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Exact stability regions for quartic polynomials

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Abstract. Given an arbitrary real quartic polynomial, we find the exact region containing the coefficients of the polynomial such that all roots have absolute values less than 1.

Keywords: quartic polynomial, root, stability region.

Mathematical subject classification: 26C10.

1 Introduction

Given the real quadratic polynomial

$$Q(\lambda) = \lambda^2 - \alpha\lambda - \beta,\ \alpha, \beta \in R,$$

all its roots have absolute values less than 1 if, and only if, (α, β) lies in the plane triangular region defined by $|\beta| < 1$, $1 - \alpha - \beta > 0$ and $1 + \alpha - \beta > 0$. Such a result is well known and has many applications in macroeconomic models and population models (see e.g. [1]) as well as stability of dynamical systems. It is also shown in [6] that for the real cubic polynomial

$$P(\lambda) = \lambda^3 - (\alpha + 1)\lambda^2 - \beta\lambda - \gamma,\ \alpha, \beta, \gamma \in R,$$

all its roots have absolute values less than 1 if, and only if (α, β, γ) lies in the three dimensional region defined by $|\alpha + 1| < 3$, $\alpha+\beta+\gamma < 0$, $-\alpha+\beta-\gamma < 2$ and $\beta > \gamma^2 - (\alpha + 1)\gamma - 1$.

Although necessary and sufficient conditions (in terms of determinants) are known for all roots of a real polynomial to have absolute values less than 1 (see e.g. [7]), there is just a short list [2, 3, 4, 5] of results that describe explicit 'stability' regions for other polynomials. One reason may be explained as follows. By

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the Schur-Cohn condition, a real polynomial $f(z) = z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n$ is a Schur polynomial (i.e. all its roots have absolute values less than 1) if, and only if, the polynomial

$$g(w) = 2^{-n/2}(w-1)^n f\left(\frac{w+1}{w-1}\right) \tag{1}$$

is a Hurwitz polynomial, and a polynomial $g(w) = w^n + b_1 w^{n-1} + ... + b_{n-1} w + b_n$ is a Hurwitz polynomial (i.e. all its roots have negative real parts) if, and only if, for each $k = 1, ..., n$,

$$\begin{vmatrix} b_1 & b_3 & b_5 & \cdots & b_{2k-1} \\ 1 & b_2 & b_4 & \cdots & b_{2k-2} \\ 0 & b_1 & b_3 & \cdots & b_{2k-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_k \end{vmatrix} > 0$$

where $b_j = 0$ for $j > n$. If we were to apply the above condition to yield stability criteria for a polynomial, we will be considering inequalities involving a large number of terms. For instance, for a quartic polynomial $g(w) = z^4 + b_1 z^3 + b_2 z^2 + b_3 z + b_4$, the case $k = 4$ will lead us to consider nonlinear inequality of the form

$$10 + 5b_1 - 6b_2 - 9b_3 + b_1^2 - b_1 b_2 + 9b_1 b_4 + b_2 b_3 + 6b_2 b_4 - b_3^2 - 5b_3 b_4 - 10b_4^2 > 0.$$

It certainly is not easy, if not impossible, to extract good information about the stability regions of our original quartic polynomial, not to mention that we have not even incoporated the transformation (1) to our problem yet.

In this paper, we will find the stability region for an arbitrary real quartic polynomial of the form

$$\Phi(\lambda\,|a, b, c, d) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d, \;\; a, b, c, d \in R.$$

For the sake of convenience, a root λ of a polynomial is said to be subnormal, normal or supernormal if $|\lambda| < 1$, $|\lambda| = 1$ or $|\lambda| > 1$ respectively. We will also let

$$\rho\,(a, b, c, d) = \max\{|\lambda| : \Phi(\lambda\,|a, b, c, d) = 0\}$$

and

$$\Omega = \{(a, b, c, d) \in R^4 : \rho\,(a, b, c, d) < 1\}.$$

The set Ω is called the stability region for the polynomial Φ. A point in Ω is also called a point of stability.

We will prove the following result.

Theorem 1. *All roots of* Φ *are subnormal if, and only if,*

$$|d| < 1,\ |a| < d + 3,$$

$$a + b + c + d > -1,\ -a + b - c + d > -1,$$

and

$$(1 - d)^2 b < -c^2 + a(1 + d)c + (1 + d)(1 - d)^2 - a^2 d.$$

We remark that when a and d are fixed numbers, the equation

$$b = \frac{1}{(1 - d)^2}\left\{-c^2 + a(1 + d)c + (1 + d)(1 - d)^2 - a^2 d\right\}$$

defines a parabola in the c, b-plane. Therefore, the conditions in Theorem 1 yield a geometrical region in R^4. Such a set of geometrical conditions is quite different from the recursive algebraic conditions in [7].

2 Proof

First of all, let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the four roots of $\Phi(\lambda\,|a, b, c, d)$. Then $a = -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$ and $d = \lambda_1\lambda_2\lambda_3\lambda_4$. If $|a| \geq 4$ or $|d| \geq 1$, then clearly at least one of $\lambda_1, \lambda_2, \lambda_3$ or λ_4 must be normal or supernormal.

