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A characterization of isometries on an open convex set

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Abstract. Let X be a real Hilbert space with $\dim X \geq 2$ and let Y be a real normed space which is strictly convex. In this paper, we generalize a theorem of Benz by proving that if a mapping f, from an open convex subset of X into Y, has a contractive distance ρ and an extensive one $N\rho$ (where $N \geq 2$ is a fixed integer), then f is an isometry.

Keywords: Aleksandrov problem, isometry, distance preserving mapping.

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1 Introduction

Let X and Y be normed spaces. A mapping $f : X \to Y$ is called an isometry (or a congruence) if f satisfies

$$\| f(x) - f(y)\| = \|x - y\|$$

for all $x, y \in X$. A distance $\rho > 0$ is said to be contractive (or non-expanding) by $f : X \to Y$ if $\|x - y\| = \rho$ always implies $\| f(x) - f(y)\| \leq \rho$. Similarly, a distance ρ is said to be extensive (or non-shrinking) by f if the inequality $\| f(x) - f(y)\| \geq \rho$ is true for all $x, y \in X$ with $\|x - y\| = \rho$. We say that ρ is preserved (conserved or conservative) by f if ρ is contractive and extensive by f simultaneously.

If f is an isometry, then every distance $\rho > 0$ is preserved by f, and conversely. We can now raise a question:

Is a mapping that preserves certain distances an isometry?

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In 1970, A. D. Aleksandrov [1] had raised a question whether a mapping $f : X \to X$ preserving a distance $\rho > 0$ is an isometry, which is now known to us as the Aleksandrov problem. Without loss of generality, we may assume $\rho = 1$ when X is a normed space (see [15]).

Indeed, earlier than Aleksandrov, F. S. Beckman and D. A. Quarles [2] solved the Aleksandrov problem for finite-dimensional real Euclidean spaces $X = E^n$:

> *If a mapping $f : E^n \to E^n$ $(2 \le n < \infty)$ preserves distance 1, then f is a linear isometry up to translation.*

For $n = 1$, they suggested the mapping $f : E^1 \to E^1$ defined by

$$f(x) = \begin{cases} x + 1 & \text{for } x \in \mathbb{Z}, \\ x & \text{otherwise} \end{cases}$$

as an example for a non-isometric mapping that preserves distance 1. For $X = E^\infty$, Beckman and Quarles also presented an example for a unit distance preserving mapping that is not an isometry (*cf*. [12]).

We may find a number of papers on a variety of subjects in the Aleksandrov problem (see [5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and also the references cited therein).

In 1985, W. Benz [3] introduced a sufficient condition under which a mapping, with a contractive distance ρ and an extensive one $N\rho$, is an isometry (see also [4]):

> *Let X and Y be real normed spaces such that $\dim X \ge 2$ and Y is strictly convex. Suppose $f : X \to Y$ is a mapping and $N > 1$ is a fixed integer. If a distance $\rho > 0$ is contractive and $N\rho$ is extensive by f, then f is a linear isometry up to translation.*

Recently, the author and Th. M. Rassias proved in [9] that the theorem of Benz is also true when the relevant domain is restricted to a half space of a real Hilbert space with dimension ≥ 3.

In this paper, we will generalize the above theorem of Benz; More precisely, let X be a real Hilbert space with $\dim X \ge 2$ and let Y be a real normed space which is strictly convex. We prove that if a mapping, from an open convex subset of X into Y, has a contractive distance ρ and an extensive one $N\rho$ (where $N \ge 2$ is a fixed integer), then the restriction of f to an open convex subset of the domain is an isometry.

2 Preliminaries

From now on, let X be a real Hilbert space with dim $X \geq 2$. For a fixed integer $N \geq 2$ and a constant $\rho > 0$, let us define a sequence (d_n) by

$$d_1 = N\rho \quad \text{and} \quad d_n = N^{3-n}\rho \quad \text{for} \quad n \in \{2, 3, \ldots\}.$$

Let (X_n) be a sequence of open convex subsets of X with

$$X_0 \supset X_1 \supset \cdots \supset X_n \supset X_{n+1} \supset \cdots \quad \text{and} \quad d(X_{n+1}, \partial X_n) \geq d_{n+1}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $d(X_{n+1}, \partial X_n) = \inf\{\|x - y\| : x \in X_{n+1}, y \in \partial X_n\}$ and ∂X_n denotes the boundary of X_n. (If one of X_{n+1} and ∂X_n is unbounded, we will set $d(X_{n+1}, \partial X_n) = \infty$.)

