9. The initial value in (37) follows by applying

Lemma 10. Finally, since

$$\int_0^1 \frac{2w-1}{\left(4\rho w^2 - 4\rho w + 1\right)^{\alpha+1/2}} dw = 0,$$

it follows that $J(1) = (1/2)J(0)$. □

5 Beta Stacy

Theorems 11 and 12 derive the pdfs of $R = X + Y$ and $W = X/(X + Y)$ when X and Y are distributed according to (4).

Theorem 11. *If X and Y are jointly distributed according to* (4) *then*

$$f_R(r) = \frac{cr^{bc-1}}{a^{bc}\Gamma(b)} \sum_{k=0}^{\infty} \frac{(-1)^k r^{kc}}{a^{kc}} \, {}_2F_1\left(p, p+q-bc-kc; p+q; \frac{1}{2}\right) \tag{38}$$

for $0 < r < \infty$.

Proof. From (4), the joint pdf of $(R, W) = (X + Y, X/R)$ becomes

$$f(r, w) = \frac{cr^{bc-1}}{a^{bc}\Gamma(b)B(p, q)} w^{p-1}(1-2w)^{q-1}(1-w)^{bc-p-q} \exp\left\{-\frac{r^c(1-w)^c}{a^c}\right\}. \tag{39}$$

Using the series expansion

$$\exp(-x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!},$$

the pdf of R can be written as

$$f_R(r) = \frac{cr^{bc-1}}{a^{bc}\Gamma(b)B(p, q)} w^{p-1}(1-2w)^{q-1}(1-w)^{bc-p-q} \left\{\sum_{k=0}^{\infty} \frac{(-1)^k r^{ck}(1-w)^{ck}}{a^{ck}k!}\right\} dw$$

$$
\begin{aligned}
&= \frac{c r^{bc-1}}{a^{bc}\Gamma(b)B(p,q)} \sum_{k=0}^{\infty} \frac{(-1)^k r^{kc}}{a^{kc}} \\
&\quad \int_0^{1/2} w^{p-1}(1-w)^{bc+kc-p-q}(1-2w)^{q-1} dw \\
&= \frac{c r^{bc-1}}{2^p a^{bc}\Gamma(b)B(p,q)} \sum_{k=0}^{\infty} \frac{(-1)^k r^{kc}}{a^{kc}} \\
&\quad \int_0^{1} u^{p-1}(1-u/2)^{bc+kc-p-q}(1-u)^{q-1} du \\
&= \frac{c r^{bc-1}}{2^p a^{bc}\Gamma(b)} \sum_{k=0}^{\infty} \frac{(-1)^k r^{kc}}{a^{kc}} \, {}_2F_1\left(p, p+q-bc-kc; p+q; \frac{1}{2}\right),
\end{aligned}
$$

where we have set $u = 2w$ and applied Lemma 2 (in the last step). The proof of the theorem is complete. □

Theorem 12. *If X and Y are jointly distributed according to* (4) *then*

$$
f_W(w) = \frac{w^{p-1}(1-2w)^{q-1}}{B(p,q)(1-w)^{p+q}} \tag{40}
$$

for $0 < w < 1/2$.

Proof. Using (39), one can write

$$
\begin{aligned}
f_W(w) &= \frac{c}{a^{bc}\Gamma(b)B(p,q)} w^{p-1}(1-2w)^{q-1}(1-w)^{bc-p-q} \\
&\quad \int_0^{\infty} r^{bc-1} \exp\left\{-\frac{r^c(1-w)^c}{a^c}\right\} dr \\
&= \frac{1}{a^{bc}\Gamma(b)B(p,q)} w^{p-1}(1-2w)^{q-1}(1-w)^{bc-p-q} \\
&\quad \int_0^{\infty} y^{b-1} \exp\left\{-\frac{y(1-w)^c}{a^c}\right\} dy \\
&= \frac{w^{p-1}(1-2w)^{q-1}}{B(p,q)(1-w)^{p+q}},
\end{aligned}
$$

where we have set $y = r^c$ and used the definition of gamma function. The proof of the theorem is complete. □

Using special properties of the Gauss hypergeometric function, one can derive simpler forms for the pdf in (38). This is illustrated in the corollary below.