Lemma 1. *The region of stability* Ω *is contained in the set*

$$\Psi = \{(a, b, c, d) \in R^4 : |a| < 4, |d| < 1\}.$$

Since the function ρ is continuous, the boundary of Ω is contained in the set of points (a, b, c, d) such that $\Phi(\lambda\,|a, b, c, d)$ has a normal root. This prompts us to consider

$$\Phi(1\,|a, b, c, d) = 1 + a + b + c + d = 0, \tag{2}$$

$$\Phi(-1\,|a, b, c, d) = 1 - a + b - c + d = 0, \tag{3}$$

and

$$\Phi(e^{\pm i\theta}\,|a, b, c, d) = 0,\ \theta \in (0, \pi). \tag{4}$$

The condition (4) can be rewritten as

$$\begin{aligned} &\cos 4\theta + a\cos 3\theta + b\cos 2\theta + c\cos\theta + d = 0,\\ &\sin 4\theta + a\sin 3\theta + b\sin 2\theta + c\sin\theta = 0, \end{aligned} \tag{5}$$

for $\theta \in (0, \pi)$. By well known trigonometric identities, we may further write

$$8\cos^4\theta + 4a\cos^3\theta + (2b-8)\cos^2\theta + (c-3a)\cos\theta + d - b + 1 = 0,$$
$$\{8\cos^3\theta + 4a\cos^2\theta + (2b-4)\cos\theta + c - a\}\sin\theta = 0,$$

and

$$8\cos^4\theta + 4a\cos^3\theta + (2b-8)\cos^2\theta + (c-3a)\cos\theta + d - b + 1 = 0,$$
$$8\cos^3\theta + 4a\cos^2\theta + (2b-4)\cos\theta + c - a = 0.$$

As a consequence,

$$4\cos^2\theta + 2a\cos\theta + b - d - 1 = 0 \tag{6}$$

and

$$(2d-2)\cos\theta + (c-a) = 0. \tag{7}$$

Under the condition $|d| < 1$, (7) can be written as

$$\cos\theta = \frac{c-a}{2(1-d)}. \tag{8}$$

Under the condition $\theta \in (0, \pi)$, $|\cos\theta| < 1$ so that

$$|c-a| < 2\,|1-d|\,. \tag{9}$$

By (6), (8) and (9),

$$(1-d)^2 b = -c^2 + a(1+d)c + (1+d)(1-d)^2 - a^2 d. \tag{10}$$

The equation defined by (2) separates R^4 into two parts:

$$\{(a,b,c,d) \in R^4 : a+b+c+d > -1\}$$

and

$$\{(a,b,c,d) \in R^4 : a+b+c+d < -1\}.$$

We assert that $\Omega \subseteq \{(a,b,c,d) \in R^4\,|a+b+c+d > -1\}$. To see this, note that

$$\lim_{\lambda\in R, \lambda\to\infty} \Phi(\lambda\,|a,b,c,d) = +\infty$$

and

$$\lim_{\lambda\in R, \lambda\to -\infty} \Phi(\lambda\,|a,b,c,d) = +\infty.$$

If $a+b+c+d+1 \le 0$, then $\Phi(1\,|a,b,c,d) \le 0$. Thus there exists a real root $\lambda^* \ge 1$ such that $\Phi(\lambda^*\,|a,b,c,d) = 0$. This is contrary to the definition of Ω.

Similarly, we can show that $\Omega \subseteq \{(a,b,c,d) \in R^4\,|-a+b-c+d > -1\}$.

We summarize these as follows.

Lemma 2. *Under the condition $|d| < 1$, if $\Phi(\lambda|a, b, c, d)$ has a normal root, then (a, b, c, d) satisfies*

$$1 + a + b + c + d = 0, \tag{11}$$

or,

$$1 - a + b - c + d = 0, \tag{12}$$

or,

$$(1-d)^2 b = -c^2 + a(1+d)c + (1+d)(1-d)^2 - a^2 d \tag{13}$$

and

$$a - 2(1-d) < c < a + 2(1-d). \tag{14}$$

Furthermore, the region of stability Ω is contained in the set

$$\Gamma = \Big\{(a, b, c, d) \in R^4 : |a| < 4, |d| < 1, a + b + c + d > -1, \\ -a + b - c + d > -1\Big\}. \tag{15}$$

In order to visualize the four dimensional region Ω, we will consider its level sets at each given pair $(a, d) \in R^2$. In view of Lemma 1, we may also restrict our attention to the set

$$\Theta = \{(a, d) : |a| < 4, |d| < 1\},$$

and the corresponding level set

$$\Omega_{da} = \{(c, b) \in R^2 \,|(a, b, c, d) \in \Omega\}$$

in the c, b-plane.

In view of Lemma 2, we will let

$$\Gamma_{da} = \{(c, b) \in R^2 : a + b + c + d > -1, -a + b - c + d > -1\}$$

be the level set of Γ corresponding to $(a, d) \in \Theta$. Note that under the condition $(a, d) \in \Theta$, the relation (13) defines a parabola in the c, b-plane which can be described by the function $b = f(c|a, d)$ defined by

$$f(c|a, d) = \frac{-c^2 + a(1+d)c + (1+d)(1-d)^2 - a^2 d}{(1-d)^2},$$

and the relation (14) further restricts its domain of definition.

Similarly, the relations (11) and (12) define two straight lines which can respectively be described by the functions $b = L_*(c|a, d)$ and $b = L^*(c|a, d)$:

$$\begin{aligned} L_*(c|a, d) &= -a - c - d - 1, \\ L^*(c|a, d) &= c + a - d - 1. \end{aligned}$$

We will need the points of intersection of the parabola (13) and the straight lines L_* and L^*. First, we consider the function

$$\begin{aligned} g^*(c|a, d) &= f(c|a, d) - L^*(c|a, d) \\ &= \frac{-c^2 + a(1+d)c + (1+d)(1-d)^2 - a^2 d}{(1-d)^2} - a - c + d + 1, \end{aligned}$$

which describes a parabola in the c, b-plane with roots

$$c_1 = ad + 1 - d^2 \text{ and } c_2 = a - 2 + 2d.$$

Hence the points of intersection of the parabola f and the straight line L^* are

$$\begin{aligned} (c_1, b_1) &= (ad + 1 - d^2, ad - d^2 + a - d), \\ (c_2, b_2) &= (a - 2 + 2d, 2a + d - 3). \end{aligned}$$