Furthermore, we assume

$$X_\infty := \left(\bigcap_{n=0}^{\infty} X_n \right)^{\circ} \neq \emptyset.$$

We know that the intersection of any family of convex subsets of a topological vector space is convex. Moreover, the interior of any convex subset of a topological vector space is a convex set. Thus, X_∞ is an open convex subset of X.

Let Y be a real normed space with the following property:

(P1) If unit vectors $a, b \in Y$ satisfy $\|a + b\| = 2$, then $a = b$.

Using (P1) and an idea from (c) in the proof of the theorem in [3], we may easily prove the following lemma.

Lemma 1. *For all $a, b, c \in Y$, $\|b - a\| = \beta = \|c - b\|$ and $\|c - a\| = 2\beta$ imply $c = 2b - a$, where β is a positive real number.*

In the proof of the following lemma, we apply the mathematical induction many times.

Lemma 2. *Suppose a mapping $f : X_0 \to Y$ satisfies both the following properties:*

(P2) *ρ is contractive by f;*

(P3) *$N\rho$ is extensive by f.*

The following assertions are true for any given $n \in \mathbb{N}$ *and for all* $x, y \in X_n$:

(A1) *If* $\|x - y\| = N^{1-n}\rho$, *then* $\|f(x) - f(y)\| = N^{1-n}\rho$;

(A2) *If* $\|x - y\| = 2N^{1-n}\rho$, *then* $\|f(x) - f(y)\| = 2N^{1-n}\rho$;

(A3) *If* $\|x - y\| = N^{1-n}\rho$ *and* $x + m(y - x) \in X_n$ *for some* $m \in \mathbb{N}$, *then we have* $f(x + i(y - x)) = f(x) + i(f(y) - f(x))$ *for* $i \in \{0, 1, \ldots, m\}$.

Proof. (a) We first prove $(A1)$ for $n = 1$, i.e., we show that for all $x, y \in X_1$, $\|x - y\| = \rho$ implies $\|f(x) - f(y)\| = \rho$. Define $p_i = y + i(x - y)$ for $i \in \{0, 1, \ldots, N\}$. It then follows that $\|p_N - y\| = N\rho$, $p_i \in X_0$ for $i \in \{0, \ldots, N\}$, and that $\|p_i - p_{i-1}\| = \rho$ for $i \in \{1, \ldots, N\}$. Using $(P2)$ and $(P3)$ we have

$$N\rho \le \|f(p_N) - f(y)\| \le \sum_{i=1}^{N} \|f(p_{N+1-i}) - f(p_{N-i})\| \le N\rho.$$

Hence, we conclude that $\|f(x) - f(y)\| = \|f(p_1) - f(p_0)\| = \rho$.

We prove (A2) for $n = 1$, i.e., we prove that for all $x, y \in X_1$, $\|x - y\| = 2\rho$ implies $\|f(x) - f(y)\| = 2\rho$. Let $p_i = y + (i/2)(x - y)$ for $i \in \{0, 1, \ldots, N\}$. Then, it follows that $\|p_N - y\| = N\rho$, $p_i \in X_0$ for $i \in \{0, \ldots, N\}$, and that $\|p_i - p_{i-1}\| = \rho$ for $i \in \{1, \ldots, N\}$. Now, we make use of (P2) and (P3) to get

$$N\rho \le \|f(p_N) - f(y)\| \le \sum_{i=1}^{N} \|f(p_{N+1-i}) - f(p_{N-i})\| \le N\rho,$$

i.e.,

$$\|f(p_N) - f(y)\| = \sum_{i=1}^{N} \|f(p_{N+1-i}) - f(p_{N-i})\|. \tag{1}$$

If we assume $\|f(p_2) - f(p_0)\| < \|f(p_2) - f(p_1)\| + \|f(p_1) - f(p_0)\|$, then it should be $N \ge 3$ in view of (1) and further

$$\begin{aligned} \|f(p_N) - f(y)\| &\le \sum_{i=1}^{N-2} \|f(p_{N+1-i}) - f(p_{N-i})\| + \|f(p_2) - f(p_0)\| \\ &< \sum_{i=1}^{N} \|f(p_{N+1-i}) - f(p_{N-i})\|, \end{aligned}$$

which is contrary to (1). Therefore, we conclude by using (A1) for $n = 1$ that

$$\begin{aligned}\| f(x) - f(y)\| &= \| f(p_2) - f(p_0)\| = \\ &= \| f(p_2) - f(p_1)\| + \| f(p_1) - f(p_0)\| = 2\rho,\end{aligned}$$

since $p_0 = y \in X_1$, $p_2 = x \in X_1$ and $p_1 = (x + y)/2 \in X_1$ (X_1 is a convex set).