Corollary 6. *If X and Y are jointly distributed according to* (4) *and if $q = 1$ then*

$$f_R(r) = \frac{cr^{bc-1}}{a^{bc}\Gamma(b)} \sum_{k=0}^{\infty} \frac{(-1)^k r^{kc}}{a^{kc}} B_{1/2}\left(p, bc + kc - p\right)$$

for $0 < r < \infty$.

Now, we derive the moments of $R = X + Y$ and $W = X/(X + Y)$ when X and Y are distributed according to (4). We need the following lemma.

Lemma 16. *If X and Y are jointly distributed according to* (4) *then*

$$E\left(X^m Y^n\right) = \frac{a^{m+n}\Gamma\left(b + (m+n)/c\right) B\left(p + m, q\right)}{\Gamma(b) B(p, q)}$$

for $m \geq 1$ and $n \geq 1$.

Proof. The result follows by writing

$$\begin{aligned} E\left(X^m Y^n\right) &= \frac{c}{a^{bc}\Gamma(b)B(p,q)} \int_0^\infty \int_x^\infty x^{p+m-1}(y-x)^{q-1} y^{bc+n-p-q} \\ &\quad \exp\left\{-\left(\frac{y}{a}\right)^c\right\} dydx \\ &= \frac{c}{a^{bc}\Gamma(b)B(p,q)} \int_0^\infty \int_x^\infty x^{p+m-1}(y-x)^{q-1} \\ &\quad \times y^{-(p+m)-q+c(b+(m+n)/c)} \exp\left\{-\left(\frac{y}{a}\right)^c\right\} dydx \\ &= \frac{c}{a^{bc}\Gamma(b)B(p,q)} \frac{a^{bc+m+n}\Gamma\left(b+(m+n)/c\right) B(p+m,q)}{c}. \end{aligned}$$

The proof of the theorem is complete. □

The moments of $R = X + Y$ are now simple consequences of this lemma as illustrated in Theorem 13. The moments of $W = X/(X + Y)$ require a separate treatment as shown by Theorem 14.

Theorem 13. *If X and Y are jointly distributed according to* (4) *then*

$$E\left(R^n\right) = \frac{a^n \Gamma\left(b+n/c\right)}{\Gamma(b) B(p,q)} \sum_{k=0}^{n} \binom{n}{k} B\left(p+n-k, q\right) \tag{41}$$

for $n \geq 1$.

Proof. The result in (41) follows by writing

$$E\left((X+Y)^n\right) = \sum_{k=0}^{n} \binom{n}{k} E\left(X^{n-k} Y^k\right)$$

and applying Lemma 16 to each expectation in the sum. □

Theorem 14. *If X and Y are jointly distributed according to* (4) *then*

$$E\left(W^n\right) = \frac{B(p+n,q)}{2^{p+n} B(p,q)} \, {}_2F_1\left(p+n, \, p+q; \, p+q+n; \, \frac{1}{2}\right) \tag{42}$$

for $n \neq 1$.

Proof. It follows from (40) that one can write

$$\begin{aligned} E\left(W^n\right) &= \int_0^{1/2} \frac{w^{p+n-1}(1-2w)^{q-1}}{B(p,q)(1-w)^{p+q}} dw \\ &= \int_0^{1} \frac{u^{p+n-1}(1-u)^{q-1}}{2^{p+n} B(p,q)(1-u/2)^{p+q}} du \\ &= \frac{B(p+n,q)}{2^{p+n} B(p,q)} \, {}_2F_1\left(p+n, \, p+q; \, p+q+n; \, \frac{1}{2}\right), \end{aligned}$$

where we have set $u = 2w$ and applied Lemma 2 in the last step. □

Using special properties of the Gauss hypergeometric function, one can derive elementary forms for (42) when p and q are integers. This is shown in the corollary below.