Similarly, we consider the function

$$\begin{aligned} g_*(c|a, d) &= f(c|a, d) - L_*(c|a, d) \\ &= \frac{-c^2 + a(1+d)c + (1+d)(1-d)^2 - a^2 d}{(1-d)^2} + a + c + d + 1, \end{aligned}$$

which describes a parabola in the c, b-plane with roots

$$c_3 = ad - 1 + d^2 \text{ and } c_4 = a + 2 - 2d.$$

Hence the points of intersection of the parabola f and the straight line L_* are

$$\begin{aligned} (c_3, b_3) &= (ad - 1 + d^2, -ad - d^2 - a - d), \\ (c_4, b_4) &= (a + 2 - 2d, -2a + d - 3). \end{aligned}$$

Further, the point of intersection of the lines L_* and L^* is

$$(c_5, b_5) = (-a, -1 - d).$$

To proceed further, it is necessary to divide Θ into five mutually disjoint parts Θ_1, Θ_2, Θ_3, Θ_4 and Θ_5 (see Figure 1). They are formed by intersections of Θ with the half planes defined by $a - d - 3 \geq 0$, $a + d + 3 \leq 0$, $a - d + 1 > 0$ and $a + d - 1 < 0$, $a + d - 1 \geq 0$ and $a - d - 3 < 0$, $a - d + 1 \leq 0$ and $a + d + 3 > 0$:

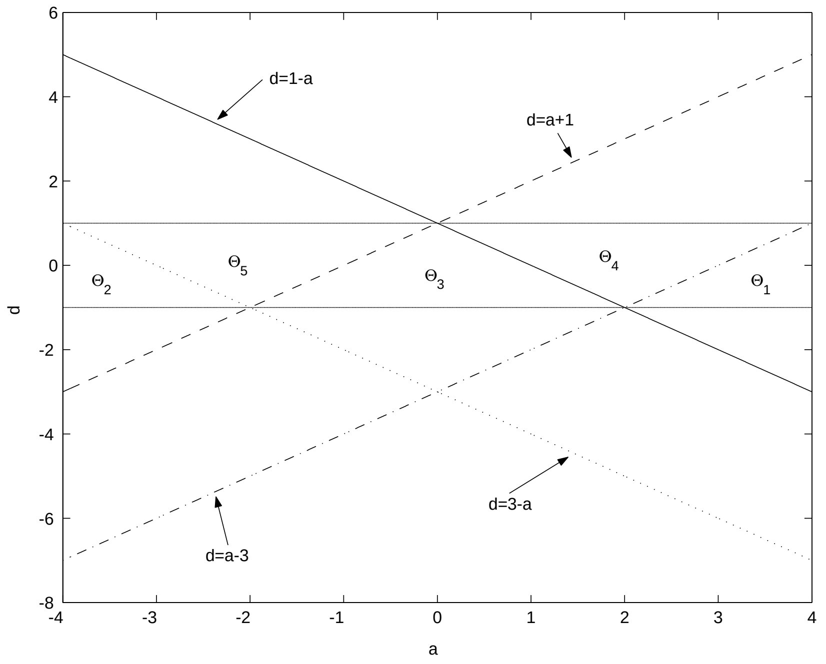

Figure 1

(i) $\Theta_1 : |a| < 4,\ |d| < 1$ and $a - d - 3 \geq 0$;

(ii) $\Theta_2 : |a| < 4,\ |d| < 1$ and $a + d + 3 \leq 0$;

(iii) $\Theta_3 : |a| < 4,\ |d| < 1$ and $|a| < 1 - d$;

(iv) $\Theta_4 : |a| < 4,\ |d| < 1,\ a + d - 1 \geq 0$ and $a - d - 3 < 0$; and

(v) $\Theta_5 : |a| < 4,\ |d| < 1,\ a - d + 1 \leq 0$ and $a + d + 3 > 0$.

Lemma 3. *Let $c = 0$ and $b = 7$. Then for any $(a, d) \in \Theta$, we have $(c, b) \in \Gamma_{da}$ and $\rho(a, b, c, d) > 1$.*

Proof. Indeed, when $|a| < 4$ and $|d| < 1$,

$$a + b + c + d = a + d + 7 > -5 + 7 > 1$$

and

$$-a + b - c + d = d - a + 7 > -5 + 7 > 1.$$

Furthermore, let λ_1, λ_2, λ_3, λ_4 be the four roots of $\Phi(\lambda|a, b, c, d)$. Then

$$\begin{aligned} 7 &= b = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 \\ &\leq |\lambda_1\lambda_2| + |\lambda_1\lambda_3| + |\lambda_1\lambda_4| + |\lambda_2\lambda_3| + |\lambda_2\lambda_4| + |\lambda_3\lambda_4|, \end{aligned}$$

so that there exists some $|\lambda_i| > 1$. □

2.1 Case 1

Suppose $(a, d) \in \Theta_1$ defined by $|a| < 4$, $|d| < 1$ and $a - d - 3 \geq 0$. We assert that the graph of $b = f(c|a, d)$ with $c \in (a - 2(1 - d), a + 2(1 - d))$ and the level set Γ_{da} are disjoint.

