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Nonexistence of invariant graphs in all supercritical energy levels of mechanical Lagrangians in T^2

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Abstract. Let (T^2, g) be a smooth Riemannian structure in the torus T^2. We show that given $\epsilon > 0$ and any C^∞ function $U\colon T^2 \longrightarrow \mathbb{R}$ there exists a C^1 function U_ϵ with Lipschitz derivatives that is ϵ-C^0 close to U for which there are no continuous invariant graphs in any supercritical energy level of the mechanical Lagrangian $L_\epsilon\colon TT^2 \longrightarrow \mathbb{R}$ given by $L(p, v) = \frac{1}{2}g(v, v) - U_\epsilon(p)$. We also show that given $n \in \mathbb{N}$, the set of C^∞ potentials $U\colon T^2 \longrightarrow \mathbb{R}$ for which there are no continuous invariant graphs in any supercritical energy level $E \leq n$ of $L(p, v) = \frac{1}{2}g(v, v) - U(p)$ is C^0 dense in the set of C^∞ functions.

Keywords: invariant graph, mechanical Lagrangian, critical level.

Mathematical subject classification: 37J40, 37J50, 53D25.

Introduction

In a previous article [20], we showed that the set of smooth Riemannian metrics in the two-torus whose geodesic flows have no continuous invariant graphs is open and dense in the set of metrics endowed with the C^1 topology. Motivated by this result, it is natural to ask whether the set of mechanical Lagrangians in the torus without invariant graphs in any supercritical level of energy is dense in some C^k topology. Namely, given a mechanical Lagrangian in the torus, does there exists a smooth, C^k-close mechanical Lagrangian without invariant graphs in any supercritical level?. The purpose of this article is to show that the answer to this question is positive, provided that the considered topology is the C^0 topology. Given a mechanical Lagrangian $L(p, v) = \frac{1}{2}g(v, v) - U(p)$, where g is a smooth Riemannian metric in T^2 and $U\colon T^2 \longrightarrow \mathbb{R}$ is a smooth positive potential, what we call the critical value of the Lagrangian is the absolute

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critical value $C = \max_{p \in T^2} U(p)$.We say that a submanifold S in the tangent bundle TM of a smooth manifold M is called a graph if the canonical projection $\pi : TM \longrightarrow M$, $\pi(p, v) = p$ restricted to S is a homeomorphism. Our main result is the following.

Theorem 1. *Let (T^2, g) be a C^∞ Riemannian structure in T^2, and let $C^k(T^2, \mathbb{R})$ be the set of C^k functions from T^2 to the real numbers. Then, given $n > 0$, the set of $U \in C^\infty(T^2, \mathbb{R})$ for which there are no continuous invariant graphs of the Euler-Lagrange flow in any supercritical energy level $E \leq n$ of the Lagrangian $L : TT^2 \longrightarrow \mathbb{R}$ given by $L(p, v) = \frac{1}{2}g(v, v) - U(p)$ is dense in the C^0 topology. Moreover, given $\epsilon > 0$ and $U \in C^\infty(T^2, \mathbb{R})$, there is a function $U_\epsilon \in C^1(T^2, \mathbb{R})$ with Lipschitz first derivatives, such that*

(1) $\| U - U_\epsilon \|_\infty \leq \epsilon$,

(2) *There are no continuous invariant graphs in any supercritical energy level of $L_\epsilon(p, v) = \frac{1}{2}g(v, v) - U_\epsilon(p)$.*

Let us comment briefly some of the main difficulties and ideas concerning the proof of Theorem 1. The Euler-Lagrange flow in energy levels whose energy E is greater than C is, up to reparametrization, the geodesic flow of a Riemannian metric $g_p^E = (E - U(p))g_p$, called Maupertuis' metric. This is the well known Maupertuis' Principle of reduced action. The elimination of invariant tori of the geodesic flow of a Riemannian metric by perturbations of the metric [20] allows us to eliminate continuous invariant graphs in small open subsets of energy levels: under certain C^1 perturbations of g^E we obtain an open subset of metrics close to g^E in the C^1 topology having no invariant tori. However, the set of Maupertuis' metrics is infinite, and although Maupertuis' metrics are all conformal to each other, they might have very different geometric features. We would like to point out that the Gaussian curvature of Maupertuis' metrics tends to ∞ at certain points as the energy gets close to the critical value. With all these problems it seems unlikely that with a single perturbation of a metric we could succeed in eliminating invariant tori in all Maupertuis' metrics simultaneously. The definition of Maupertuis' metrics suggests that perturbing the potential could be more convenient than perturbing a particular Maupertuis' metric, and perturbations of the potential are somehow more natural from the point of view of physics. The key idea of the proof of Theorem 1 is to show the existence of perturbations of the potential which provide what we call in Section 2 uniformly geodesic neighborhoods: open balls in T^2 where the exponential map of the metric g is a diffeomorphism, where the radial g-geodesics are geodesics of all Maupertuis' metrics, and where the Gaussian curvatures of Maupertuis' metrics are uniformly bounded above. The existence of uniformly geodesic balls allows us to control the geometry of all the Maupertuis' metrics simultaneously in a certain ball. Then, with the help of uniformly geodesic balls and some special C^0 bumps we construct in this article (Appendix), we show that given $e > C$ we can perturb the potential in order to eliminate invariant graphs of constant energy E simultaneously for all $E \in (C, e]$.

We would like to point out that Theorem 1 is the first result, as far as we know, about the destruction of invariant graphs in all supercritical levels, and that the proof of Theorem 1 cannot be extended to the C^1 topology. We also think that without much extra work we can extend Theorem 1 to n-dimensional tori.

1 Conformal metrics, Maupertuis' principle

The goal of the section is to show some basic facts concerning the geodesics and curvature of Maupertuis' metrics. Although some of these results might be well known we decided to include them in a preliminary section for the sake of completeness. Let (T^2, g) be a C^∞ Riemannian structure in T^2, let $U\colon T^2 \longrightarrow \mathbb{R}$ be a smooth function (that can be assumed to be positive without loss of generality), and consider the mechanical Lagrangian $L(p, v) = \frac{1}{2}g(v, v) - U(p)$, where (p, v) are the canonical coordinates of the tangent bundle of T^2. The energy function of L is given by $E(p, v) = \frac{1}{2}g(v, v)+U(p)$, and the number $C = C(U) = \max_{p\in T^2} U(p)$ is called the absolute critical value of the energy. We shall refer to C simply as the critical value of the energy. The well known Maupertuis' principle tells us that the integral curves of the Euler-Lagrange flow of L in a level of constant energy $E > C$ are the geodesics of the Riemannian metric g^E in T^2 given by

$$g_p^E(z, w) = (E - U(p))g_p(z, w),$$

which is usually called a Maupertuis' metric. All these metrics are conformal to the metric g, and hence the formulae of conformal geometry can be applied to study the surfaces (T^2, g^E). Let us recall briefly the conformal connection formula and the conformal curvature formula, we follow [7], [16]. Let $\bar{g}_p = f(p)g_p$ be two conformal metrics, where $f\colon T^2 \longrightarrow \mathbb{R}$ is a smooth positive function, and let ∇ be the Levi-Civita connection of the metric g. Writing $f(p) = e^{2\sigma(p)}$, so $\sigma(p) = \frac{1}{2}ln(f(p))$, we have that the Levi-Civita connection of $\bar{g}$ can be written in the following way:

Lemma 1.1. *The Levi-Civita connection $\bar{\nabla}$ of the metric $\bar{g}$ evaluated in smooth,* *local vector fields X, Y of T^2 at the point $p \in T^2$ is given by*

$$\begin{aligned}\bar{\nabla}_X Y|_p \ &= \nabla_X Y|_p + g_p(\mathrm{grad}(\sigma), \\ &\quad X(p))Y(p) + g_p(\mathrm{grad}(\sigma), \\ &\quad Y(p))X(p) - g_p(X, Y)\,\mathrm{grad}(\sigma)_p,\end{aligned}$$

where grad(σ) *is the gradient vector field of the function σ.*

The following lemma that is straightforward from the conformal connection formula is essentially proved in [16] (see also [17], [19] for instance).

Lemma 1.2. *If the gradient* grad(σ) *is parallel to the field of vectors tangent to a geodesic $\gamma\colon (0, 1) \longrightarrow T^2$ of the metric g, then the geodesic $\gamma(0, 1)$ is a geodesic of the metric $\bar{g} = fg$.*

Now, let R be the curvature tensor of the metric g. The curvature tensor of $\bar{g}$ is given by the well known conformal curvature formula.

Lemma 1.3. *The curvature tensor $\bar{R}$ of the metric $\bar{g}$ evaluated in smooth, local vector fields X, Y, Z, W at the point $p \in T^2$ is given by*

$$\begin{aligned}e^{-2\sigma(p)}\bar{R}_p(X, Y, Z, W) \ &= \ R_p(X, Y, Z, W) \\ &+ \ [Q(Y, Z)(\sigma) + g_p(Y, Z) \parallel \mathrm{grad}(\sigma)_p \parallel^2] g_p(X, W) \\ &- \ [Q(X, Z)(\sigma) + g_p(X, Z) \parallel \mathrm{grad}(\sigma)_p \parallel^2] g_p(Y, W) \\ &+ \ g_p(Y, Z) Q(X, W)(\sigma) - g_p(X, Z) Q(Y, W)(\sigma),\end{aligned}$$

where $\parallel v \parallel^2 = g(v, v)$, and $Q(X, Y)$ is the vector field which applied to a smooth function $h\colon T^2 \longrightarrow \mathbb{R}$ gives the function

$$Q(X, Y)(h) = X(Y(h)) - (\nabla_X Y)(h) - X(h)Y(h).$$

The main lemma of Section 1 contains some elementary properties of the geodesics and the Gaussian curvature of Maupertuis' metrics which are very important for the forthcoming sections. Let $K(p)$ and $K^E(p)$ be respectively the Gaussian curvatures of the metrics g and g^E at $p \in T^2$. We shall use the notation d_g to designate the distance associated to the metric g, a ball of g-radius r centered at p will be denoted by $B_r(p)$, and a g^E-ball of radius r centered at p by $B_r^E(p)$.

Lemma 1.4.

(1) *If the energy E tends to $+\infty$, then the Gaussian curvature K^E tends uniformly to* 0.

(2) *If x_0 is a local minimum of the potential U such that $U(x_0) < C = \max_{p\in T^2} U(p)$, then there exist $r_0 > 0$, $A > 0$, $K_\infty > 0$ such that*

$$K^E(p) \leq \frac{A}{\min_{q\in B_{r_0}(x_0)}(E - U(q))} \leq K_\infty$$

for every $p \in B_{r_0}(x_0)$.

