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Quadrature formulas for the generalized Riemann-Stieltjes integral

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Abstract. In this paper we propose a technique of approximation for the generalized Riemann-Stieltjes integral and we found an analogue for Newton-Cotes formulas in the case $n = 2$ and $n = 3$.

Keywords: Riemann Stieltjes integral, Weyl derivative, quadrature formula.

Mathematical subject classification: 65D32.

1 Introduction

The study of nonlinear and integral equations leads to consider the fixed point problem

$$u = Tu \tag{1}$$

with T a completely continuous operator acting on a Hilbert space H. I. Moret & P. Omari in [3] were concerned with the numerical solution of (1) by iterative techniques based on linearization. These procedures consist of approximating (1) by linear equations of the form

$$(I - A_m)(u - u_m) = -u_m + Tu_m \tag{2}$$

where u_m is the current iterative and A_m is a suitable linear model of T at u_m. The next iterate u_{m+1} is defined as the unique solution of (2). In the financial mathematics, for examples, there are studied the equations of the form

$$X(t,\omega) = X_0(\omega) + \int_0^t \sigma(s, X(s,\omega))dB(s,\omega) + \int_0^t b(s, X(s,\omega))ds \tag{3}$$

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where $\int_0^t \sigma(s, X(s,\omega))dB(s,\omega)$ is the stochastic integral which can be defined in various manner depending on the exponent of the Brownian motion and $\int_0^t b(s, X(s,\omega))ds$ is the classical integral. If the exponent of the Brownian motion is bigger than $\frac{1}{2}$ (situation requested by practical necessities) the stochastic integral can be defined by trajectory, i.e. for all ω fixed, we have

$$I = \int_0^t \sigma(s, X(s))dB(s). \tag{4}$$

In this situation, the Brownian motion is a process β-Hölder continuous, with $\frac{1}{2} < \beta < 1$. In order to apply the algorithm described by Moret & Omari, given the iteration at the step m, we should be able to compute the right hand side of (3), more precisely to approximate the stochastic integral.

The hypothesis concerning the function σ are: σ is $\alpha-$ Hölder continuous with respect to the first argument and σ is Lipschitz with respect to the second argument, by consequence, the integral (4) is a Riemann-Stieltjes integral (RS) $\int_a^b f dg$ where f is $\alpha-$Hölder continuous and g is $\beta-$ Hölder continuous. D. Nualart & A. Răşcanu proved a global existence and uniqueness of the solutions for integral equations containing integrals of type (RS) (see for details [4]). In the following we use the definition of the Riemann-Stieltjes integral given by M. Zähle [5] and we propose a technique of the approximation. More precisely, we give the following results:

Theorem A. *Let f and g be two Hölder continuous functions on $[a, b]$, namely $|f(x) - f(y)| \leq M_f |x-y|^\alpha$ and $|g(x)-g(y)| \leq M_g|x-y|^\beta$ for all $x, y \in [a, b]$ where M_f, M_g are positive constants and $\alpha + \beta > 1$. Let us consider the nodes $a = x_1 < x_2 = b$ and denote $h = b - a$. Then*

$$\int_a^b f(x)dg(x) = \frac{1}{2}\left(g(x_2) - g(x_1)\right)\left(f(x_1) + f(x_1) + f(x_2)\right) + R_2 \tag{5}$$

where $|R_2| \leq cM_f M_g h^{\alpha+\beta}$ with c a positive constant.

Theorem B. *Consider as above f, g and the nodes $a = x_1 < x_2 < x_3 = b$ and denote $h = \frac{b-a}{2}$. We have*

$$\begin{aligned}\int_a^b f(x)dg(x) \;=\; & \frac{1}{6}\left\{[f(x_1) + 4f(x_2) + f(x_3)][g(x_3) - g(x_1)]\right. \\ & \left. -\; [g(x_1) + 4g(x_2) + g(x_3)][f(x_3) - f(x_1)]\right\} \\ & +\; \frac{1}{2}[g(x_3) + g(x_1)][f(x_3) - f(x_1)] + R_3\end{aligned} \tag{6}$$

where $|R_3| \leq cM_f M_g h^{\alpha+\beta}$ with c a positive constant.

2 Preliminaries

In this section we present the definition of Riemann Stieltjes integral by using Weyl derivatives and the method of approximation. Let $\gamma \in (0, 1)$ be an arbitrary number and denote

$$f_{a+}(x) = (f(x) - f(a+))1_{(a,b)}(x)\,, \quad g_{b-}(x) = (b(x) - g(b-))1_{(a,b)}(x)$$
$$(-1)^\gamma = e^{i\pi\gamma}$$

(throughout of this paper we denote $f(a+) = \lim_{\delta\downarrow 0} f(a+\delta)$, $g(b-) = \lim_{\delta\downarrow 0} g(b-\delta)$ supposing that the limits exist).