To see the proof, note that the points of intersection of the parabola (13) and the straight lines L_* and L^* have been found as (c_i, b_i), $1 \leq i \leq 5$. They can be ordered by their first coordinates. Indeed, since $|a| < 4$, $|d| < 1$ and $a - d - 3 \geq 0$,

$$\begin{aligned} &c_4 - c_2 = 4 - 4d > 0, \\ &c_1 - c_3 = 2 - 2d^2 > 0, \\ &c_2 - c_1 = a - 2 + 2d - ad - 1 + d^2 = (1 - d)(a - d - 3) \geq 0, \\ &c_3 - c_5 = ad + d^2 - 1 + a = (1 + d)(a + d - 1) > 0, \end{aligned} \tag{16}$$

so that we have the relation

$$c_5 < c_3 < c_1 \leq c_2 < c_4.$$

We now only need to show that the graph of the function f (with domain defined by (14)) lies below the region Γ_{da} (see Figure 2). To see this, it suffices to show that the domain of f is between c_2 and c_4 and $f(c|a, d) < L^*(c|a, d)$. Indeed, this follows from

$$c_2 = a - 2(1 - d) < c < a + 2(1 - d) = c_4$$

and

$$L^*(c|a, d) - f(c|a, d) = -g^*(c|a, d)$$

because $-g^*(c|a, d)$ is a parabola and $-g^*(c_1|a, d) = -g^*(c_2|a, d) = 0$, $-g^*(c|a, d) > 0$, for $c_2 < c < c_4$.

We now show that for each (a, b, c, d) such that $(a, d) \in \Theta_1$ and $(c, b) \in \Gamma_{da}$, $\rho(a, b, c, d) > 1$. To see this, note that in view of Lemma 3, the point $(a', b', c', d') = (a', 6, 0, d')$ with $(a', d') \in \Theta_1$ satisfies $(c', b') \in \Gamma_{da}$ and

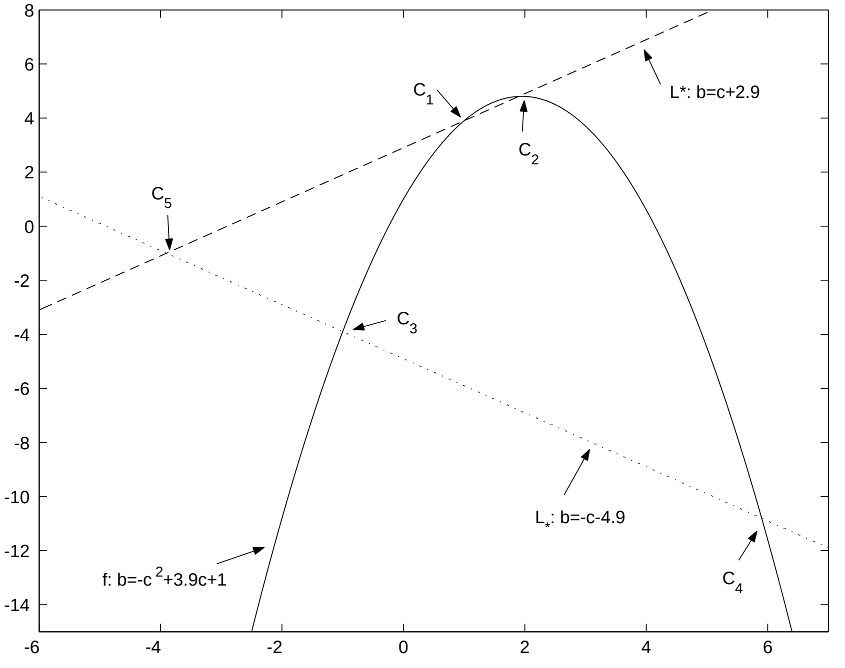

Figure 2: $a = 3.9$, $d = 0$.

$\rho(a', b', c', d') > 1$. By the continuity of ρ and the obvious fact that $\Gamma_{d'a'}$ is path-wise connected, the contrary conclusion would imply there exists (a', b'', c'', d') such that $(c'', b'') \in \Gamma_{da}$ and $\Phi(\lambda \left| a', b'', c'', d' \right)$ has a normal root. But then by Lemma 2, $b'' = L_*(c''|a', d')$, or $b'' = L^*(c''|a', d')$, or $b'' = f(c''|a', d')$ and $\left|c'' - a'\right| < 2\left|1 - d'\right|$, which cannot be true since we have assumed that $(a', d') \in \Theta_1$ and $(c'', b'') \in \Gamma_{d'a'}$.

We may now assert that Ω_{da} is empty, for otherwise, Ω_{da} is contained in Γ_{da} by Lemma 2, which is contrary to what we have just shown.

2.2 Case 2

Suppose $(a, d) \in \Theta_2$ defined by $|a| < 4$, $|d| < 1$ and $a + d + 3 \leq 0$. Since Θ_1 and Θ_2 in the a, d-plane are symmetric with respect to the d axis, we may follow the arguments in Case 1 closely and show that the points (c_i, b_i), $i = 1, ..., 5$, can be ordered by

$$c_2 < c_4 \leq c_3 < c_1 < c_5.$$

Then we may show that the function $b = f(c|a, d)$ with $c \in (a - 2(1 - d), a + 2(1-d))$ lies entirely below the line L_* (see Figure 3). Then for each (a, b, c, d) such that $(a, d) \in \Theta_2$ and $(c, b) \in \Gamma_{da}$, we may show by continuity of ρ and the pathwise connectedness of Γ_{da} that $\rho(a, b, c, d) > 1$. In view of Lemma 2, Ω_{da} is empty.

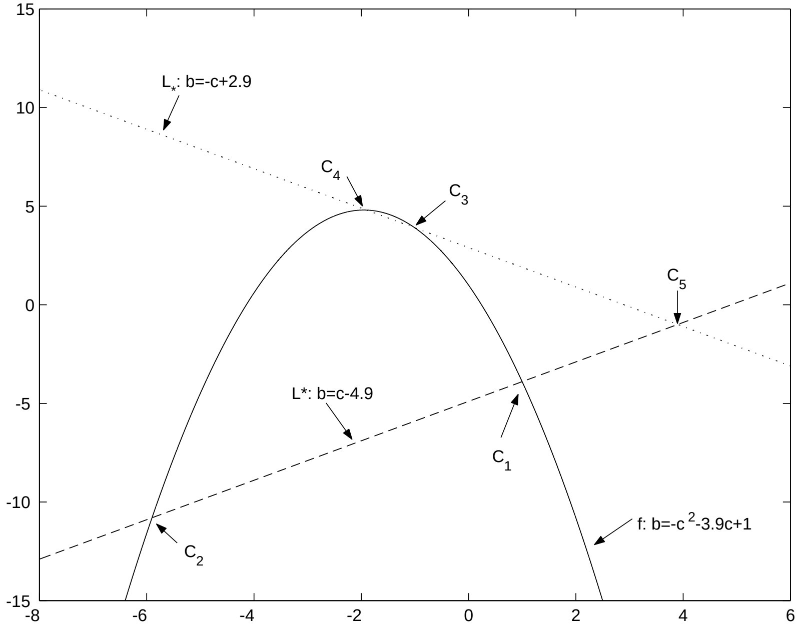

Figure 3: $a = -3.9$, $d = 0$.