(3) *Assume that the minimum point x_0 of U is of Morse type. If the curvature $K(x_0)$ of g at x_0 is nonnegative, then there exists $r_1 > 0$ such that the Gaussian curvatures $K^E(p)$ are positive for every $p \in B_{r_1}(x_0)$ and $E > C$.*

Proof. Let $p \in T^2$, and let $\gamma\colon (-\epsilon, \epsilon) \longrightarrow T^2$ be a geodesic of g parametrized by g-arc length such that $\gamma(0) = p$. Let $\phi\colon (-\epsilon, \epsilon) \times (-\delta, \delta) \longrightarrow V_p$ be a Fermi coordinate system defined in an open neighborhood V_p of p, such that $\phi(t, 0) = \gamma(t)$ for every $t \in (-\epsilon, \epsilon)$. Let us denote by $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial s}$ respectively, the coordinate vector fields of ϕ, so $\frac{\partial}{\partial t}|_{\gamma(t)} = \gamma'(t)$. Let us recall that the coordinate vector fields are perpendicular along the geodesic γ and have g-norm equal to 1 along γ (for details see for instance [6]). This implies that the Gaussian curvature $K(p)$ can be calculated by

$$K(p) = R_p\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right).$$

Clearly, the coordinate vector fields are perpendicular in any metric conformal to g. So let $\bar{g}_p = f(p)g_p$ be conformal to g, where f is a positive smooth function, according to Lemma 1.3 we can calculate the curvature tensor of $\bar{g}$ evaluated in the coordinate fields by

$$\begin{aligned} e^{-2\sigma(p)}\bar{R}_p\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) &= K(p) \\ &+ \left[Q\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)(\sigma) + g_p\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) \| \operatorname{grad}(\sigma)_p \|^2\right] g_p\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) \\ &- \left[Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)(\sigma) + g_p\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) \| \operatorname{grad}(\sigma)_p \|^2\right] g_p\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) \\ &+ g_p\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right)(\sigma) - g_p\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) Q\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right)(\sigma). \end{aligned}$$

Since $g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) = 0$ along $\gamma(t)$, and $g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = 1$ along $\gamma(t)$, we get

$$\begin{aligned} e^{-2\sigma(p)}\bar{R}_p\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) &= K(p) - \left[Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)(\sigma) + \parallel \mathrm{grad}(\sigma)_p \parallel^2\right] \\ &\quad - Q\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right)(\sigma). \end{aligned}$$

Equivalently,

$$\begin{aligned} \bar{R}_p\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right) &= f(p)K(p) \\ &\quad - f(p)\left[Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)(\sigma) + \parallel \mathrm{grad}(\sigma)_p \parallel^2\right] \\ &\quad - f(p)Q\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right)(\sigma). \end{aligned}$$

Let us calculate $Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)(\sigma)$. Since $\frac{\partial}{\partial t}$ is the field of vectors $\gamma'(t)$, we have that $\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = 0$, and since $\sigma = \frac{1}{2}ln(f)$ we get that

$$\frac{\partial \sigma}{\partial t} = \frac{1}{2f}\frac{\partial f}{\partial t} \quad \text{and} \quad \frac{\partial^2 \sigma}{\partial t^2} = \frac{1}{2f^2}\left[\frac{\partial^2 f}{\partial t^2} f - \left(\frac{\partial f}{\partial t}\right)^2\right].$$

This yields,

$$Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)(\sigma) = \frac{1}{2f^2}\left[\frac{\partial^2 f}{\partial t^2} f - \left(\frac{\partial f}{\partial t}\right)^2\right] - \frac{1}{4f^2}\left(\frac{\partial f}{\partial t}\right)^2.$$

Recall that by the definition of a Fermi coordinate system, the vector field $\frac{\partial}{\partial s}$ is tangent to the geodesics in the tubular neighborhood V_p which are perpendicular to γ. Moreover, we can assume without loss of generality that s is the arc length parameter of such geodesics, so $\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} = 0$. Hence, it is easy to show that the formula for $Q\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right)(\sigma)$ is obtained from the formula for $Q\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)(\sigma)$ just by interchanging the parameters t and s, i.e.,

$$Q\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right)(\sigma) = \frac{1}{2f^2}\left[\frac{\partial^2 f}{\partial s^2} f - \left(\frac{\partial f}{\partial s}\right)^2\right] - \frac{1}{4f^2}\left(\frac{\partial f}{\partial s}\right)^2.$$

Thus we get,

$$\begin{aligned}\bar{R}_p\left(\frac{\partial}{\partial t},\frac{\partial}{\partial s},\frac{\partial}{\partial t},\frac{\partial}{\partial s}\right) &= f(p)K(p)-\frac{1}{2f}\left[\frac{\partial^2 f}{\partial t^2}f-\frac{3}{2}(\frac{\partial f}{\partial t})^2\right]\\ &\quad -\frac{1}{4f}\parallel \mathrm{grad}(f)\parallel^2-\frac{1}{2f}\left[\frac{\partial^2 f}{\partial s^2}f-\frac{3}{2}\left(\frac{\partial f}{\partial s}\right)^2\right].\end{aligned}$$

Since the area in the metric $\bar{g}$ of the parallelogram whose sides are the vectors $\frac{\partial}{\partial t}_p$, $\frac{\partial}{\partial s}_p$, is $f(p)$ at $p=\gamma(0)$ we have that the Gaussian curvature of $\bar{g}$ at p is

$$\bar{K}(p)=\frac{1}{f(p)^2}\bar{R}_p\left(\frac{\partial}{\partial t},\frac{\partial}{\partial s},\frac{\partial}{\partial t},\frac{\partial}{\partial s}\right),$$

and hence we obtain

$$\begin{aligned}\bar{K}(p) &= \frac{1}{f(p)}K(p)-\frac{1}{2f^3}\left[\frac{\partial^2 f}{\partial t^2}f-\frac{3}{2}\left(\frac{\partial f}{\partial t}\right)^2\right]\\ &\quad -\frac{1}{4f^3}\parallel \mathrm{grad}(f)\parallel^2-\frac{1}{2f^3}\left[\frac{\partial^2 f}{\partial s^2}f-\frac{3}{2}\left(\frac{\partial f}{\partial s}\right)^2\right].\end{aligned}$$

To calculate the curvature of the metrics $g_p^E=(E-U(p))g_p$ we just take $f(p)=E-U(p)$, so $\frac{\partial f}{\partial t}=-\frac{\partial U}{\partial t}$, $\frac{\partial f}{\partial s}=-\frac{\partial U}{\partial s}$, and hence we get

$$\begin{aligned}\bar{K}(p) &= \frac{1}{E-U(p)}K(p)-\frac{1}{2(E-U(p))^3}\\ &\quad \left[-\frac{\partial^2 U}{\partial t^2}(p)(E-U(p))-\frac{3}{2}\left(\frac{\partial U}{\partial t}(p)\right)^2\right]\\ &\quad -\frac{1}{4(E-U(p))^3}\parallel \mathrm{grad}_p(U)\parallel^2\ \frac{1}{2(E-U(p))^3}\\ &\quad \left[-\frac{\partial^2 U}{\partial s^2}(p)(E-U(p))-\frac{3}{2}\left(\frac{\partial U}{\partial s}(p)\right)^2\right]\\ &= \frac{1}{E-U(p)}K(p)+\frac{1}{2(E-U(p))^2}\left[\frac{\partial^2 U}{\partial t^2}(p)+\frac{\partial^2 U}{\partial s^2}(p)\right]\\ &\quad +\frac{3}{2(E-U(p))^3}\left[\left(\frac{\partial U}{\partial t}(p)\right)^2+\left(\frac{\partial U}{\partial s}(p)\right)^2\right]\\ &\quad -\frac{1}{4(E-U(p))^3}\parallel \mathrm{grad}_p(U)\parallel^2 .\end{aligned}$$

To show item (1) in Lemma 1.3, notice first that there exists a constant $Q > 0$ such that $\|$ grad(U) $\|_\infty \leq Q$, and such that in any Fermi coordinate system defined in an open neighborhood of T^2 the first and second partial derivatives of U with respect to the coordinates are bounded above by Q. This follows from the fact that U is C^∞ and the compactness of T^2. So we deduce from the above formula that

$$|K^E(p)| \leq \frac{1}{(E-U(p))} K(p) + \frac{Q}{(E-U(p))^2} + \frac{4Q^2}{(E-U(p))^3},$$

which clearly implies item (1).

The proof of item (2) follows from the above formula for $\bar{K}$. Indeed, let $r_0 > 0$ be such that $E - U(p) > \frac{C}{2}$ for every $p \in B_{r_0}(x_0)$. From the curvature formula we get

$$K^E(p) \leq \frac{1}{(E-U(p))} \left(K(p) + \frac{Q}{(E-U(p))} + \frac{4Q^2}{(E-U(p))^2} \right),$$

which implies that

$$K^E(p) \leq \frac{1}{(E-U(p))} \left(K(p) + \frac{2Q}{C} + \frac{8Q^2}{C^2} \right)$$

for every $p \in B_{r_0}(x_0)$.

So letting $A = sup_{q \in T^2} K(q) + \frac{2Q}{C} + \frac{8Q^2}{C^2}$, $L = \frac{2}{C} A$, we prove item (2).

To prove item (3) notice first of all that the curvature formula and the fact that x_0 is a critical point of U imply that

$$\bar{K}(p) \geq \frac{1}{E-U(p)} K(p) + \frac{1}{2(E-U(p))^2} \left[\frac{\partial^2 U}{\partial t^2}(p) + \frac{\partial^2 U}{\partial s^2}(p) \right].$$

So if x_0 is a Morse type minimum of U, and the curvature $K(x_0)$ is nonnegative, there exists $r_1 > 0$ such that:

(1) the ball $B_{r_1}(x_0)$ is contained in a Fermi chart around x_0,

(2) the second derivatives of U in $B_{r_1}(x_0)$ with respect to the Fermi coordinates are positive,

(3) the curvature $K(p)$ is positive for every $p \in B_{r_1}(x_0)$.

The number r_1 suits the requirements of item (3) of Lemma 1.4, thus finishing the proof of the lemma. □

2 Uniformly geodesic balls for Maupertuis' metrics

We continue with the notation of Section 1, (T^2, g) is a smooth Riemannian structure in T^2, $U\colon T^2 \longrightarrow \mathbb{R}$ is a smooth positive function and $L(p, v) = \frac{1}{2}g(v, v) - U(p)$ is the mechanical Lagrangian defined by g and U. The goal of the section is to study Lagrangians with the property that there exists a ball $B_r(p)$ of g-radius $r > 0$ centered at some point $p \in T^2$ such that the radial geodesics of g in $B_r(p)$ starting at p are in fact g^E-minimizing geodesics for all the Maupertuis' metrics $g^E = (E - U)g$. This construction will allow us to control the behaviour of radial geodesics in $B_r(p)$ under perturbations of U simultaneously in all the Maupertuis' metrics. We shall use the notations d_g to designate the distance associated to the metric g, d_E for the distance of the metric g^E, $\| v \|$ for the norm of a vector v in the metric g, and $\| v \|_E$ for the norm of v in the metric g^E. Metric balls with respect to g, with g-radius r and centered at p will be denoted by $B_r(p)$, and metric balls with respect to g^E with g^E-radius r and centered at p will be denoted by $B^E_r(p)$.

Definition 2.1. *Given a family of metrics G in T^2, we say that a ball $B_r(p)$ of g-radius $r > 0$ for some $g \in G$ is uniformly geodesic for the family G if the g-geodesics in $B_r(p)$ containing the point p are geodesics for every metric $h \in G$.*

Our first result is an elementary remark that plays a key role in the proof of the main theorem.