Corollary 7. *If X and Y are jointly distributed according to* (4) *and if both* $p \geq 1$ *and* $q \geq 1$ *are integers then*

$$E\left(W^n\right) = \frac{1}{B(p,q)} \sum_{i=0}^{n+p-1} \sum_{j=0}^{q-1} \binom{n+p-1}{i} \binom{q-1}{j} (-1)^{i+j} 2^{q-1-j} \delta(q-1+i-j),$$

for $n \geq 1$*, where* $\delta(m) = (1-2^{-m-1})/(m+1)$ *if* $m \neq -1$ *and* $\delta(-1) = \log 2$.

6 Becker and Roux's Bivariate Gamma

Theorems 15 and 16 derive the pdfs of $R = X + Y$ and $W = X/(X+Y)$ when X and Y are distributed according to (5).

Theorem 15. *If X and Y are jointly distributed according to* (5) *then*

$$\begin{aligned} f_R(r) &= \frac{\alpha^a (\beta')^b r^{a+b-1} \exp(-\beta' r)}{2^a \Gamma^2(a)\Gamma(b)} \Phi_1 \\ &\quad \left(a, 1-b, 1+a; -\frac{\beta - 2\beta'}{2\beta'}, \frac{(\alpha+\beta-2\beta')r}{2}\right) \\ &\quad + \frac{(\alpha')^a \beta^b r^{a+b-1} \exp(-\alpha' r)}{2^b \Gamma(a)\Gamma^2(b)} \Phi_1 \\ &\quad \left(b, 1-a, 1+b; -\frac{\alpha - 2\alpha'}{2\alpha'}, \frac{(\alpha+\beta-2\alpha')r}{2}\right) \end{aligned} \tag{43}$$

for $0 < r < \infty$.

Proof. From (5), the joint pdf of $(R, W) = (X+Y, X/R)$ can be written as

$$f(r,w) = \begin{cases} \dfrac{\beta' \alpha^a}{\Gamma(a)\Gamma(b)} r^{a+b-1} w^{a-1} \left\{\beta'(1-2w) + \beta w\right\}^{b-1} \\ \quad \times \exp\left\{-\beta' r(1-w) - (\alpha+\beta-\beta')rw\right\}, & \text{if } w < 1/2, \\ \dfrac{\alpha' \beta^b}{\Gamma(a)\Gamma(b)} r^{a+b-1} (1-w)^{b-1} \left\{\alpha'(2w-1) + \alpha(1-w)\right\}^{a-1} \\ \quad \times \exp\left\{-\alpha' rw - (\alpha+\beta-\alpha')r(1-w)\right\}, & \text{if } w > 1/2. \end{cases} \tag{44}$$

Thus, the pdf of R can be written as

$$\begin{aligned} f_R(r) &= \frac{\beta' \alpha^a (\beta - 2\beta')^{b-1} r^{a+b-1} \exp(-\beta' r)}{\Gamma(a)\Gamma(b)} I_1(r) \\ &\quad + \frac{\alpha' \beta^b (\alpha - 2\alpha')^{a-1} r^{a+b-1} \exp(-\alpha' r)}{\Gamma(a)\Gamma(b)} I_2(r), \end{aligned}$$

where $I_1(r)$ and $I_2(r)$ are the integrals given by

$$I_1(r) = \int_0^{1/2} w^{a-1} \left(\frac{\beta'}{\beta - 2\beta'} + w \right)^{b-1} \exp\left\{ -(\alpha + \beta - 2\beta') r w \right\} dw$$

and

$$I_2(r) = \int_0^{1/2} w^{b-1} \left(\frac{\alpha'}{\alpha - 2\alpha'} + w \right)^{a-1} \exp\left\{ -(\alpha + \beta - 2\alpha') r w \right\} dw.$$