2.3 Case 3

Suppose $(a, d) \in \Theta_3$ defined by $|a| < 4$, $|d| < 1$ and $|a| < 1 - d$. We first assert that the graph G_f of $b = f(c|a, d)$ with $c \in (a - 2(1 - d), a + 2(1 - d))$ separates the level set Γ_{da} into three parts

$$D_{da} = \{(c, b) \in \Gamma_{da} : b < f(c|a, d)\}, \tag{17}$$

$$E_{da} = \{(c, b) \in \Gamma_{da} : b > f(c|a, d)\} \tag{18}$$

and

$$F_{da} = \{(c, b) \in \Gamma_{da} : b = f(c|a, d)\}. \tag{19}$$

To verify our assertion, consider the points of intersections (c_i, b_i), $i = 1, \ldots, 5$. Since

$$\begin{aligned}
&c_4 - c_1 = a + 2 - 2d - ad - 1 + d^2 = (1-d)(a-d+1) > 0,\\
&c_1 - c_5 = a - d^2 + 1 + ad = (1+d)(a-d+1) > 0,\\
&c_5 - c_3 = -a - ad + 1 - d^2 = (1+d)(-a-d+1) > 0,\\
&c_3 - c_2 = ad + d^2 - 1 - a + 2 - 2d = (d-1)(a+d-1) > 0,
\end{aligned}$$

we have the relation

$$c_2 < c_3 < c_5 < c_1 < c_4.$$

Then since

$$\begin{aligned}
f(c|a,d) - L_*(c|a,d) &= g_*(c|a,d),\\
f(c|a,d) - L^*(c|a,d) &= g^*(c|a,d)
\end{aligned}$$

and $g_*(c|a,d)$ is a parabola in the c, b-plane, and $g_*(c_3|a,d) = g_*(c_4|a,d) = 0$, $g_*(c|a,d) > 0$, for $c \in (c_3, c_1)$, we see that the graph G_f lies above the line L_*. Similarly, we may show that the graph G_f lies above the line L^*, the graph of $b = f(c|a,d)$ for $c \in (c_3, c_1)$ lies inside the region Γ_{da} (see Figure 4).

Now that we have shown the graph G_f separates Γ_{da} into three parts D_{da}, E_{da} and F_{da}. We assert further that $D_{da} = \Omega_{da}$, that is, the region D_{da} is the desired stability region. It suffices to show that for each point (c, b) in D_{da}, $\rho(a,b,c,d) < 1$ and for each point (c,b) in $E_{da} \cup F_{da}$, $\rho(a,b,c,d) \geq 1$.

To see that the former statement holds, we take an arbitrary point (a,d) in Θ_3, and take

$$b' = -d^2 - d,\ c' = ad.$$

Then $(c', b') \in D_{da}$ for

$$\begin{aligned}
a - d^2 - d + ad + d &= a(1+d) - d^2 > d^2 - 1 - d^2 > -1,\\
-a - d^2 - d - ad + d &= -a(1+d) - d^2 > d^2 - 1 - d^2 > -1,
\end{aligned}$$

which imply $(c', b') \in \Gamma_{da}$, and

$$\frac{-a^2d^2 + a^2d(1+d) + (1+d)(1-d)^2 - (a)^2(d)}{(1-d)^2} = 1 + d > -d^2 - d,$$

which implies $b' < f(c'|a,d)$. Furthermore, the corresponding quartic polynomial is

$$\Phi(\lambda|a,b',c',d) = \lambda^4 + a\lambda^3 + (-d^2-d)\lambda^2 + ad\lambda + d.$$

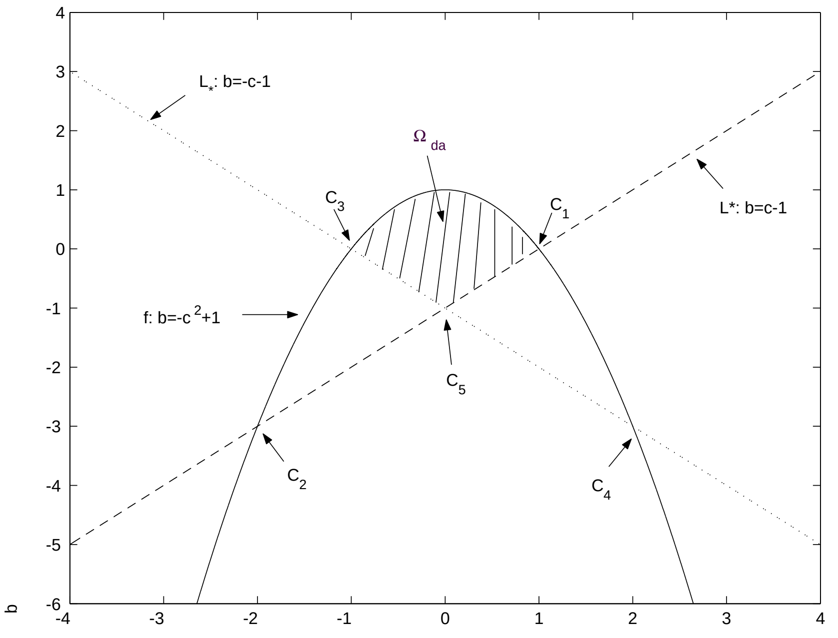

Figure 4: $a = 0, d = 0$.