Lemma 2.1. *Let $L(p, v) = \frac{1}{2}g(v, v) - U(p)$ be the mechanical Lagrangian defined by the metric g and a smooth potential U. Suppose that the point x_0 is a Morse minimum of U, and that there exists $\delta > 0$ such that*

(1) *x_0 is the only singularity of U in the ball $B_\delta(x_0)$,*

(2) *the integral curves of the gradient of U in $B_\delta(x_0)$ are geodesics of the metric g.*

Then the integral curves of the gradient of U in $B_\delta(x_0)$ are geodesics for all the Maupertuis metrics $g^E_p = (E - U(p))g_p$ for every $E > C$.

Proof. The proof is straightforward from Lemma 1.2: if the gradient of f is parallel to the geodesics of g in $B_\delta(x_0)$ through x_0, then the integral curves of the gradient are geodesics of the metric $\bar{g}_p = f(p)g_p$. Since the gradients of the functions $f_E(p) = E - U(p)$ are all the same, we get that the g-geodesics through x_0 in $B_\delta(x_0)$ are in fact g^E-geodesics for every $E > C$. $\square$

Next, we proceed to study g^E-minimizing properties of the geodesics through x_0 in the ball $B_\delta(x_0)$. We can suppose that δ is a normal radius for g, i.e., the ball $B_\delta(x_0)$ is a normal ball for g: given x, y in $B_\delta(x_0)$ the geodesic joining x and y is unique, minimizing, and contained in $B_\delta(x_0)$. In particular, the exponential map $\exp_{x_0}: \{\| v \| < \delta\} \longrightarrow B_\delta(x_0)$ is a diffeomorphism, and δ is less than or equal to the injectivity radius of (T^2, g). We shall denote by $l_g(c)$ the length of a curve c in the metric g, and by $l_E(c)$ the length of c in the metric g^E.

Lemma 2.2. *Let (T^2, g) be a smooth Riemannian structure in T^2, $U: T^2 \longrightarrow \mathbb{R}$ be a C^∞ function, and $L(p, v) = \frac{1}{2}g(v, v) - U(p)$ be a C^∞ Lagrangian defined in TT^2. Suppose that there exist a point $x_0 \in T^2$, and a normal radius $r > 0$ for the metric g, such that*

(1) $U(q) < \max_{p\in T^2} U(p)$ *for every* $q \in B_r(x_0)$,

(2) *the point x_0 is an isolated critical point of U in $B_r(x_0)$, and the level curves of U in this ball are the g-spheres of radius $\rho \leq r$ centered at x_0.*

Let $E = C$ be the critical energy level of the Lagrangian L. Then for every $E \geq C$ there exists $r(E) > 0$ such that the g-geodesics in $B_r(x_0)$ through x_0 are g^E-minimizing in the ball $B^E_{r(E)}(x_0) = \{p \in T^2, d_{g^E}(p, x_0) \leq r(E)\}$. Moreover, there exists $D > 0$ such that:

$$\left(\min_{p\in B_r(x_0)} \sqrt{E - U(p)}\right) D \leq r(E).$$

Proof. The proof is a consequence of Lemma 2.1 and comparison Theorems of basic Riemannian geometry. Since the level curves of U in $B_r(x_0)$ are the g-spheres centered at x_0, the gradient of U in $B_r(x_0)$ is tangent to the g-geodesic rays through x_0. So by Lemma 2.1, the ball $B_r(x_0)$ is uniformly geodesic for all the Maupertuis' metrics g^E. By Rauch's comparison Theorem, we have that if the curvature K^E satisfies $K^E \leq H$ then the injectivity radius $\rho(E)$ of (T^2, g^E) satisfies $\rho(E) \geq \frac{\pi}{\sqrt{H}}$. Choose an energy $E > C$. According to Lemma 1.4, item (2), we get

$$\rho(E) \geq \frac{\pi}{\sqrt{A}} \min_{q\in B_r(x_0)} \sqrt{E - U(q)},$$

for every $E > C$. If $E = C$, although the Maupertius' principle does not hold in the energy level, the quadratic forms g^E define indeed metrics in $B_r(x_0)$ by the choice of x_0 and r. Therefore, by Lemma 1.4 the above bounds for $K^E|_{B_r(x_0)}$ still hold for the curvature $K^C|_{B_r(x_0)}$. Let $M_E = \max_{p\in B_r(x_0)} \sqrt{E - U(p)}$, and

$m_E = \min_{p\in B_r(x_0)} \sqrt{E - U(p)}$. The injectivity radius of the metrics g^E restricted to $B_r(x_0)$ for $C \le E \le E_0$ have a common lower bound $\rho_0 > 0$. Writing

$$\rho_0 = \rho_0\sqrt{E - U(p)}\frac{1}{\sqrt{E - U(p)}} \ge \frac{\rho_0}{M_E} m_E$$

we have a lower bound for the injectivity radius of all the metrics $g^E|_{B_r(x_0)}$ for $C \le E \le E_0$ analogous to the lower bound of $\rho(E)$ for $E \ge E_0$. Since in (T^2, g^E) for $E \ge E_0$ all geodesics whose length is at most $r(E) = \frac{1}{2}\rho(E)$ are minimizers, this concludes the proof of the Lemma in the case $E \ge E_0 > C$. If $C \le E \le E_0$, since the curvatures K^E of the ball $B_r(x_0)$ are uniformly bounded above, there is a lower bound for the normal radius $\nu_E(p)$ of $p \in B_r(x_0)$, i.e., there exists $\nu > 0$ such that $\nu_E(p) \ge \nu$ for every $p \in B_r(x_0)$. Hence, the g^E-geodesics through x_0 whose length is at most $r(E) = \frac{1}{2}\nu = \frac{\nu}{\rho_0}\rho_0$ are minimizers, thus proving the Lemma in the case $C \le E \le E_0$. □

Lemma 2.3. *Let (T^2, g) be a smooth Riemannian structure in T^2, U, x_0, $r > 0$ and $r(E) > 0$ be as in Lemma 2.2. Given $0 < \epsilon \le r(E)$, the ball $B^E_\epsilon(x_0)$ is contained in the ball $B^{E'}_\epsilon(x_0)$ for every $C \le E' \le E$. More precisely,*

$$B^E_\epsilon(x_0) \subset B^{E'}_{\epsilon \max_{p\in B_r(x_0)} \sqrt{\frac{E'-U(p)}{E-U(p)}}}$$

for every $C \le E' \le E$.

Proof. The proof is an easy calculation using the definitions of the metrics g^E. Given a g-geodesic $\gamma\colon (-\epsilon, \epsilon) \longrightarrow B^E_\epsilon(x_0)$ with $\gamma(0) = x_0$, parametrized by g^E-arc length, we obtain its $g^{E'}$-length by the following formula:

$$\begin{aligned} l_{E'}(\gamma) &= \int_0^\epsilon \| \gamma'(t) \|_{E'}\, dt \\ &= \int_0^\epsilon \sqrt{E' - U(\gamma(t))}\, \| \gamma'(t) \|\, dt \\ &= \int_0^\epsilon \sqrt{E - U(\gamma(t))}\, \| \gamma'(t) \|\, \frac{\sqrt{E' - U(\gamma(t))}}{\sqrt{E - U(\gamma(t))}} dt \\ &= \int_0^\epsilon \| \gamma'(t) \|_E \sqrt{\frac{E' - U(\gamma(t))}{E - U(\gamma(t))}} dt \\ &\le l_E(\gamma) \max_{p\in B_r(x_0)} \sqrt{\frac{E' - U(p)}{E - U(p)}} < l_E(\gamma) = \epsilon. \end{aligned}$$

Hence, a subset $\gamma[0, t_{E'}]$ of $g^{E'}$-length ϵ contains the curve $\gamma[0, \epsilon]$ of g^E-length ϵ and parametrized by g-arc length. □

The next result will be very useful in the proof of the main Theorem. It shows that there exists a system of "nested" uniformly geodesic neighborhoods where the radial geodesics are minimizing for all the Maupertuis' metrics.

Lemma 2.4. *Let* (T^2, g) *be a smooth Riemannian structure in* T^2, U, x_0, $r > 0$ *and* $r(E) > 0$ *be as in Lemma 2.2. There exists* $r_1 > 0$, $0 < \alpha(E) \leq r(E)$, *such that:*

(1) *The balls* $B^E_{\alpha(E)}(x_0)$ *are subsets of* $B_r(x_0)$, *and the radial* g*-geodesics through* x_0 *are* g^E*-minimizers.*

(2) *For every* $E > C$ *and* $C \leq E' \leq E$ *there exists* $0 < \alpha_E(E')$ *such that* $B^E_{\alpha(E)}(x_0) = B^{E'}_{\alpha_E(E')}(x_0) \subset B^{E'}_{r(E')}(x_0)$,

(3) $B^E_{\alpha(E)}(x_0)$ *contains a ball* $B_{r_1}(x_0)$ *of* g*-radius* r_1 *for every* $E > C$.

Proof. Let us recall the notations

$$M_E = \max_{p \in B_r(x_0)} \sqrt{E - U(p)} = \sqrt{E - U(x_0)}, \; m_E = \min_{p \in B_r(x_0)} \sqrt{E - U(p)}.$$

We begin by observing that there exists a constant $P > 0$ such that $P > \frac{M_E}{m_E}$ for every $C \leq E$. This is straightforward from the following two facts:

(1) $\lim_{E \to +\infty} \frac{M_E}{m_E} = 1$,

(2) by the definition of $B_r(x_0)$ there exists $\sigma > 0$ such that $C - U(p) > \sigma$ for every $p \in B_r(x_0)$, $C = \max_{p \in T^2} U(p)$ which means that

$$\frac{1}{m_E} < \frac{1}{\sqrt{\sigma}}$$

for every $E \leq C$.

By Lemma 2.2, there exists $D > 0$ such that $r(E) \geq Dm_E$. For our purposes, we can assume without loss of generality that $r(E) = Dm_E$ and $P \geq 1$.

Claim. The number $\alpha(E) = \frac{D}{P} m_E$ satisfies the requirements of Lemma 2.4.