We now prove (A3) for $n = 1$, i.e., we prove by induction that if $x, y \in X_1$ satisfy $\|x - y\| = \rho$ and $x + m(y - x) \in X_1$ for some $m \in \mathbb{N}$, then $f(x + i(y - x)) = f(x) + i(f(y) - f(x))$ for $i \in \{0, 1, \ldots, m\}$. There is nothing to prove for $i = 0$ or 1. We now assume that our assertion is true for $i \in \{0, 1, \ldots, k\}$ $(1 \leq k < m)$. Put $p_l = x + l(y - x)$ for $l \in \mathbb{N}$. Then, since X_1 is convex, we have $p_2, \ldots, p_{k+1} \in X_1$ and we get

$$\| p_k - p_{k-1}\| = \rho = \| p_{k+1} - p_k\| \text{ and } \| p_{k+1} - p_{k-1}\| = 2\rho.$$

According to (A1) and (A2) for $n = 1$, we have

$$\| f(p_k) - f(p_{k-1})\| = \rho = \| f(p_{k+1}) - f(p_k)\| \text{ and } \| f(p_{k+1}) - f(p_{k-1})\| = 2\rho.$$

Hence, it follows from Lemma 1 that

$$f(p_{k+1}) = 2 f(p_k) - f(p_{k-1}) = f(x) + (k + 1)(f(y) - f(x)),$$

which completes the proof of (A3) for $n = 1$.

(b) We now assume that for any $n \in \{1, \ldots, q\}$, our assertions (A1), (A2) and (A3) are true for all $x, y \in X_n$, where q is a given positive integer.

(c) We consider (A1) for $n = q + 1$. Assume that $x, y \in X_{q+1}$ are given with $\|x - y\| = N^{-q}\rho$. It is to show that $\| f(x) - f(y)\| = N^{-q}\rho$. Choose $z, x', y' \in X_q$ such that $\|x - z\| = \|y - z\| = N^{1-q}\rho$, $\|x' - z\| = \|y' - z\| = N^{2-q}\rho$, $\|x' - y'\| = N^{1-q}\rho$, and such that x and y lie on the segments $\overline{x'z}$ and $\overline{y'z}$, respectively. In view of (A1), (A2) and (A3) for $n = q - 1$ and q, we see that

$$\| f(x) - f(z)\| = \| f(y) - f(z)\| = N^{1-q}\rho,$$

$$\| f(x') - f(z)\| = \| f(y') - f(z)\| = N^{2-q}\rho,$$

$$\| f(x') - f(y')\| = N^{1-q}\rho,$$

and that $f(x)$ and $f(y)$ lie on the segments $\overline{f(x') f(z)}$ and $\overline{f(y') f(z)}$, respectively. These facts imply that the triangles $f(x) f(z) f(y)$ and $f(x') f(z) f(y')$ are similar. Therefore, we conclude that $\| f(x) - f(y)\| = N^{-q}\rho$.

Let us consider (A2) for $n = q+1$. Assume that $x, y \in X_{q+1}$ are given with $\|x - y\| = 2N^{-q}\rho$. Similarly as the proof of (A1) for $n = q+1$ (see the last paragraph), choose $z, x', y' \in X_q$ such that $\|x - z\| = \|y - z\| = N^{1-q}\rho$, $\|x' - z\| = \|y' - z\| = N^{2-q}\rho$, $\|x' - y'\| = 2N^{1-q}\rho$, and such that x and y lie on the segments $\overline{x'z}$ and $\overline{y'z}$, respectively. By a similar argument as in the proof of (A1) for $n = q+1$, we get $\|f(x) - f(y)\| = 2N^{-q}\rho$.

Finally, we consider (A3) for $n = q+1$, i.e., we prove that if $x, y \in X_{q+1}$ satisfy $\|x - y\| = N^{-q}\rho$ and $x + m(y - x) \in X_{q+1}$ for some $m \in \mathbb{N}$, then $f(x + i(y - x)) = f(x) + i(f(y) - f(x))$ for $i \in \{0, 1, \ldots, m\}$. There is nothing to prove for $i = 0$ or 1. Suppose our assertion is true for $i \in \{0, 1, \ldots, k\}$ $(1 \le k < m)$. Put $p_i = x + i(y - x)$ for $i \in \mathbb{N}$. By the convexity of X_{q+1}, we see that $p_2, \ldots, p_{k+1} \in X_{q+1}$ and

$$\|p_k - p_{k-1}\| = N^{-q}\rho = \|p_{k+1} - p_k\| \text{ and } \|p_{k+1} - p_{k-1}\| = 2N^{-q}\rho.$$