If we take [7, p.203]

$$\hat{\Phi}(\lambda|a,b,c,d) = -[1-d^2][\lambda^4 + a\lambda^3 - d\lambda^2] = -[1-d^2]\lambda^2[\lambda^2 + a\lambda - d],$$

then we have

$$\begin{aligned}
& \left|\Phi(e^{i\theta}|a,b,c,d) + \hat{\Phi}(e^{i\theta}|a,b,c,d)\right| \\
= & \left|d^2 e^{i4\theta} + ad^2 e^{i3\theta} + (-d^2 - d^3)e^{i2\theta} + ade^{i\theta} + d\right| \\
= & |d| \left|e^{i4\theta}\right| \left|d + ade^{-i\theta} + (-d^2 - d)e^{-2i\theta} + ae^{-3i\theta} + e^{-4i\theta}\right| \\
= & |d| \left|d + ade^{i\theta} + (-d^2 - d)e^{2i\theta} + ae^{3i\theta} + e^{4i\theta}\right| \\
= & |d| \left|\Phi(e^{i\theta}|a,b,c,d)\right| \\
< & \left|\Phi(e^{i\theta}|a,b,c,d)\right|
\end{aligned}$$

for all $\theta \in [0, 2\pi)$. By Rouche's theorem $\Phi(\lambda|a,b',c',d)$ and $\hat{\Phi}(\lambda|a,b,c,d)$ have an equal number of zeros inside the unit circle $\{\lambda : |\lambda| = 1\}$ in the plane. In view of $|d| < 1$, $1 + a - d > 0$ and $1 - a - d > 0$, the very first statement in the introduction implies $\hat{\Phi}(\lambda|a,b,c,d)$ have four zeros inside the unit circle

$\{\lambda : |\lambda| = 1\}$. Thus $\Phi(\lambda|a, b', c', d)$ have four zeros inside $\{\lambda : |\lambda| = 1\}$. So $\rho(a, b', c', d) < 1$, in other words, $(c', b') \in \Omega_{da}$.

For any point $(c, b) \in D_{da}$, we assert that $\rho(a, b, c, d) < 1$. Suppose not, there would exist $(c'', b'') \in D_{da}$ such that $\rho(a, b'', c'', d) \geq 1$. But in view of the continuity of ρ and the obvious fact that D_{da} is pathwise connected, there exists $(c_0, b_0) \in D_{da}$ such that $\Phi(\lambda|a', b_0, c_0, d')$ has a normal root, but this is contrary to Lemma 2.

Next, we show that for each point (c, b) in E_{da}, $\rho(a, b, c, d) > 1$. Indeed we take an arbitrary point $(a, d) \in \Theta_3$ and $(c', b') = (0, 7)$. Then by Lemma 3, $\rho(a, b', c', d) > 1$. If there exists $(c, b) \in E_{da}$ such that $\rho(a, b, c, d) < 1$, then by the continuity of ρ and the obvious fact that $E_{d'a'}$ is pathwise connected, there would exist $(c_0, b_0) \in E_{da}$ such that $\Phi(\lambda|a, b_0, c_0, d)$ has a normal root, but this is contrary to Lemma 2.

Finally, we show that for each point (c, b) in F_{da}, $\rho(a, b, c, d) \geq 1$. Indeed, if not, by continuity of ρ and the pathwise connectedness of $E_{da} \cup F_{da}$, there would exist a point (c_0, b_0) in F_{da} such that $\rho(a, b_0, c_0, d) = 1$, which is contrary to Lemma 2.

2.4 Case 4

Suppose $(a, d) \in \Theta_4$defined by $|a| < 4$, $|d| < 1$, $a + d - 1 \geq 0$ and $a - d - 3 < 0$. We assert that the graph G_f of the function $b = f(c|a, d)$ with $c \in (a - 2(1 - d), a + 2(1 - d))$ separates Γ_{da} into three parts

$$G_{da} = \{(c, b) \in \Gamma_{da} : b < f(c|a, d)\} \tag{20}$$

and

$$H_{da} = \{(c, b) \in \Gamma_{da} : b > f(c|a, d)\}. \tag{21}$$

and

$$I_{da} = \{(c, b) \in \Gamma_{da} : b = f(c|a, d)\}. \tag{22}$$

To verify our assertion, consider the points of intersections (c_i, b_i), $i = 1, ..., 5$. Since

$$\begin{aligned}
c_4 - c_1 &= a + 2 - 2d - ad - 1 + d^2 = (1 - d)(a - d + 1) > 0,\\
c_1 - c_2 &= ad - d^2 + 1 - a + 2 - 2d = (d - 1)(a - d - 3) > 0,\\
c_2 - c_3 &= -ad - d^2 + 1 + a - 2 + 2d = (1 - d)(a + d - 1) \geq 0,\\
c_3 - c_5 &= ad - 1 + d^2 + a = (1 + d)(a + d - 1) \geq 0,
\end{aligned}$$

we have the relation

$$c_5 < c_3 < c_2 < c_1 < c_4,$$

or

$$c_5 = c_3 = c_2 < c_1 < c_4.$$

Then since

$$f(c|a,d) - L^*(c|a,d) = g^*(c|a,d)$$

and $g^*(c|a,d)$ is a parabola in the c, b-plane, and $g^*(c_1|a,d) = g^*(c_2|a,d) = 0$, $g^*(c|a,d) > 0$, for $c \in (c_2, c_1)$, we see that the graph G_f lies above line L^*. This shows that the graph G_f separates Γ_{da} into three parts G_{da}, H_{da} and I_{da} (see Figure 5).