Indeed, the choice of P implies that $\alpha(E) \leq Dm_E = r(E)$, so according to Lemma 2.2 the radial geodesics through x_0 in $B^E_{\alpha(E)}(x_0)$ are g^E-minimizing and this proves partially item (1). To show item (2) notice that the spheres of the metrics g^E in $B_r(x_0)$ are the level curves of the potential U, so in fact the ball

$B^E_{\alpha(E)}(x_0)$ corresponds to a certain ball $B^{E'}_{\alpha_E(E')}(x_0)$ with $g^{E'}$-radius $\alpha_E(E') > 0$ for every $C \leq E' \leq E$. By Lemma 2.3, we have that

$$\begin{aligned}\alpha_E(E') \ &\leq \ \alpha(E) \max_{p \in B_r(x_0)} \sqrt{\frac{E' - U(p)}{E - U(p)}} \leq \alpha(E) \frac{M_{E'}}{m_E} \\ &\leq \ \frac{D}{P} M_{E'} = \frac{D}{P} m_{E'} \frac{M_{E'}}{m_{E'}} < D m_{E'} = r(E'),\end{aligned}$$

which shows item (2). To show item (3) observe that by the definition of the metric g^E we have that

$$m_E d_g(x, y) \leq d_E(x, y) \leq M_E d_g(x, y),$$

for every $x, y \in B_r(x_0)$, and hence

$$\alpha(E) = d_E(x_0, \partial B^E_{\alpha(E)}(x_0)) \leq M_E d_g(x_0, \partial B^E_{\alpha(E)}(x_0)).$$

This implies

$$d_g(x_0, \partial B^E_{\alpha(E)}(x_0)) \geq \frac{\alpha(E)}{M_E} = \frac{D}{P} \frac{m_E}{M_E} \geq \frac{D}{P^2},$$

by the choice of P. Therefore, the ball $B^E_{\alpha(E)}(x_0)$ contains the ball $B_{\frac{D}{P^2}}(x_0)$ of g-radius $r_1 = \frac{D}{P^2}$, proving item (3). We are left to complete the proof of item (1), namely, that $B^E_{\alpha(E)}(x_0) \subset B_r(x_0)$. The comparison inequality between the metrics g^E and g yields

$$d_g(x_0, \partial B^E_{\alpha(E)}(x_0)) \leq \frac{\alpha(E)}{m_E},$$

so if we show that $\frac{\alpha(E)}{m_E} \leq r$ we are done. In fact, the number $\frac{\alpha(E)}{m_E}$ might be larger than r. However, since all the statements proved by now hold if we multiply $\alpha(E)$ by a constant $0 < \lambda \leq 1$, we can rescale $\alpha(E)$ in order to get $\lambda \frac{\alpha(E)}{m_E} = r$. Thus, considering $\lambda\alpha(E)$ instead of $\alpha(E)$ we get a number that fullfils the requirements of Lemma 2.4. $\square$

3 Perturbations of the potential preserving uniformly geodesic balls

We follow the notations of Section 2. Given a Lagrangian $L(p, v) = \frac{1}{2} g(v, v) - U(p)$ where (T^2, g) has a uniformly geodesic ball $B_r(x_0)$ for all the Maupertuis' metrics, we shall show how we can construct perturbations $\tilde{U}$ of U such that

$B_r(x_0)$ is still uniformly geodesic for the Maupertuis' metrics of the Lagrangian $\tilde{L}(p, v) = \frac{1}{2}g(v, v) - \tilde{U}(p)$. The idea is based in the theory of conformal perturbations of a metric g which preserve some prescribed subset of geodesics of g (see for instance [16], [17]).

Proposition 3.1. *Let (T^2, g) be a smooth Riemannian structure in T^2, $U : T^2 \longrightarrow \mathbb{R}$ be a smooth positive function having a Morse minimum x_0, such that there exists an embedded ball $B_r(x_0)$ of g-radius $r > 0$ centered at x_0 where*

(1) *The level curves of U in $B_r(x_0)$ are the g-spheres centered at x_0,*

(2) *$U(p) < C$ for every $p \in B_r(x_0)$.*

Given $\epsilon > 0$, $n > 0$, there exist ϵ-C^n perturbations $\tilde{U}$ of U such that:

(1) *The support of $\tilde{U}$ is $B_r(x_0)$,*

(2) *$\tilde{U}(p) < U(p)$ for every p in the interior of $B_r(x_0)$, and $\min_{p\in B_r(x_0)} \tilde{U}(p) = \tilde{U}(x_0)$,*

(3) *The level curves of $\tilde{U}$ are the g-spheres centered at x_0, and hence the ball $B_r(x_0)$ is uniformly geodesic for all the Maupertuis' metrics of the Lagrangian $\tilde{L}(p, v) = \frac{1}{2}g(v, v) - \tilde{U}(p)$,*

(4) *Let g^E, $\tilde{g}^E$ be respectively the Maupertuis' metrics associated to the energy $E > C$ of L and $\tilde{L}$. Then the difference between the arc-lengths of g^E and $\tilde{g}^E$ can be estimated as follows:*

$$\begin{aligned} \frac{\min_{p\in B_{\frac{r}{2}}(x_0)} \Delta U(p)}{2(E - \tilde{U}(x_0))} l_{g^E}(\gamma) &\leq l_{\tilde{g}^E}(\gamma) - l_{g^E}(\gamma) \\ &\leq \frac{\max_{p\in B_r(x_0)} \Delta U(p)}{\min_{p\in B_r(x_0)}(E - U(p))} l_{g^E}(\gamma), \end{aligned}$$

where $\Delta U(p) = U(p) - \tilde{U}(p)$.

Proof. By hypothesis, the radial g-geodesics in $B_r(x_0)$ through x_0 are geodesics for all the Maupertuis' metrics of L, according to Lemma 2.1: the ball $B_r(x_0)$ is uniformly geodesic for these metrics. The spheres $S^E_\delta(x_0)$ of g^E-radius $\delta > 0$ contained in $B_r(x_0)$ are the level curves of the potential U for every $E \geq C$. This special feature of the metric g allows us to apply some ideas of [17] to obtain perturbations $\tilde{U}$ of the potential that preserve the geodesics of the new Maupertuis' metrics of $\tilde{L}$. Indeed, if we perturb the potential in a way that the perturbation $\tilde{U}$ preserves the system of level curves of U in $B_r(x_0)$ we get that the gradient of $\tilde{U}$ is parallel to the gradient of U. Since the gradient of U is parallel to the radial geodesics of g in $B_r(x_0)$ we apply Lemma 2.1 to conclude that the radial g-geodesics in $B_r(x_0)$ are also geodesics for every metric $\tilde{g}^E_p = (E - \tilde{U}(p))g_p$ where $E \geq \tilde{C}$, $\tilde{C}$ being the critical value of $\tilde{L}$. Moreover, if we decrease the value of U pointwise with the perturbation $\tilde{U}$, then the critical values of L and $\tilde{L}$ coincide.

To construct such a perturbation, we proceed as in [17]. Take polar coordinates (ρ, θ) in $B_r(x_0)$ with $\rho = 0$ corresponding to the point x_0, and $\rho \in [0, r]$. Consider a C^∞ bump function $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ that is even, $f(t) = f(0)$ for every $|t| \geq \frac{r}{2}$ and attains its maximum value at $t = 0$, $f(t)$ is strictly increasing in the interval $[-r, -\frac{r}{2}]$, and $f(t) = 0$ for $|t| > r$. Now, define $\tilde{U}(p) = U(p) - f(\rho(p))$, that gives a function of the same differentiability class of f. If f is a perturbation of the zero function in the C^k topology, then $\tilde{U}$ is a perturbation of U in the same topology. Item (1) in the proposition obviously holds. Since f is positive and attains its maximum value at $t = 0$ item (2) is trivially true. Since the curves $\rho = r_0$ represent the spheres of g around x_0 which are level curves of U, then it is clear that the level curves of $\tilde{U}$ are also these spheres and hence, the gradient of $\tilde{U}$ in $B_r(x_0)$ is parallel to the g-geodesics through x_0. The same is true for the gradient of the functions $E - \tilde{U}(p)$ and therefore the radial g-geodesics in $B_r(x_0)$ are $\tilde{g}^E$-geodesics for every $E \geq C$. This proves item (3). The proof of item (4) is a calculation. Let $\gamma : [a, b] \longrightarrow B_r(x_0)$ be a differentiable curve. Then,

$$\begin{aligned}
l_{\tilde{g}^E}(\gamma) - l_{g^E}(\gamma) &= \int_a^b [\sqrt{E - \tilde{U}(\gamma(t))} - \sqrt{E - U(\gamma(t))}] \parallel \gamma'(t) \parallel dt \\
&= \int_a^b \frac{U(p) - \tilde{U}(p)}{\sqrt{E - \tilde{U}(\gamma(t))} + \sqrt{E - U(\gamma(t))}} \parallel \gamma'(t) \parallel dt \\
&= \int_a^b \frac{\Delta U(p)\sqrt{E - U(\gamma(t))} \parallel \gamma'(t) \parallel}{E - U(p) + \sqrt{(E - \tilde{U}(\gamma(t)))(E - U(\gamma(t)))}} dt \\
&= \int_a^b \frac{\Delta U(p) \parallel \gamma'(t) \parallel_{g^E}}{E - U(p) + \sqrt{(E - \tilde{U}(\gamma(t)))(E - U(\gamma(t)))}} dt.
\end{aligned}$$

Since $\tilde{U}(p) < U(p)$ for every $p \in B_r(x_0)$ we have $E - \tilde{U}(p) > E - U(p)$ and hence

$$E - U(p) < \sqrt{(E - \tilde{U}(p))(E - U(p))} < E - \tilde{U}(p).$$

Consider a subinterval $[a', b'] \subset [a, b]$. Replacing in the above inequalities we get

$$\frac{\min_{t\in[a',b']} \Delta U(\gamma(t))}{2\max_{t\in[a',b']}(E - \tilde{U}(\gamma(t)))} l_{g^E}(\gamma) \leq l_{\tilde{g}^E}(\gamma) - l_{g^E}(\gamma)$$
$$\leq \frac{\max_{t\in[a',b']} \Delta U(\gamma(t))}{\min_{p\in B_r(x_0)}(E - U(p))} l_{g^E}(\gamma).$$

In particular, taking a g-geodesic $\gamma[-r, r]$ parametrized by g-arc length with $\gamma(0) = x_0$ we obtain the inequalities in item (4), just replacing $[a', b'] = [-\frac{r}{2}, \frac{r}{2}]$, $[a, b] = [-r, r]$. $\square$

4 Elimination of invariant graphs in large subsets of energy levels

We shall proof Theorem 1 in two steps. The first step, which is the goal of this section, consists in showing that given $E > C$, where C is the critical value of the Lagrangian, there exist a C^0 perturbation of the potential creating a bump for the Lagrangian action which is avoided by minimizers of the action **in all energy levels** $C \leq E' \leq E$. The second step, that is the subject of the next section, uses this fact to show the density of Lagrangians with no invariant graphs in any regular level of energy. So we start with a characterization of a family of C^0 perturbations of the metric g which create bumps.

Proposition 4.1. *Let (T^2, g) be a smooth Riemannian structure in T^2, let $\rho > 0$ be a normal radius of (T^2, g), and let $G_g > 1$ be an upper bound for the Gaussian curvature of g. Let $p \in T^2$ and $L > 0$. Then there exist $0 < \delta(\rho, G_g) < \rho$ such that if for some $0 < \delta \leq \delta(\rho, G_g)$, and g_δ is a metric in T^2 satisfying:*

(1) *The metric g_δ coincides with g outside the ball $B_\delta(p)$ and $\| g - g_\delta \|_\infty \leq L\delta$,*

(2) *The radial g-geodesics through p in $B_\rho(p)$ are also g_δ-geodesics,*

(3) *The g_δ-length of each radial geodesic γ in the ball $B_\delta(p)$ of g-radius δ around p exceeds the g-length of γ according to the following formula:*

$$l_{g_\delta}(\gamma[0, \delta]) - l_g(\gamma[0, \delta]) \geq 8G_g\delta^2,$$

where γ is parametrized by g-arc length, then no radial geodesic through p is g_δ-minimizing.