According to (A1) and (A2) for $n = q+1$, we have

$$\|f(p_k) - f(p_{k-1})\| = N^{-q}\rho = \|f(p_{k+1}) - f(p_k)\|$$

and

$$\|f(p_{k+1}) - f(p_{k-1})\| = 2N^{-q}\rho.$$

Hence, it follows from Lemma 1 that

$$f(p_{k+1}) = 2f(p_k) - f(p_{k-1}) = f(x) + (k+1)(f(y) - f(x)),$$

which completes the proof of $(A3)$ for $n = q+1$.

3 Generalization of a theorem of Benz

In this section, let X, X_0, X_n, X_∞, Y, N and ρ be the same ones as in the previous section. We are ready to prove the main theorem of this paper.

Theorem 3. *If a mapping $f : X_0 \to Y$ satisfies both the properties* (P2) *and* (P3), *then $f|_{X_\infty}$ is an isometry.*

Proof. (a) We assert that if $x, y \in X_\infty$ are separated from each other by a distance $mN^{1-n}\rho$ then $\|f(x) - f(y)\| = mN^{1-n}\rho$, where m and n are arbitrary positive integers. Since X_∞ is convex, we can choose a $z \in X_\infty$ on the segment $\overline{xy}$ such that $\|x - z\| = N^{1-n}\rho$. Define $p_i = x + i(z - x)$ for $i \in \{0, 1, \ldots, m\}$. The convexity of X_∞ again implies that $p_i \in X_\infty \subset X_n$ for $i \in \{0, 1, \ldots, m\}$. By

(A3) of Lemma 2, we get $f(p_i) = f(x) + i(f(z) - f(x))$ for $i \in \{0, 1, \ldots, m\}$. Hence, using (A1) of Lemma 2, we have

$$\|f(x) - f(y)\| = \|f(x) - f(p_m)\| = m\,\|f(z) - f(x)\| = mN^{1-n}\rho.$$

(b) Assume that $x, y \in X_\infty$ are distinct. For these x and y, choose the sequences, (k_i), (m_i) and (n_i), of non-negative integers with the following properties:

(K) $k_i N^{1-n_i}\rho \le \|x - y\| < (k_i + 1)N^{1-n_i}\rho$ for all sufficiently large integers i;

(M) $(m_i - 1)N^{1-n_i}\rho < \|x - y\| \le m_i N^{1-n_i}\rho$ for all sufficiently large integers i;

(N) (n_i) increases strictly (to infinity.)

Since X_∞ is open, we can select a z_i on the segment $\overline{xy}$ and a $w_i \in X_\infty$ such that

$$\|x - z_i\| = k_i N^{1-n_i}\rho \quad \text{and} \quad \|z_i - w_i\| = \|w_i - y\| = N^{1-n_i}\rho$$

for any sufficiently large i. It then follows from (a) that

$$\|f(x) - f(z_i)\| = k_i N^{1-n_i}\rho$$

and

$$\|f(z_i) - f(w_i)\| = \|f(w_i) - f(y)\| = N^{1-n_i}\rho$$

for any sufficiently large integer i. Thus, it follows from (K) that

$$\begin{aligned}\|f(x) - f(y)\| &\le \|f(x) - f(z_i)\| + \|f(z_i) - f(w_i)\| + \|f(w_i) - f(y)\| \\ &\le \|x - y\| + 2N^{1-n_i}\rho\end{aligned}$$

for any sufficiently large integer i, i.e., we get $\|f(x) - f(y)\| \le \|x - y\|$.

On the other hand, since X_∞ is open, we can choose a $v_i \in X_\infty$ such that

$$\|x - v_i\| = m_i N^{1-n_i}\rho \quad \text{and} \quad \|y - v_i\| = N^{1-n_i}\rho$$

for all sufficiently large integers i. From (a) we get

$$\|f(x) - f(v_i)\| = m_i N^{1-n_i}\rho \quad \text{and} \quad \|f(y) - f(v_i)\| = N^{1-n_i}\rho.$$

Hence, it follows from (M) that

$$\|f(x) - f(y)\| \ge \|f(x) - f(v_i)\| - \|f(y) - f(v_i)\| \ge \|x - y\| - N^{1-n_i}\rho$$

for all sufficiently large integers i, i.e., we get $\|f(x) - f(y)\| \ge \|x - y\|$, which completes the proof. $\square$

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