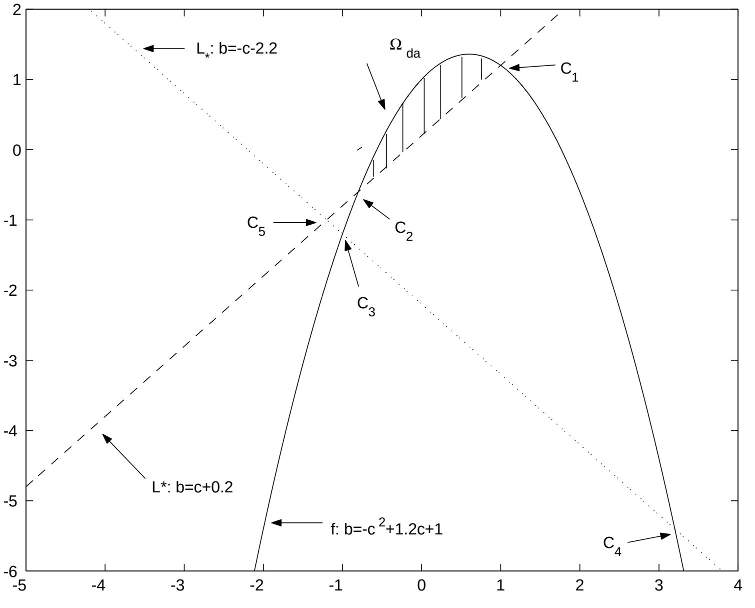

Figure 5: $a = 1.2, d = 0$.

Now that we have shown G_f separates Γ_{da} into three parts G_{da}, H_{da}, and I_{da}. We assert further that $G_{da} = \Omega_{da}$, that is, the region G_{da} is the desired stability region. It suffices to show that for each point (c, b) in G_{da}, $\rho(a, b, c, d) < 1$ and for each point (c, b) in $H_{da} \cup I_{da}$, $\rho(a, b, c, d) \geq 1$.

To see that the former statement holds, we take an arbitrary point $(a, d) \in \Theta_4$, and take

$$c' = \frac{c_1 + c_2}{2} = \frac{a(1+d) - (1-d)^2}{2},$$

$$b' = \frac{L^*(c'|a,d) + f(c'|a,d)}{2} = \frac{a^2 - 3(1-d)^2 + 2a(1+d) + 4a}{8},$$

Then $(b, c) \in G_{da}$ for

$$\begin{aligned} & a+\frac{a^2-3(1-d)^2+2a(1+d)+4a}{8}+\frac{a(1+d)-(1-d)^2}{2}+d \\ = & \frac{a^2-7(1-d)^2+6a(1+d)+12a+8d}{8} \\ \geq & \frac{(1-d)^2-7(1-d)^2+6(1-d)(1+d)+12(1-d)+8d}{8} \\ = & \frac{-12d^2+8d+12}{8} \\ = & \frac{-3d^2+2d+3}{2} > -1, \end{aligned}$$

and

$$\begin{aligned} & -a+\frac{a^2-3(1-d)^2+2a(1+d)+4a}{8}-\frac{a(1+d)-(1-d)^2}{2}+d \\ = & \frac{a^2+(1-d)^2-2a(1+d)-4a+8d}{8} \\ = & \frac{a^2+(1-d)^2-2a(3+d)+8d}{8} \\ > & \frac{(3+d)^2+(1-d)^2-2(3+d)^2+8d}{8} = -1, \end{aligned}$$

imply $(c', b') \in \Gamma_{da}$, and

$$b' = \frac{L^*(c'|a,d)+f(c'|a,d)}{2} < \frac{f(c'|a,d)+f(c'|a,d)}{2} = f(c'|a,d)$$

implies $b' < f(c'|a,d)$. Let [7, p.203]

$$\hat{\Phi}(\lambda|a,b',c',d) = -(1-d^2)\lambda^4-(a-dc')\lambda^3-b'(1-d)\lambda^2-(c'-ad)\lambda.$$

Since

$$\begin{aligned} & \left|\Phi(e^{i\theta}|a,b',c',d)+\hat{\Phi}(e^{i\theta}|a,b',c',d)\right| \\ = & \left|d^2e^{4i\theta}+dc'e^{3i\theta}+b'de^{2i\theta}+ade^{i\theta}+d\right| \\ = & |d|\left|e^{4i\theta}\right|\left|d+c'e^{-i\theta}+b'e^{-2i\theta}+ae^{-3i\theta}+e^{-4i\theta}\right| \\ = & |d|\left|d+c'e^{i\theta}+b'e^{2i\theta}+ae^{3i\theta}+e^{4i\theta}\right| \\ = & |d|\left|\Phi(e^{i\theta}|a,b',c',d)\right| \\ < & \left|\Phi(e^{i\theta}|a,b',c',d)\right| \end{aligned}$$

for $\theta \in [0, 2\pi)$, by Rouche's theorem, $\Phi(\lambda|a, b', c', d)$ and $\hat{\Phi}(\lambda|a, b', c', d)$ have an equal number of zeros inside the unit circle $\{\lambda : |\lambda| = 1\}$ in the plane. So we only need to consider

$$\begin{aligned}\hat{\Phi}(\lambda|a, b', c', d) &= -(1-d^2)\lambda^4 - (a - dc')\lambda^3 - b'(1-d)\lambda^2 - (c' - ad)\lambda \\ &= -(1-d^2)\lambda\left\{\lambda^3 + \frac{(a-dc')}{1-d^2}\lambda^2 + \frac{b'(1-d)}{1-d^2}\lambda + \frac{(c'-ad)}{1-d^2}\right\}.\end{aligned}$$