The proof of Proposition 4.1 will be given in the Appendix, we prefer to show how Proposition 4.1 applies to eliminate invariant graphs in large subsets of energy levels.

Recalling the notation of the previous sections, let $U : T^2 \longrightarrow \mathbb{R}$ be a smooth potential with an isolated minimum x_0 in $B_r(x_0)$ with $U(x_0) < \max_{p \in T^2} U(p) = C$, then the equation $g^E_x = (E - U(x))g$ defines a Riemannian metric in the ball $B_r(x_0)$ for every $E \geq C$. Assume that the level curves of the restriction to $B_r(x_0)$ of U are the g-spheres centered at x_0. So the g-geodesic rays in $B_r(x_0)$ through x_0 are the integral curves of ∇U in $B_r(x_0)$, and this ball is uniformly geodesic for the family of metrics g^E, $E \geq C$. Let $r(E) > 0$, $\alpha(E) > 0$ be the constants defined in Lemmas 2.2 and 2.4: radial g-geodesics in $B^E_{r(E)}(x_0)$ are g^E-minimizing, and the ball $B^E_{\alpha(E)}(x_0)$ of g^E-radius $\alpha(E)$ is contained in $B^{E'}_{r(E')}(x_0)$ for every $C \leq E' \leq E$. Let us denote by ρ_E a normal radius for the Maupertuis' metric g^E in $B_r(x_0)$, $E \geq C$, G_E will denote the supremum of the Gaussian curvature of g^E in $B_r(x_0)$, and observe that we can suppose that $\rho_E = \alpha(E)$. Moreover, according to Lemma 1.4, there exists an upper bound $G > 0$ for the Gaussian curvatures of the g^E's in $B_r(x_0)$: $G \geq G_E$ for every $E \geq C$. Let us denote by $\delta(\rho_E) = \delta(\rho_E, G)$ the constant given in Proposition 4.1 corresponding to the metric g^E. We can apply Proposition 4.1 to each metric g^E in the ball $B_r(x_0)$ taking $G_{g^E} = G$ for every $E \geq C$. This provides us a sufficient criterion to decide whether a metric h in $B_r(x_0)$ with the same radial g^E-geodesics has the property that radial geodesics in $B^E_{\rho_E}(x_0)$ are not h-minimizing.

Lemma 4.1. *Let (T^2, g), $U : T^2 : \longrightarrow \mathbb{R}$, $x_0 \in T^2$, $r > 0$ be as above. Assume that the Gaussian curvature of (T^2, g) is nonnegative in $B_r(x_0)$. Given $\epsilon > 0$, $E > C$, there exist $0 < r_{E,\epsilon} < r$, and a ϵ-C^0 perturbation $\bar{U}$ of the potential U such that:*

(1) *The support of $\bar{U}$ is contained in the ball $B_{r_{E,\epsilon}}(x_0) = \{q \in T^2, d_g(q, x_0) \leq r_{E,\epsilon}\}$,*

(2) *The ball $B_r(x_0)$ is uniformly geodesic for the metrics $\bar{g}^E_p = (E - \bar{U}(p))g_p$,*

(3) *No radial g-geodesic in $B_r(x_0)$ is $g^{E'}$-minimizer for every $C \leq E' \leq E$.*

Proof. The idea is to use Propositions 3.1 and 4.1 to construct a perturbation of the potential enjoying the properties of assertions (1) and (2) in the statement which at the same time satisfies item (3). So let $E > C$ and consider $\delta < \delta(\rho_E)$, where $\delta(\rho_E)$ is the constant defined in Proposition 4.1. We restrict our study to the ball $B_r(x_0)$ where the metrics g^E are all well defined and have curvatures uniformly bounded from above. Without loss of generality, we can replace G_E

by the supremum G of the Gaussian curvatures of the metrics g^E restricted to $B_r(x_0)$. Let $\gamma : [-\rho_E, \rho_E] \longrightarrow B^E_{\rho_E}(x_0)$ be a geodesic parametrized by g^E-arc length such that $\gamma(0) = x_0$. Let us recall that by Proposition 3.1, the change in the arc length of the Maupertuis' metrics induced by a perturbation $\tilde{U}$ of U with support in the ball $B^E_\delta(x_0)$ can be estimated as follows:

$$\begin{aligned}\frac{\min_{p\in B^E_{\frac{\delta}{2}}(x_0)} \Delta U(p)}{2(E-\tilde{U}(x_0))} l_{g^E}\left(\gamma[0,\frac{\delta}{2}]\right) &\le l_{\tilde{g}^E}(\gamma[0,\delta]) - l_{g^E}(\gamma[0,\delta]) \\ &\le \frac{\max_{p\in B^E_\delta(x_0)} \Delta U(p)}{\min_{p\in B^E_\delta(x_0)}(E-U(p))} l_{g^E}(\gamma[0,\delta]),\end{aligned}$$

where $\Delta U(p) = U(p) - \tilde{U}(p)$. Since $l_{g^E}(\gamma[0,\frac{\delta}{2}]) = \frac{1}{2} l_{g^E}(\gamma[0,\delta])$ we can rewrite the left inequality in the following way:

$$\frac{\min_{p\in B^E_{\frac{\delta}{2}}(x_0)} \Delta U(p)}{4(E-\tilde{U}(x_0))} l_{g^E}(\gamma[0,\delta]) \le l_{\tilde{g}^E}(\gamma[0,\delta]) - l_{g^E}(\gamma[0,\delta]).$$

According to Proposition 4.1, if we had that

$$\frac{\min_{p\in B^E_{\frac{\delta}{2}}(x_0)} \Delta U(p)}{4(E-\tilde{U}(x_0))} l_{g^E}(\gamma[0,\delta]) \ge 8G_E\delta^2$$

then the radial geodesics through x_0 in $B^E_{\rho(E)}(x_0)$ would be not g^E-minimizing, and hence, these geodesics would not be g^E-minimizing in $B_r(x_0)$. The last inequality can be reduced to

$$\frac{\min_{p\in B^E_{\frac{\delta}{2}}(x_0)} \Delta U(p)}{4(E-\tilde{U}(x_0))} \delta \ge 8G_E\delta^2,$$

or equivalently,

$$\frac{\min_{p\in B_{\frac{\delta}{2}}(x_0)} \Delta U(p)}{E-\tilde{U}(x_0)} \ge 32G_E\delta.$$

Now, recall that by Lemma 1.4 (1), (2) there exists a constant $A > 0$ such that

$$G_E \le \frac{A}{\min_{p\in B_r(x_0)}(E-U(p))}$$

for every $C \le E$. Therefore, if we had that

$$\frac{\min_{p\in B^E_{\frac{\delta}{2}}(x_0)} \Delta U(p)}{E-\tilde{U}(x_0)} \ge 32\frac{A}{\min_{p\in B_r(x_0)}(E-U(p))}\delta$$

then we would have that the radial geodesics in $B_r(x_0)$ are not g^E-minimizing. Or equivalently,

$$\min_{p \in B^E_{\frac{\delta}{2}}(x_0)} \Delta U(p) \geq 32A \frac{E - \tilde{U}(x_0)}{\min_{p \in B_r(x_0)}(E - U(p))}\delta,$$

for some $\delta < \delta(\rho_E)$. Notice that there exists a constant $B > 0$ such that

$$\frac{E - \tilde{U}(x_0)}{\min_{p \in B_r(x_0)}(E - U(p))} \leq B,$$

since $\min_{p \in B_r(x_0)}(E - U(p)) > a > 0$ and $\lim_{E \to +\infty} \frac{E - \tilde{U}(x_0)}{\min_{p \in B_r(x_0)}(E - U(p))} = 1$. Hence, we can replace the above inequalities by

$$\min_{p \in B^E_{\frac{\delta}{2}}(x_0)} \Delta U(p) \geq 32BA\delta$$

without loss of generality.

Claim 1. Given $\epsilon > 0$ there exists $\delta_{\epsilon,E} \in (0, \delta(\rho_E))]$ such that for every $\delta \leq \delta_{\epsilon,E}$ there exist $\sigma = \sigma(\delta) > 0$, and a smooth potential $U_{\epsilon,\delta,E} \colon T^2 \longrightarrow \mathbb{R}$ such that

(1) The support of $U_{\epsilon,\delta,E}$ is $B_\sigma(x_0) = \{p \in T^2, d_g(p, x_0) \leq \sigma\}$, and the level curves of $U_{\epsilon,\delta,E}$ are the g-spheres centered at x_0,

(2) $\| U_{\epsilon,\delta,E} - U \|_\infty = 32BA\delta \leq \epsilon$,

(3) There exists $0 < \lambda = \lambda(\sigma) < \sigma$ such that:

$$\min_{p \in B_\lambda(x_0)} \Delta U(p) = 32BA\delta,$$

where $\Delta U(p) = U(p) - U_{\epsilon,\delta,E}(p)$.

Indeed, using the ideas in the proof of Proposition 3.1, let us first construct a positive bump function $f \colon (-1, 1) \longrightarrow \mathbb{R}$ satisfying

(1) $\min_{|t| \leq \frac{1}{2}} f(t) \geq 32BA\delta$,

(2) $\| f \|_\infty \leq \epsilon$.

Let us start with a smooth even function $f(t)$ supported in $[-1, 1]$ such that $f(t)$ is increasing in $[-1, 0]$ and takes its maximum value in the interval $|t| \leq \frac{\delta}{2}$, where $f(t) = 32BA\delta$ for every $|t| \leq \frac{\delta}{2}$. In this way we have that $\| f \|_\infty = 32BA\delta$, and then we calculate δ such that this norm is at most ϵ: for this purpose we must have that $32BA\delta \leq \epsilon$ which means that $\delta \leq \delta_{\epsilon,E} = \frac{\epsilon}{32BA}$. Now, recall that the number δ represents a length in the metric g^E, and we want a ball $B_{r_E(\delta)}(x_0)$ in the metric g with g-radius $r_E(\delta)$ which represents the ball $B^E_{\delta_{\epsilon,E}}(x_0)$. As we mentioned in the previous sections, the balls in $B_r(x_0)$ of the Maupertuis' metrics coincide, their boundaries are the level curves of the potential U. So in fact, there exists $r_E(\delta) > 0$ such that $B^E_\delta(x_0) = B_{r_E(\delta)}(x_0)$ for every $\delta \leq \delta_{E,\epsilon}$. So let $r_{E,\epsilon} = r_E(\delta_{E,\epsilon})$, and given $\delta < \delta_{E,\epsilon}$ let $f_{\epsilon,\delta,E} : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f_{\epsilon,\delta,E}(t) = f(\frac{t}{r_E(\delta)})$ for $\delta \leq \delta_{\epsilon,E}$, whose support is the interval $[-r_E(\delta), r_E(\delta)]$ and has the same image as the function f. Clearly, $r_E(\delta) < r_{E,\epsilon}$ for every $\delta < \delta_{E,\epsilon}$; $\| f_{\epsilon,\delta,E} \|_\infty \leq \epsilon$, and $f_{\epsilon,\delta,E}(t) = 32BA\delta$ for every $|t| \leq \frac{r_E(\delta)}{2}$. Finally, take polar coordinates (ρ, θ) in $B_r(x_0)$, where ρ is the g-radius of points in $B_r(x_0)$, and define

$$U_{\epsilon,\delta,E}(p) = U(p) - f_{\epsilon,\delta,E}(\rho(p)),$$

for every $p \in B_r(x_0)$. The potential $U_{\epsilon,\sigma,E}$ satisfies the requirements of Claim 1.