Application of Lemma 11 shows that one can calculate $I_1(r)$ and $I_2(r)$ as

$$\begin{aligned} I_1(r) &= \frac{(\beta')^{b-1}}{a 2^a (\beta - 2\beta')^{b-1}} \Phi_1 \\ &\quad \left(a, 1-b, 1+a; -\frac{\beta - 2\beta'}{2\beta'}, \frac{(\alpha + \beta - 2\beta') r}{2} \right) \end{aligned} \tag{45}$$

and

$$\begin{aligned} I_2(r) &= \frac{(\alpha')^{a-1}}{b 2^b (\alpha - 2\alpha')^{a-1}} \Phi_1 \\ &\quad \left(b, 1-a, 1+b; -\frac{\alpha - 2\alpha'}{2\alpha'}, \frac{(\alpha + \beta - 2\alpha') r}{2} \right). \end{aligned} \tag{46}$$

The result of the theorem follows by substituting (45) and (46) into (44). □

Theorem 16. *If X and Y are jointly distributed according to* (5) *then*

$$f_W(w) = \begin{cases} \dfrac{\beta' \alpha^a \Gamma(a+b)}{\Gamma(a)\Gamma(b)} \dfrac{w^{a-1} \left\{ \beta'(1-2w) + \beta w \right\}^{b-1}}{\left\{ \beta'(1-w) + (\alpha + \beta - \beta') w \right\}^{a+b}}, \\ \quad \text{if } w < 1/2, \\ \dfrac{\alpha' \beta^b \Gamma(a+b)}{\Gamma(a)\Gamma(b)} \dfrac{(1-w)^{b-1} \left\{ \alpha'(2w-1) + \alpha(1-w) \right\}^{a-1}}{\left\{ \alpha' w + (\alpha + \beta - \alpha')(1-w) \right\}^{a+b}}, \\ \quad \text{if } w > 1/2, \end{cases} \tag{47}$$

for $0 < w < 1$.

Proof. Using (44), one can write

$$
f_W(w) = \begin{cases} \dfrac{\beta' \alpha^a}{\Gamma(a)\Gamma(b)} w^{a-1} \{\beta'(1-2w) + \beta w\}^{b-1} \\ \qquad \times \displaystyle\int_0^\infty r^{a+b-1} \exp\{-\beta' r(1-w) - (\alpha+\beta-\beta') r w\} dr, \\ \qquad \text{if } w < 1/2, \\ \dfrac{\alpha' \beta^b}{\Gamma(a)\Gamma(b)} (1-w)^{b-1} \{\alpha'(2w-1) + \alpha(1-w)\}^{a-1} \\ \qquad \times \displaystyle\int_0^\infty r^{a+b-1} \exp\{-\alpha' r w - (\alpha+\beta-\alpha') r(1-w)\} dr, \\ \qquad \text{if } w > 1/2. \end{cases}
$$

The result of the theorem follows by elementary integration of the above integrals. □

Using special properties of the hypergeometric function of two variables, one can derive simpler forms for the pdf in (43). This is illustrated in the corollary below.

Corollary 8. *If X and Y are jointly distributed according to* (5) *and if both* $a \geq 1$ *and* $b \geq 1$ *are integers then*

$$
\begin{aligned}
f_R(r) = \; & K_1(r) \sum_{k=0}^{b-1} \binom{b-1}{k} B(k+a, b-k) \left(\frac{\beta}{2\beta'}\right)^k \\
& {}_1F_1\left(k+a; a+b; -\frac{r(\alpha+\beta-2\beta')}{2}\right) \\
& + K_2(r) \sum_{k=0}^{a-1} \binom{a-1}{k} B(k+b, a-k) \left(\frac{\alpha}{2\alpha'}\right)^k \\
& {}_1F_1\left(k+b; a+b; -\frac{r(\alpha+\beta-2\alpha')}{2}\right),
\end{aligned}
$$

where

$$
K_1(r) = \frac{\alpha^a (\beta')^b r^{a+b-1} \exp(-\beta' r)}{2^a \Gamma(a)\Gamma(b)}
$$

and

$$K_2(r) = \frac{\beta^b (\alpha^{'})^a r^{a+b-1} \exp(-\alpha^{'} r)}{2^b \Gamma(a)\Gamma(b)}.$$

Now, we derive the moments of $R = X + Y$ and $W = X/(X+Y)$ when X and Y are distributed according to (5). We need the following lemma.