Note that

$$\begin{aligned}a^* &= -\frac{(a-dc')}{1-d^2} = -\frac{a(2+d)+d(1-d)}{2(1+d)} \\ b^* &= -\frac{b'(1-d)}{1-d^2} = -\frac{a^2+2a(d+3)-3(1-d)^2}{8(1+d)} \\ c^* &= -\frac{(c'-ad)}{1-d^2} = -\frac{a+d-1}{2(1+d)}\end{aligned}$$

satisfy

$$\begin{aligned}-1+d \geq a^* &> -3, \\ 0 \geq c^* &> -1, \\ a^* + b^* + c^* &= -\frac{a^2+6a(d+3)-7(1-d)^2}{8(1+d)} \leq -\frac{3(1-d)}{2} < 1, \\ -a^* + b^* - c^* &= -\frac{a^2-2a(d+3)+(1-d)^2}{8(1+d)} < 1, \\ -(c^*)^2 + a^*c^* + b^* &= \frac{[a(2+d)+d(1-d)][a+d-1]-[a+d-1]^2}{4(1+d)^2} + b^* \\ &= \frac{[a+d-1][a+1-d]}{4(1+d)} + b^* \\ &= \frac{a^2-2a(d+3)+(1-d)^2}{8(1+d)} > -1,\end{aligned}$$

by the result mentioned above for cubic polynomials, we see that $\hat{\Phi}(\lambda|a, b', c', d)$ have four zeros inside the unit circle $\{\lambda : |\lambda| = 1\}$ in the plane. Thus $\Phi(\lambda|a, b', c', d)$ have four zeros inside the unit circle $\{\lambda : |\lambda| = 1\}$ in the plane. So $\rho(a, b', c', d) < 1$, in other words, $(c', b') \in \Omega_{da}$.

As in the proof of the previous Case 3, we may now show that for any point $(c, b) \in G_{da}$, $\rho(a, b, c, d) < 1$. We may also show that for any point $(c, b) \in H_{da} \cup I_{da}$, $\rho(a, b, c, d) \geq 1$. The proof of our assertion is complete.

2.5 Case 5

Suppose $(a, d) \in \Theta_5$ defined by $|a| < 4$, $|d| < 1$, $a-d+1 \leq 0$ and $a+d+3 > 0$. Since Θ_4 and Θ_5 in the a, d-plane are symmetric with respect to the d axis, we may follow the arguments in Case 4 closely and show that the points (c_i, b_i), $i = 1, ..., 5$, can be ordered by

$$c_2 < c_3 < c_4 < c_1 < c_5 \quad \text{or} \quad c_2 < c_3 < c_4 = c_1 = c_5$$

Then we may show that the function $b = f(c|a, d)$ with $c \in (c_3, c_4)$ lies entirely above the line L_* and separates Γ_{da} into three parts

$$J_{da} = \{(c, b) \in \Gamma_{da} : b < f(c|a, d)\} \tag{23}$$

and

$$K_{da} = \{(c, b) \in \Gamma_{da} : b > f(c|a, d)\} \tag{24}$$

and

$$L_{da} = \{(c, b) \in \Gamma_{da} : b = f(c|a, d)\} \tag{25}$$

(see Figure 6). Then we may show that $J_{da} = \Omega_{da}$ as in the Case 4.

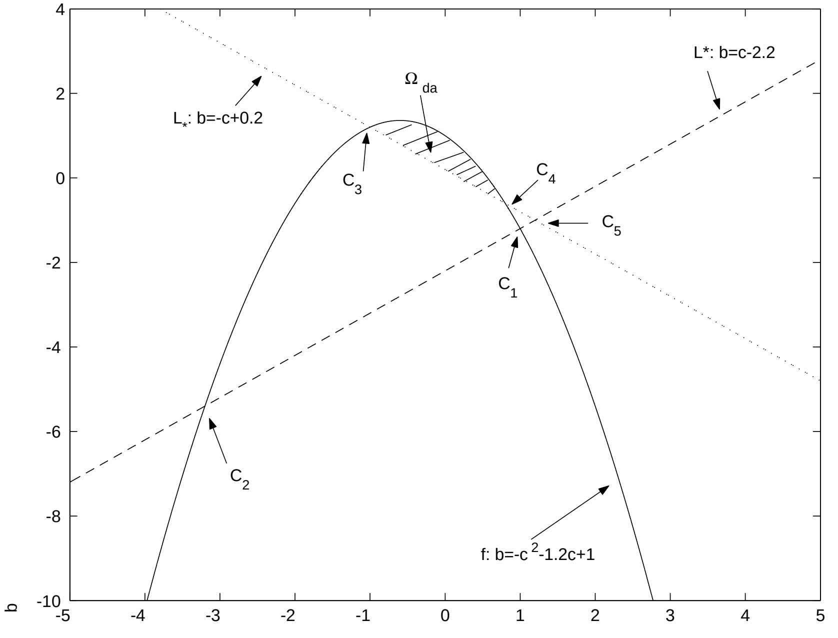

Figure 6: $a = -1.2$, $d = 0$.

3 Second and Third Order Polynomials

As a check for Theorem 1, consider second order polynomials of the form

$$Q(\lambda) = \lambda^2 + a\lambda + b, \; a, b \in R.$$

Theorem 1 asserts that all roots of Q are subnormal if, and only if,

$$|a| < 3, a + b > -1, -a + b > -1, b < 1,$$

or equivalently,

$$a + b > -1, -a + b > -1, b < 1.$$

This is exactly the same result stated at the very beginning.

As another check, consider third order polynomials of the form

$$Q(\lambda|a, b, c) = \lambda^3 + a\lambda^2 + b\lambda + c, \; a, b, c \in R.$$

Then Theorem 1 asserts that all roots of Q are subnormal if, and only if,

$$|a| < 3, a + b + c > -1, -a + b - c > -1, b < -c^2 + ac + 1. \tag{26}$$

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