Following the notation in Claim 1, let $r_E(\delta)$ be the g-radius of the ball around x_0 which coincides with the ball $B^E_\delta(x_0)$, i.e., $B^E_\delta(x_0) = B_{r_E(\delta)}(x_0)$. Let us call by $r_{E,E'}(\delta)$ the $g^{E'}$-radius of the ball $B^E_\delta(x_0)$ for $C \leq E' \leq E$, i.e., $B^E_\delta(x_0) = B^{E'}_{r_{E,E'}(\delta)}(x_0)$. Notice that $r_{E,E}(\delta) = \delta$, and by Lemma 2.3 we have that $r_{E,E'}(\delta) < \delta$ for every $E' < E$.

Claim 2. Let $U_{\epsilon,\delta,E}$ be the potential defined in Claim 1 for $\delta \leq \delta_{\epsilon,E}$. Then we have that

$$\min_{p \in B^{E'}_{r_{E,E'}(\frac{\delta}{2})}(x_0)} \Delta U(p) \geq 32BAl_{E'}(\gamma[0, \delta]),$$

where $\Delta U(p) = U(p) - U_{\epsilon,\delta,E}(p)$, and γ is a radial geodesic parametrized by g^E-arc length.

The proof of Claim 2 is a straightforward calculation.

Observe that

$$\begin{aligned} l_E(\gamma[0,\delta]) &= \delta \\ &= l_{E'}(\gamma[0,\delta])\frac{l_E(\gamma[0,\delta])}{l_{E'}(\gamma[0,\delta])} \\ &= l_{E'}(\gamma[0,\delta])\frac{\delta}{r_{E,E'}(\delta)} \geq l_{E'}(\gamma[0,\delta]). \end{aligned}$$

Combining this estimate with Claim 2 we get that for each $C \leq E' \leq E$, every $g^{E'}$-geodesic $\gamma[0, r_{E,E'}(\frac{\delta}{2})]$ parametrized by $g^{E'}$-arc length in the ball $B^{E'}_{r_{E,E'}(\frac{\delta}{2})}(x_0)$ with $\gamma(0) = x_0$, satisfies the inequality in Proposition 4.1:

$$(*)\ l_{\bar{g}^{E'}}(\gamma) - l_{g^{E'}}(\gamma) \geq 8G_g l_{g^{E'}}(\gamma)^2.$$

Since we chose $\delta < \alpha(E)$, and $r_{E,E'}(\alpha(E)) = \alpha_E(E') \leq \alpha(E')$ according to Lemma 2.4, we get that

$$r_{E,E'}\left(\frac{\delta}{2}\right) < r_{E,E'}(\delta) < r_{E,E'}(\alpha(E)) \leq \alpha(E'),$$

which means that the ball $B^{E'}_{r_{E,E'}(\frac{\delta}{2})}(x_0)$ is contained in the $g^{E'}$-normal ball $B^{E'}_{\alpha(E')}(x_0)$. The final step of the proof of Claim 2 is to apply Proposition 4.1 to the metric $g^{E'}$ in the ball $B^{E'}_{\alpha(E')}(x_0)$, where we take $\alpha(E') = \rho(E')$ as normal radius of $g^{E'}$ in $B_r(x_0)$. By the inequality (*) it would be enough to show that $r_{E,E'}(\delta) \leq \delta(\alpha(E'))$.

Claim 3. We can choose $\delta \leq \delta(\alpha(E'))$ such that $r_{E,E'}(\delta) \leq \delta(\alpha(E'))$ for every $C \leq E' \leq E$.

In fact, there exists a minimum δ_0 of the values $\delta(\alpha(E'), G_E) = \delta(\alpha(E'))$ in $C \leq E' \leq E$ defined in Proposition 4.1, simply because the set of metrics $\{g^{E'}|_{B_r(x_0)}, C \leq E' \leq E\}$ is co-compact with bounded curvatures $K^{E'} \leq G_{E'} \leq G$. Hence, we can take $0 < \delta \leq \delta_0$, and this implies by Lemma 2.3

$$r_{E,E'}(\delta) \leq \delta \leq \delta_0 \leq \delta(\alpha(E'))$$

for every $C \leq E' \leq E$, thus proving the Claim and Lemma 4.1. □

5 The proof of the main Theorem

Let us start the proof of the main Theorem by the following remark.

Lemma 5.1. *Let g be a smooth Riemannian metric in T^2. Let $U : T^2 \longrightarrow \mathbb{R}$ be a smooth function whose C^2 norm is $\bar{A} > 0$. Given $\epsilon > 0$, $D \geq 2\bar{A}$, and a point $p \in T^2$, there exist $\tau = \tau(\epsilon, U, D) > 0$, $0 < \delta = \delta(\epsilon, U, D) < \tau$, $W = W(\bar{A}) > 0$ and a function $\bar{U} : T^2 \longrightarrow \mathbb{R}$ such that:*

(1) *The function $\bar{U}$ is ϵ-C^0 close to U,*

(2) *The function $\bar{U}$ has a Morse minimum at p,*

(3) *The functions U and $\bar{U}$ coincide in the complement of the ball of g-radius τ centered at p,*

(4) *The C^2 norm of $\bar{U}$ in the ball of g-radius δ around p is bounded above by W, and the eigenvalues of the Hessian of $\bar{U}$ in this ball are bounded below by D,*

(5) *The level curves of $\bar{U}$ in $B_\delta(p)$ are the g-spheres centered at p and hence integral curves of $\nabla\bar{U}$ in $B_\delta(p)$ are the g-geodesics through p.*

Proof. The proof is quite elementary, we shall just sketch the reasoning for the sake of completeness. Let $p \in T^2$, and let $f : B_r(p) \longrightarrow \mathbb{R}$ be the square of the distance from p, $f(x) = d_g(x, p)^2$. This function is smooth and strictly convex and the point p is a Morse minimum. The level curves of f are the spheres around p and the integral curves of the gradient of f are the geodesics of g in $B_r(p)$. Moreover, by compactness, there exists $T > 0$ such that the C^2 norm of f is bounded above by T regardless of the point p. Consider the family of functions $f_a : B_r \longrightarrow \mathbb{R}$ given by $f_a(x) = a + Bf(x)$, where $a \leq U(p)$, and $B > 0$ is a constant such that the eigenvalues of the Hessian of f_a in $B_r(p)$ are bounded below by $2A = D$. Hence, the function f_a is more convex than the function U, and it is clear that there exists $W = W(A) > 0$ such that the C^2 norm of f_a is bounded above by W. Let us consider the sets $Q_a = \{x \in B_r(p),\ f_a(x) \leq U(p)\}$. By the implicit function theorem, there exist $\mu = \mu(A) > 0$, $\nu = \nu(A)$ such that for every $\mu < a < U(p)$ the graphs of f_a and U meet tranversally at a compact set C_a diffeomorphic to a circle, having diameter $d(a) \leq \nu$ which is the boundary of a subset Σ_a of the graph of f_a containing $(p, f_a(p))$. Moreover, the numbers $d(a)$ tend to zero as a tends to $U(p)$. Let Σ_a be the graph of $f_a(V(a))$, where $V(a)$ is an open subset of $B_r(p)$ containing a ball of maximal radius of the form $B_{r(a)}(p)$. Clearly, $V(a)$

is diffeomorphic to an open ball, and let us define the function $U_a : T^2 \longrightarrow \mathbb{R}$ by $U_a(z) = U(z)$ if $z \notin V(a)$, $U_a(z) = f_a(z)$ if $z \in V(a)$. The function U_a is continuous, and U_a is within C^0 distance $|a - U(p)|$ from U. U_a has a Morse minimum at p, and we can smooth the function U_a by changing it a little in a tubular neighborhood of the boundary of $V(a)$ of radius $\frac{r(a)}{5}$ for instance, which does not meet the ball $B_{\frac{r(a)}{2}}(p)$. Let $\bar{U}_a$ be such a function. It is clear that p is still a local minimum of $\bar{U}_a$, and with this construction we get Lemma 5.1 taking

$$\epsilon = |a - U(p)|, \delta = \frac{r(a)}{2} \quad \text{and} \quad \tau = d(a) + \frac{r(a)}{5}. \qquad \square$$

Proof of Theorem 1. First of all, let $B(p) \in T^2$ be an open ball where the curvature of (T^2, g) is nonnegative. Let us choose a sequence of disjoint balls $B_{\delta_n}(p_n)$ contained in $B(p)$. Given $\epsilon > 0$, $\bar{A} > 0$, p_n, let $W = W(\bar{A})$, τ_n, δ_n be the constants defined in Lemma 5.1. According to Lemmas 5.1 and 2.1, there exists a family of potentials $U_n : T^2 \longrightarrow \mathbb{R}$ such that:

(1) $U_n(x) - U(x) = 0$ for every x in the complement of $B_{\tau_n}(p_n)$, for every $n > C$,

(2) $\| U_n - U \|_\infty < \epsilon$ for every $n > C$,

(3) the point p_n is a Morse minimum of U_n, which is unique in $B_{\tau_n}(p_n)$,

(4) The C^2 norm of U_n in $B_{\delta_n}(p_n)$ is at most W,

(5) the critical value of $L_n(p, v) = \frac{1}{2}g(v, v) - U_n(p)$ is C, the critical value of L, for every $n > C$,

(6) the level sets of U_n in $B_{\tau_n}(p_n)$ are the g-spheres centered at p_n, and therefore the balls $B_{\tau_n}(p_n)$ are uniformly geodesic for all the Maupertuis' metrics $g_n^E = (E - U_n)g$ for $E \geq C$.

We can assume, by shrinking $B(p)$ if necessary, that $\max_{q \in T^2} U(q) - \max_{x \in B(p)} U_n(x) = m > 0$ for every $n > C$.

Applying Claim 1 in Lemma 4.1, we have that there exist $A, B > 0$, $N = N(A, B) > 0$, such that for every $n > C$, there exist $0 < \sigma(\delta_n) \leq \delta_n$, and a new Lagrangian $\bar{L}_n(p, v) = \frac{1}{2}g(v, v) - \bar{U}_n(p)$ such that

(1) The support of $\bar{U}_n$ is $B_{\sigma(\delta_n)}(p_n) = \{p \in T^2, d_g(p, p_n) \leq \sigma(\delta_n)\}$, and the level curves of $\bar{U}_n$ are the g-spheres centered at p_n,

(2) $\| \bar{U}_n - U_n \|_\infty = 32BA\delta_n$,

(3) $\| \bar{U}_n - U_n \|_{C^2} \leq N(A, B)$, for every n,

(4) There exists $0 < \lambda_n < \sigma(\delta_n)$ such that:

$$\Delta \bar{U}_n(p) = \bar{U}_n(p) - U_n(p) = 32BA\delta_n,$$

for every $p \in B_{\lambda_n}(p_n)$.