Lemma 17. *If X and Y are jointly distributed according to* (5) *then*

$$E\left(X^m Y^n\right) =$$

$$\frac{\alpha^a n! \Gamma(m+n+a+b)}{\beta^{m+a} (\beta^{'})^n \Gamma(a)\Gamma(b)\Gamma(1-b)} \sum_{k=0}^{n} \frac{(\beta^{'})^k \Gamma(m+a+k)\Gamma(1-a-b-m-k)}{k!(n-k)!}$$

$$+ \frac{\beta^b m! \Gamma(m+n+a+b)}{\alpha^{n+b} (\alpha^{'})^m \Gamma(a)\Gamma(b)\Gamma(1-a)} \sum_{k=0}^{m} \frac{(\alpha^{'})^k \Gamma(n+b+k)\Gamma(1-a-b-n-k)}{k!(m-k)!}$$

for $m \geq 1$ and $n \geq 1$.

Proof. One can write

$$\begin{aligned} E\left(X^m Y^n\right) &= \frac{\beta^{'} \alpha^a}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_x^\infty x^{m+a-1} y^n \{\beta^{'}(y-x) + \beta x\}^{b-1} \\ &\quad \times \exp\{-\beta^{'} y - (\alpha + \beta - \beta^{'})x\} dy dx \\ &\quad + \frac{\alpha^{'} \beta^b}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_y^\infty x^m y^{n+b-1} \{\alpha^{'}(x-y) + \alpha y\}^{a-1} \\ &\quad \times \exp\{-\alpha^{'} x - (\alpha + \beta - \alpha^{'})y\} dx dy \\ &= \frac{\beta^{'} \alpha^a}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty x^{m+a-1} (z+x)^n \{\beta^{'} z + \beta x\}^{b-1} \qquad (48) \\ &\quad \times \exp\{-\beta^{'} z - (\alpha+\beta)x\} dz dx \\ &\quad + \frac{\alpha^{'} \beta^b}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty y^{n+b-1} (z+y)^m \{\alpha^{'} z + \alpha y\}^{a-1} \\ &\quad \times \exp\{-\alpha^{'} z - (\alpha+\beta)y\} dz dy \\ &= \frac{\beta^{'} \alpha^a}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{n} \binom{n}{k} I_1(k) + \frac{\alpha^{'} \beta^b}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{m} \binom{m}{k} I_2(k), \end{aligned}$$

where

$$I_1(k) \;=\; \int_0^\infty \int_0^\infty x^{m+a-1+k} z^{n-k} (\beta^{'} z + \beta x)^{b-1} \exp\{-\beta^{'} z - (\alpha + \beta) x\} dz dx$$

and

$$I_2(k) \;=\; \int_0^\infty \int_0^\infty y^{n+k+b-1} z^{m-k} (\alpha^{'} z + \alpha y)^{a-1} \exp\{-\alpha^{'} z - (\alpha + \beta) y\} dz dy.$$

These two integrals can be calculated as follows: application of Lemma 12 shows that one can write $I_1(k)$ as

$$\begin{aligned} I_1(k) \;&=\; \beta^{n+b-k} (\beta^{'})^{k-n-1}\, \Gamma(n-k+1) \\ &\quad \int_0^\infty x^{m+n+a+b-1}\, \Psi\big(n-k+1, n+1-k+b; \beta x\big) dx \\ &=\; \beta^{(n+b-k-1)/2} (\beta^{'})^{k-n-1}\, \Gamma(n-k+1) \\ &\quad \times \int_0^\infty x^{m+n/2+a+b/2+k/2-3/2} \\ &\quad \exp\,(\beta x/2)\; W_{(n+1-k+b)/2,(n+b-k)/2}(\beta x) dx, \end{aligned} \tag{49}$$