(5) The radial $\bar{g}_n^E$-geodesics through p_n in $B_{\delta_n}(p_n)$ are no longer $\bar{g}_n^E$-minimizers for every $C \leq E \leq n$, where $\bar{g}_n^E = (E - \bar{U}_n)g$.

The next assertion is the proof of the first part of Theorem 1.

Claim. There are no continuous invariant graphs of the Euler-Lagrange flow of the Lagrangian $\bar{L}_n$ in any level of energy $C \leq E \leq n$.

Indeed, it is enough to show that the geodesic flow of each Maupertuis' metric $\bar{g}_n^E$ for $C \leq E \leq n$ has no invariant graphs. This is straighforward from the construction of $\bar{U}_n$: if the geodesic flow of $\bar{g}_n^E = (E - \bar{U}_n)g$ had a continuous invariant graph for some $C < E \leq n$, then there would exist a continuous flow in T^2 by globally $\bar{g}_n^E$-minimizing geodesics. In fact, the projection of the geodesic flow restricted to the invariant graph into T^2 would give a **continuous** non-singular vector field whose orbits are $\bar{g}_n$-geodesics. Such vector fields in calculus of variations are called Mayer fields, and a complete proof of the fact that the orbits of such vector fields are minimizers is made in [18]. The fact that **smooth** Mayer vector fields in surfaces have minimizing orbits is a well known fact in the theory of calculus of variations (see for instance [14], [13]). In particular, this flow would cover the ball $B_{\tau_n}(p_n)$ and hence there would exist $\bar{g}_n^E$-minimizing geodesics through p_n which is impossible by the choice of the metrics $\bar{g}_n^E$. In the critical level $E = C$ the Euler-Lagrange flow has no singularities in $B(p)$ and hence a continuous invariant graph would project into a continuous vector field without singularities in $B(p)$. The same previous argument would give that the orbits of such a vector field would be minimizers in the set of rectifiable curves contained in $B(p)$, contradicting Proposition 4.1 and Lemma 4.1. The above contradictions prove the Claim.

The proof of the second part of Theorem 1 is slightly more delicate. Let us suppose for simplicity that the points p_n lie on a single g-geodesic $\gamma_0 \colon [0, \epsilon] \longrightarrow T^2$ with $\gamma_0(0) = x = \lim_{n\to+\infty} p_n$. Let $p_n = \gamma_0(z_n)$. Since the balls $B_{\delta_n}(p_n)$ are disjoint we have that $\sum_n \delta_n \leq \epsilon$ is a convergent series.

Now, let $h_n(p) = U(p) - \bar{U}_n(p)$, and let $\bar{U}(p) = \sum_{n>C}(U(p) - h_n(p))$. Consider the Lagrangian

$$\bar{L}(p, v) = \frac{1}{2} g(v, v) - \bar{U}(p).$$

The potential $\bar{U}$ coincides with $\bar{U}_n$ in the ball $B_{\tau_n}(p_n)$, and coincides with U outside the union of the balls $B_{\tau_n}(p_n)$. Moreover, from the construction of the potentials $\bar{U}_n$ we have that $\| U - \bar{U} \|_\infty < \epsilon$, and from the Claim there are no invariant graphs in supercritical energy levels.

We have to show that $\bar{U}$ is a C^1 function. By item (3) before Claim 1 we know that the C^2 norm of the restriction of $\bar{U}$ to each ball B_{δ_n} is bounded above by a constant N for every n. Since outside the union of the balls $B_{\delta_n}(p_n)$ the C^2 norm of U is already bounded by some constant N_0, we have that for $t < s$ in $[0, \delta]$,

$$\| \operatorname{grad}_{\gamma(t)} \bar{U} - \operatorname{grad}_{\gamma(s)} \bar{U} \| \leq N d_g(\gamma_0(t), p_{n_t})$$
$$+ \sum_{i=n_t+1}^{n_s-1} N\delta_i + N d_g(\gamma_0(z_{n_s} - \delta_{n_s}), \gamma_0(s)) + N_0 T(t,s)$$

where n_t is defined by $\gamma_0(t) \in B_{\delta_{n_t}}(p_{n_t})$, and $T(t,s)$ is the g-length of the complement of

$$\cup_{n=n_t}^{n_s} B_{\delta_n}(p_n)$$

with respect to the geodesic $\gamma_0[t,s]$. From this formula it is easy to get that

$$\| \operatorname{grad}_{\gamma(t)} \bar{U} - \operatorname{grad}_{\gamma(s)} \bar{U} \| \leq (N + N_0) d_g(\gamma_0(t), \gamma_0(s)).$$

Since the potentials $\bar{U}_n$ were constructed with radial symmetries in B_{δ_n} with respect to the points p_n, we obtain a constant $N_1 > 0$ such that

$$\| \operatorname{grad}_q \bar{U} - \operatorname{grad}_z \bar{U} \| \leq N_1 d_g(q, z)$$

for every q, z in the union of the balls $B_{\delta_n}(p_n)$. Therefore, we deduce that the family of gradients of the functions

$$V_m = \sum_{n>C}^{m} (U(p) - h_n(p)),$$

where $m > C + 1$, is an equicontinuous, uniformly bounded family of functions in T^2. By Arzela-Ascoli theorem, there is a convergent subsequence of the V_m's which is C^1. But since the series of functions V_m is uniformly convergent to $\bar{U}$ in the C^0 topology, any convergent subsequence tends to $\bar{U}$ thus proving that $\bar{U}$ is C^1. Notice that the above argument grants that the gradient of $\bar{U}$ is a Lipschitz function. This finishes the proof of Theorem 1.

6 Appendix: The proof of Proposition 4.1

Let (T^2, g) be a C^∞ Riemannian structure in the two dimensional torus T^2, let $\rho > 0$ be the injectivity radius of (T^2, g), and let $p \in T^2$. Consider a normal ball $B_r(p)$ centered at p with radius $r \leq \rho$, i.e., a metric ball where each two points x, y in $B_r(p)$ determine a unique minimizing geodesic $[x, y] \subset B_r(p)$ of (T^2, g) joining x and y. The existence of normal balls is an elementary consequence of the properties of the exponential map of the metric g. Let us consider a geodesic $\gamma\colon [0, 2r] \longrightarrow B_r(p)$ parametrized by arc length, with $\gamma(r) = p$, and a geodesic $\alpha\colon (-\epsilon, \epsilon) \longrightarrow B_r(p)$ parametrized by arc length such that $\alpha(0) = p$, $g(\gamma'(r), \alpha'(0)) = 0$. Denote by $l_g(c)$ the length in the metric g of a curve c. We start with the following estimate of lengths of geodesics.

Lemma 6.1. *Let G_g be an upper bound for the Gaussian curvature of (T^2, g). Given $r > 0$, there exist $r > \delta > 0$, a constant $D = D(r) > 0$ such that for every $|s| \leq \delta$ the g-length of the broken geodesic Γ_s in $B_r(p)$ given by the union of the geodesics $[\gamma(0), \alpha(s)]$ and $[\alpha(s), \gamma(2r)]$ satisfies the following two properties:*

(1) $0 < l_g(\Gamma_s) - l_g(\gamma[0, 2r]) = l_g(\Gamma_s) - 2r \leq 4G_g s^2$,

(2) *The length of the intersection of Γ_s with the ball $B_s(p)$ can be estimated by*
$$l_g(\Gamma_s \cap B_s(p)) \leq Ds^2.$$

The inequality on the left in item (1) is obvious since the geodesic γ is minimizing in $B_r(p)$. The right inequality in item (1) is essentially a consequence of the second variation formula. Lemma 6.1 (1) can be regarded as an estimate of the increase of length of variations of $\gamma[0, 2r]$ by a certain type of broken geodesics. Although its proof is based in the well known first and second variation formulas for the length of geodesics, we include a complete proof for the sake of completeness.

We shall subdivide the proof of Lemma 6.1 in two parts. To show item (1) let us introduce some notations. Let $f\colon (-\epsilon, \epsilon) \times [0, r] \longrightarrow B_r(p)$ be the variation of γ by geodesics given by

(1) $f((-\epsilon, \epsilon) \times \{0\}) = \gamma(0)$,

(2) $f(\{s\} \times [0, r])$ is the geodesic joining $\gamma(0)$ and $\alpha(s)$ for every $s \in (-\epsilon, \epsilon)$.

Notice that the family of first derivatives of the variation f, $\frac{\partial f}{\partial s}(s_0, t) = J_{s_0}(t)$ defines a family of Jacobi vector fields along the geodesics $f_{s_0}(t) = f(s_0, t)$,

$t \in [0, r]$. We claim that the collection of first derivatives

$$\left\{ \frac{\partial}{\partial t} J_s(t),\, |s| < \epsilon,\, t \in [0, r] \right\},$$

is uniformly bounded. Namely, there exists a constant $L > 0$ such that if $\eta\colon [0, r] \longrightarrow T^2$ and $\eta_\perp\colon (-\epsilon, \epsilon) \longrightarrow T^2$ are geodesics of (T^2, g) parametrized by arc length with $\eta_\perp(0) = \eta(r)$ and $g(\eta'(0), \eta'_\perp(0)) = 0$; then the norms of the first derivatives of the Jacobi fields tangent to the variation f_η of η, constructed as above, by geodesics joining $\eta(0)$ and $\eta_\perp(s)$, are bounded above by L. Let us remind briefly the proof of this assertion. The family $\{J_s, s \in (-\epsilon, \epsilon)\}$ of such Jacobi fields is determined by the boundary conditions $J_s(0) = 0$, $J_s(r) = \eta'_\perp(s)$, so $\| J_s(r) \| = 1$ for every $|s| < \epsilon$. Since $\eta'_\perp(0) = J_0(r)$ is perpendicular to $\eta'(r)$, we have by compactness of T^2 that there exists a small constant $\sigma > 0$ such that $g(f'_s(t), J_s(t)) \leq \sigma$ for every $s \in (-\epsilon, \epsilon)$, $t \in [0, r]$, and every pair of geodesics η, $\eta_\perp$ as above. Since Jacobi fields in normal neighborhoods of (T^2, g) depend continuously on their boundary conditions, there exists $b > 0$ such that $\| J_s(t) \| \leq b$ for every $(s, t) \in (-\epsilon, \epsilon) \times [0, r]$ and η, $\eta_\perp$ as above. Now, it is easy to get a uniform bound in T^2 for the norms of the second derivatives of $J_s(t)$ by means of the Jacobi equation, and hence we can derive a uniform estimate for the first derivatives as we wished. This elementary observation yields the first step towards the proof of Lemma 6.1, which can be viewed as a local version of the Theorem of Pithagoras.

Lemma 6.2. *Let (T^2, g) be a C^∞ Riemannian structure in T^2. Let G_g be an upper bound for the Gaussian curvature. There exists a normal radius $r > 0$, $\epsilon_0 > 0$ such that given $p \in T^2$, a normal ball $B_r(p)$, geodesics $\gamma\colon [0, r] \longrightarrow B_r(p)$, $\alpha\colon (-\epsilon, \epsilon) \longrightarrow B_r(p)$ such that $\gamma(r) = \alpha(0)$, $g(\gamma'(r), \alpha'(0)) = 0$, and a variation f of γ as above, we have that*

$$|l_g(f_s) - l_g(\gamma)| \leq (2G_g)s^2,$$

for every $|s| \leq \epsilon_0$.