where the second step following by using the relationship that

$$\Psi(a, b; x) \;=\; x^{-b/2} \exp\,(x/2)\; W_{b/2-a,(b-1)/2}(x).$$

Now, an application of Lemma 13 shows that (49) can be calculated as

$$I_1(k) = \frac{(\beta^{'})^{k-n-1} \Gamma\,(m+n+a+b)\, \Gamma\,(m+a+k)\, \Gamma\,(1-m-a-b-k)}{\beta^{m+a+k} \Gamma\,(1-b)}. \tag{50}$$

One can show similarly that

$$I_2(k) = \frac{(\alpha^{'})^{k-m-1} \Gamma\,(m+n+a+b)\, \Gamma\,(n+b+k)\, \Gamma\,(1-n-a-b-k)}{\beta^{n+b+k} \Gamma\,(1-a)}. \tag{51}$$

The result of the theorem follows by substituting (50) and (51) into (48). □

The moments of $R = X + Y$ are now simple consequences of this lemma as illustrated in Theorem 17.

Theorem 17. *If X and Y are jointly distributed according to* (5) *then*

$$E\left(R^n\right) = \sum_{k=0}^{n} \binom{n}{k} \left[\frac{\alpha^a k! \Gamma\left(n+a+b\right)}{\beta^{n-k+a} (\beta^{'})^k \Gamma(a)\Gamma(b)\Gamma(1-b)} \right.$$
$$\times \sum_{\ell=0}^{k} \frac{(\beta^{'})^{\ell}\Gamma(n-k+a+\ell)\Gamma(1-a-b-n+k-\ell)}{\ell!(k-\ell)!}$$
$$+ \frac{\beta^b (n-k)!\Gamma(n+a+b)}{\alpha^{k+b}(\alpha^{'})^{n-k}\Gamma(a)\Gamma(b)\Gamma(1-a)} \tag{52}$$
$$\left. \times \sum_{\ell=0}^{n-k} \frac{(\alpha^{'})^{\ell}\Gamma(k+b+\ell)\Gamma(1-a-b-k-\ell)}{\ell!(n-k-\ell)!} \right]$$

for $n \geq 1$.

Proof. The result in (52) follows by writing

$$E\left((X+Y)^n\right) = \sum_{k=0}^{n} \binom{n}{k} E\left(X^{n-k} Y^k\right)$$

and applying Lemma 17 to each expectation in the sum. □

The moments of $W = X/(X+Y)$ require a separate treatment as shown by Theorem 18.

Theorem 18. *If X and Y are jointly distributed according to* (5) *then*

$$E\left(W^n\right) = \frac{\alpha^a \Gamma(a+b)}{(\beta^{'})^a \Gamma(a)\Gamma(b)(a+n)} F_1\left(a+n, 1-b, a+b, \right.$$
$$\left. a+n+1; -\frac{\beta-2\beta^{'}}{2\beta^{'}}, -\frac{\alpha+\beta-2\beta^{'}}{2\beta^{'}}\right)$$
$$+ \frac{\beta^b \Gamma(a+b)}{(\alpha^{'})^b \Gamma(a)\Gamma(b)(b+n)} F_1\left(b+n, 1-a, a+b, \right. \tag{53}$$
$$\left. b+n+1; -\frac{\alpha-2\alpha^{'}}{2\alpha^{'}}, -\frac{\alpha+\beta-2\alpha^{'}}{2\alpha^{'}}\right)$$

for $n \geq 1$.

Proof. One can express

$$E\left(W^n\right) = \frac{\beta^{'}\alpha^a\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^{1/2}\frac{w^{a+n-1}\left\{\beta^{'}(1-2w)+\beta w\right\}^{b-1}}{\left\{\beta^{'}(1-w)+(\alpha+\beta-\beta^{'})w\right\}^{a+b}}dw$$

$$+\frac{\alpha^{'}\beta^b\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_{1/2}^{1}\frac{w^n(1-w)^{b-1}\left\{\alpha^{'}(2w-1)+\alpha(1-w)\right\}^{a-1}}{\left\{\alpha^{'}w+(\alpha+\beta-\alpha^{'})(1-w)\right\}^{a+b}}dw \tag{54}$$

$$= \frac{\beta^{'}\alpha^a\Gamma(a+b)I_1}{\Gamma(a)\Gamma(b)}+\frac{\alpha^{'}\beta^b\Gamma(a+b)I_2}{\Gamma(a)\Gamma(b)},$$

where

$$I_1 = \int_0^{1/2}\frac{w^{a+n-1}\left\{\beta^{'}(1-2w)+\beta w\right\}^{b-1}}{\left\{\beta^{'}(1-w)+\left(\alpha+\beta-\beta^{'}\right)w\right\}^{a+b}}dw$$

and

$$I_2 = \int_0^{1/2}\frac{(1-w)^n w^{b-1}\left\{\alpha^{'}(1-2w)+\alpha w\right\}^{a-1}}{\left\{\alpha^{'}(1-w)+\left(\alpha+\beta-\alpha^{'}\right)w\right\}^{a+b}}dw.$$

Application of Lemma 3 shows that I_1 and I_2 can be calculated as

$$I_1 = \frac{1}{2^{a+n}(a+n)\left(\beta^{'}\right)^{a+1}}F_1\Bigg(a+n,\ 1-b,\ a+b,\ a+n+1; -\frac{\beta-2\beta^{'}}{2\beta^{'}}, -\frac{\alpha+\beta-2\beta^{'}}{2\beta^{'}}\Bigg) \tag{55}$$

and

$$I_2 = \frac{1}{2^{b+n}(b+n)\left(\alpha^{'}\right)^{b+1}}F_1\Bigg(b+n,\ 1-a,\ a+b,\ b+n+1; -\frac{\alpha-2\alpha^{'}}{2\alpha^{'}}, -\frac{\alpha+\beta-2\alpha^{'}}{2\alpha^{'}}\Bigg). \tag{56}$$

The result of the theorem follows by substituting (55) and (56) into (54). □

Using special properties of the hypergeometric function of two variables, one can derive simpler forms for the pdf in (53). This is illustrated in the corollary below.

Corollary 9. *If X and Y are jointly distributed according to* (5) *and if both* $a \geq 1$ *and* $b \geq 1$ *are integers then*

$$
\begin{aligned}
E\left(W^n\right) = & \frac{\alpha^a \Gamma(a+b)}{(\beta^{'})^a 2^{a+n} \Gamma(a) \Gamma(b)} \sum_{m=0}^{b-1} \binom{b-1}{m} \\
& B(a+n+m, b-m) \left(\frac{\beta}{2\beta^{'}}\right)^m I_1(m) \\
& + \frac{\beta^b \Gamma(a+b)}{(\alpha^{'})^b 2^{b+n} \Gamma(a) \Gamma(b)} \sum_{m=0}^{a-1} \binom{a-1}{m} \\
& B(b+n+m, a-m) \left(\frac{\alpha}{2\alpha^{'}}\right)^m I_2(m),
\end{aligned}
$$

for $n \geq 1$, *where*

$$
I_1(m) = {}_2F_1\left(a+n+m, a+b; a+b+n; -\frac{\alpha+\beta-2\beta^{'}}{2\beta^{'}}\right)
$$

and

$$
I_2(m) = {}_2F_1\left(b+n+m, a+b; a+b+n; -\frac{\alpha+\beta-2\alpha^{'}}{2\alpha^{'}}\right).
$$

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Saralees Nadarajah
School of Mathematics
University of Manchester
Oxford Road
Manchester M13 9PL
UNITED KINGDOM

E-mail: Saralees.Nadarajah@manchester.ac.uk

Samuel Kotz
Department of Engineering Management and
Systems Engineering
The George Washington University
Washington, D.C. 20052
U.S.A.