Proof. The proof of this lemma is straightforward from basic calculus of variations of Riemannian geometry. Indeed, by the second variation formula we have that there exists $\epsilon_1 > 0$ such that

$$l_g(f_s) = l_g(\gamma[0, r]) + \frac{1}{2} s^2 I(J_0, J_0) + O(s^3),$$

for every $|s| \leq \epsilon_1$. Here,

$$I(J_0, J_0) = \int_0^r (\| J'_0(t) \|^2 - g(K(\gamma'(t), J_0(t))\gamma'(t), J_0(t)))dt \leq r(L^2 + b^2 G_g)$$

is the index formula evaluated in the Jacobi field J_0, K is the curvature tensor of the metric g, and $L, b > 0$ are the constants defined in the proof of Lemma 6.1 (namely, upper bounds for $\| J_0'(t) \|$ and $\| J_0(t) \|$ for every $t \in [0, r]$). By the definition of the variation f_s we have that $J_s(0) = 0$ for every $s \in (-\epsilon, \epsilon)$. By the local expansion of Jacobi fields (see for instance [4]) there exists $M > 0$ such that

$$\| J_0(t) \|^2 \leq \| J_0'(0) \|^2 t^2 + MG_g \| J_0'(0) \|^2 t^4 \leq L^2 t^2 (1 + MG_g t^2).$$

So by the continuity of $\| J_s(t) \|$ we can assume without loss of generality that $b \leq 2$ (by shrinking the interval $[0, r]$ if necessary).

By standard comparison theorems in Riemannian geometry (see for instance [5]) we have that

$$\| J_0'(t) \| \leq \sqrt{G_g} \| J_0(t) \| \leq b\sqrt{G_g} \leq 2\sqrt{G_g},$$

for every $t \in [0, r]$. From this inequality and the index formula we get the lemma. □

Clearly, item (1) in Lemma 6.1 follows from Lemma 6.2. Item (2) will follow from the next result of the section, which concerns the length of intersections of the geodesics f_s with the balls $B_s(p)$, for s very small. The result is obvious for the Euclidean metric and follows from elementary trigonometry. In the general case the proof leads to some technical estimates involving lengths of geodesics.

Lemma 6.3. *Then there exists a constant $D = D(r)$ such that $l_g(f_s \cap B_s(p)) \leq Ds^2$ for every $0 < s \leq \epsilon_0$.*

Proof. Since the proof is elementary we just make a sketch of proof for the sake of completeness. Let us consider the metric g in T_pT^2, and let us consider the pullback g^* of the metric g in T_pT^2 by the exponential map $\exp_p : T_pT^2 \longrightarrow T^2$. Both metrics are equivalent in T_pT^2 in balls of radius at most one. Notice that the g^* balls of small radius are round balls by the elementary properties of the exponential map.

Let us consider a g^*-ball $E_a(p)$ of radius $a > 0$, whose interior is $E_a^o(p)$. Let $r > a$ and $q \in T_pT^2$ with $\| q \| \geq r$, let $q' \in E_a(p)$ be any of the points of intersection of $E_a(p)$ with the straight line through $(0, 0)$ which is perpendicular to the straight line determined by q and $(0, 0)$. Let q_1 be the point of intersection between $E_a(p)$ and the straight line determined by q and q'. Since $E_a(p)$ is a circle, an easy calculation shows that the g^*-length of the intersection of the segment $[q_1, q']$ (whose endpoints are q_1, q') with $E_a^o(p)$ satisfies

$$\| q_1 - q' \| \leq 2\frac{a^2}{r}.$$

Now, the equivalence between the metrics g and g^*, and the application of the above estimate to the geodesics of the variation f_s imply Lemma 6.3. $\square$

Proof of Proposition 4.1. So let (T^2, g) be a smooth Riemannian structure in T^2, let $\rho > 0$ be a normal radius of (T^2, g), and let $G_g > 0$ be an upper bound for the Gaussian curvature of g. Let $p \in T^2$ and let $L > 0$. We want to show that there exist $0 < \delta(\rho, G_g) < \rho$ such that if for some $0 < \delta \leq \delta(\rho, G_g)$, there exists a metric g_δ in T^2 with support in $B_\delta(p)$ satisfying:

(1) $\| g - g_\delta \|_\infty \leq L\delta$,

(2) The radial g-geodesics through p in $B_\rho(p)$ are also g_δ-geodesics,

(3) The g_δ-length of each radial geodesic γ in the ball $B_\delta(p)$ of g-radius δ around p exceeds the g-length of γ according to the following formula:

$$l_{\bar{g}}(\gamma[0, \delta]) - l_g(\gamma[0, \delta]) \geq 8 G_g \delta^2,$$

where γ is parametrized by g-arc length, then no radial geodesic through p is g_δ-minimizing.

Following the notation in the section, let $\rho > 0$ be a normal radius of (T^2, g), let $0 < r < \rho$, let $\gamma : [0, 2r] \longrightarrow B_r(p)$ be a g-geodesic with $\gamma(r) = p$, and let $\Gamma_s : [0, 2r] \longrightarrow B_r(p)$ be the variation by g-geodesics of γ defined in Lemma 6.1 (recall that $\Gamma_0 = \gamma$).

Let us consider a metric g_δ satisfying the assumptions of Proposition 4.1. We proceed to compare the g_δ-lengths of $\Gamma_s[0, 2r]$ and $\gamma[0, 2r]$. First of all, by the assumption of Proposition 4.1, the g_δ-length of γ can be estimated by

$$2L\delta^2 > l_{\bar{g}}(\gamma[0, 2r]) - l_g(\gamma[0, 2r]) > 8 G_g \delta^2.$$

Next, let us estimate the l_{g_δ}-length of the geodesics Γ_s:

$$l_{g_\delta}(\Gamma_s) = l_{g_\delta}(\Gamma_s \cap B_\delta(p)) + l_{g_\delta}(\Gamma_s \cap B_\delta(p)^c),$$

where $B_\delta(p)^c$ is the complement of $B_\delta(p)$. The above expression can be written as

$$l_{g_\delta}(\Gamma_s) = l_g(\Gamma_s) + l_{g_\delta}(\Gamma_s \cap B_\delta(p)) - l_g(\Gamma_s \cap B_\delta(p)),$$

and by the assumptions (item (1)), we observe that the difference

$$d_\delta = |l_{g_\delta}(\Gamma_s \cap B_\delta(p)) - l_g(\Gamma_s \cap B_\delta(p))|$$

is bounded above by

$$d_\delta \leq 2L\delta l_g(\Gamma_s \cap B_\delta(p))).$$

Hence, by Lemma 6.1, (2), we get that

$$|l_{g_\delta}(\Gamma_s) - l_g(\Gamma_s)| = d_\delta \leq 2L\delta D s^2 \leq 2LD\delta^3$$

for every $|s| < \delta$. Finally, we can write $l_{g_\delta}(\gamma) - l_{g_\delta}(\Gamma_\delta)$ in the following way:

$$l_{g_\delta}(\gamma[0, 2r]) - l_{g_\delta}(\Gamma_\delta) = l_{g_\delta}(\gamma) - l_g(\gamma) + l_g(\gamma) - l_g(\Gamma_\delta) + l_g(\Gamma_\delta) - l_{g_\delta}(\Gamma_\delta).$$

Therefore, by Lemma 6.1, (1), and the estimate of d_δ, we get

$$\begin{aligned} l_{g_\delta}(\gamma[0, 2r]) - l_{g_\delta}(\Gamma_\delta) &\geq 8G_g\delta^2 - 4G_g\delta^2 - 2LD\delta^3 \\ &\geq 4G_g\delta^2 - 2LD\delta^3 \geq G_g\delta^2, \end{aligned}$$

for every $\delta < \delta(\rho, G_g) \leq \rho$ suitably small. This inequality shows that the curve $\gamma[0, 2r]$ is no longer g_δ-minimizing: the g_δ-length of the curve Γ_δ joining the points $\gamma(0)$ and $\gamma(2r)$ is smaller than the g_δ-length of $\gamma[0, 2r]$, as we wished to show.

References

[1] V.I. Arnold, Mathematical Methods of Classical Mechanics. Second Edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York, Berlin, Heidelberg.

[2] V. Bangert, Mather sets for twist maps and geodesics on tori. Dynamics reported, Vol. 1. U. Kirchgraber, H.O. Walter Editors, 1988 John Wiley and Sons and B.G. Teubner.

[3] G.D. Birkhoff, Surface transformations and their dynamical applications. *Acta Math.*, **43** (1922), 1–119.

[4] M. Do Carmo, Geometria Riemanniana. IMPA, Projeto Euclides, 1979.

[5] P. Eberlein, When is a geodesic flow of Anosov type I. *J. Diff. Geom.*, **8** (1973), 437–463.

[6] N. Hicks, Notes on differential geometry. Van Nostrand Reinhold C. Ltd, London, 1974.

[7] R. Kulkarni, Curvature structures and conformal transformations. *J. Diff. Geom.*, **4** (1970), 425–451.

[8] R. MacKay, A criterion for the nonexistence of invariant tori of Hamiltonian systems. *Phys. D.*, **98**(1-2) (1989), 64–82.

[9] R. MacKay and I.C. Percival, Converse KAM: theory and practice. *Comm. Math. Phys.*, **98**(4) (1985), 469–512.

[10] J. Mather, Glancing billliards. *Ergod. Th. and Dyn. Sys.*, **2** (1982), 397–402.

[11] J. Mather, A criterion for the nonexistence of invariant circles. *Publications Mathématiques de l'IHES*, **63** (1986), 153–204.

[12] J. Mather, Destruction of invariant circles. *Ergod. Th. and Dynam. Sys.*, **8** (1988), 199–214. Charles Conley Memorial Volume.

[13] D. McDuff and D. Salamon, Introduction to Symplectic Topology. Claredon Press, Oxford (1995).

[14] J.M. Morse, Calculus of variations in the Large (American Mathematical Society Colloquium Publications, XVIII) American Mathematical Society, Providence, RI, 1996 (reprint of the 1932 original).

[15] J.K. Moser and C.L. Siegel, Lectures on Celestial Mechanics. Springer-Verlag, New York, Heidelberg, Berlin, 1971.

[16] R. Ruggiero, On the creation of conjugate points. *Math. Z.*, **208** (1991) 41–55.

[17] R. Ruggiero, Topological stability and Gromov hyperbolicity. *Erg. Th. Dyn. Sys.*, **19** (1999), 143–154.

[18] R. Ruggiero, On manifolds admitting continuous foliations by geodesics. *Geometriae Dedicata*, **78** (1999), 161–170.

[19] R. Ruggiero, On the generic nonexistence of rational geodesic foliations in the torus, Mather sets and Gromov hyperbolic spaces. *Bol. Soc. Bras. Mat.*, **31**(1) (2000), 93–111.

[20] R. Ruggiero, The set of smooth metrics in the torus without continuous invariant graphs in open and dense in the C^1 topology. *Bull. Braz. Math. Soc. New Series*, **35**(3) (2004), 377–385.

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