We define the integral by

$$\begin{aligned}\int_a^b f(x)dg(x) \;=& (-1)^\gamma \int_a^b D^\gamma_{a+} f_{a+}(x) D^{1-\gamma}_{b-} g_{b-}(x)dx \\ & +\; f(a+)(g(b-) - g(a+))\end{aligned} \tag{7}$$

where

$$D^\gamma_{a+} f(x) = \frac{1}{\Gamma(1-\gamma)}\left(\frac{f(x)}{(x-a)^\gamma} + \gamma\int_a^x \frac{f(x) - f(y)}{(x-y)^{\gamma+1}}\,dy\right)1_{(a,b)}(x)$$

and

$$D^\gamma_{b-} f(x) = \frac{(-1)^\gamma}{\Gamma(1-\gamma)}\left(\frac{f(x)}{(b-x)^\gamma} + \gamma\int_x^b \frac{f(x) - f(y)}{(y-x)^{\gamma+1}}\,dy\right)1_{(a,b)}(x)$$

are Weyl representations for the fractional derivatives in the sense of Riemann and Liouville (e.g. [5]). Here $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$ is Euler's function. Remark that the definition is good, namely it does not depend on the choice of the parameter $\gamma \in (0, 1)$. See [5] for a detailed analysis of integrals defined by (7). Let f and g be such that $|f(x) - f(y)| \leq M_f|x-y|^\alpha$ and $|g(x)-g(y)| \leq M_g|x-y|^\beta$, with $\alpha,\ \beta \in (1/2, 1)$. Consider the nodes $a \leq x_1 < x_2 < \cdots < x_n \leq b$. We approximate the functions f and g with the Lagrange interpolating polynomials (namely we have $f(x) = L_n f(x) + R_f(x)$ and $g(x) = L_n g(x) + R_g(x)$, respectively). By using additivity properties of the Riemann Stieltjes integral we get

$$
\begin{aligned}
\int_a^b f(x)dg(x) &= \int_a^b (L_n f(x) + R_f(x))d(L_n g(x) + R_g(x)) \\
&= \int_a^b L_n f(x)dL_n g(x) + \int_a^b R_f(x)dL_n g(x) \\
&\quad + \int_a^b L_n f(x)dR_g(x) + \int_a^b R_f(x)dR_g(x) \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

We denote by $w_n(x) = (x - x_1)\cdot\cdots\cdot(x - x_n)$. Recall that the Lagrange interpolating polynomial for the function f is

$$
L_n f(x) = \sum_{i=1}^n f(x_i)\frac{w_n(x)}{(x - x_i)w_n'(x_i)}\ ,
$$

and similarly for g. Consequently,

$$
\begin{aligned}
I_1 &= \int_a^b L_n f(x)dL_n g(x) = \int_a^b L_n f(x)L_n' g(x)dx \\
&= \sum_{i=1}^n\sum_{k=1}^n \frac{f(x_i)g(x_k)}{w_n'(x_i)w_n'(x_k)}\int_a^b \frac{w_n(x)}{x - x_i}\cdot\frac{w_n'(x)(x - x_k) - w_n(x)}{(x - x_k)^2}\,dx.
\end{aligned}
$$

Denoting by

$$
I_1^{ik} = \frac{1}{w_n'(x_i)w_n'(x_k)}\int_a^b \frac{w_n(x)}{x - x_i}\cdot\frac{w_n'(x)(x - x_k) - w_n(x)}{(x - x_k)^2}\,dx, \tag{8}
$$

we can approximate the integral by the sum

$$I_1 = \sum_{i=1}^{n}\sum_{k=1}^{n} I_1^{ik} f(x_i)g(x_k) \tag{9}$$

and the rest of the quadrature formula is $R_n = I_2 + I_3 + I_4$.

3 Quadrature formulas – proofs of theorems

In this section we give an analogue for Newton-Cotes formulas in the cases $n = 2$ and $n = 3$.

3.1 Case n=2 (trapezoidal method)

We have $x_1 = a$, $x_2 = b$, $w_2(x) = (x-a)(x-b)$ and $w_2'(x) = 2x - a - b$. The coefficients of the quadrature formula I_1^{ik}, $i, k \in \{1, 2\}$ are:

$$I_1^{11} = -\frac{1}{2}, \quad I_1^{12} = \frac{1}{2}, \quad I_1^{21} = -\frac{1}{2}, \quad I_1^{22} = \frac{1}{2}\,.$$

Consequently,

$$\begin{aligned} I_1 &= -\frac{1}{2} f(x_1)g(x_1) + \frac{1}{2} f(x_1)g(x_2) - \frac{1}{2} f(x_2)g(x_1) + \frac{1}{2} f(x_2)g(x_2) \\ &= \frac{1}{2}(g(x_2) - g(x_1))(f(x_1) + f(x_2)). \end{aligned}$$

In order to estimate the rest R_2 we will evaluate R_f and R_g (the rests of the Lagrange interpolation of the two functions). By taking into account that g is Hölder continuous, we have

$$\begin{aligned} \left|R_g(x)\right| &= |[x_1, x_2, x]_g (x - x_1)(x - x_2)| \\ &= \left|\left(\frac{g(x) - g(x_2)}{(x - x_2)(x - x_1)} - \frac{g(x_2) - g(x_1)}{(x_2 - x_1)(x - x_1)}\right)(x - x_1)(x - x_2)\right| \\ &\leq \left(\frac{M_g|x - x_2|^\beta}{(x_2 - x)(x - x_1)} + \frac{M_g|x_2 - x_1|^\beta}{(x_2 - x_1)(x - x_1)}\right)(x - x_1)(x - x_2) \\ &= M_g\,\left((x_2 - x)^\beta + (x_2 - x_1)^\beta\right) \leq 2M_g h^\beta \end{aligned}$$

In the same manner we obtain $|R_f(x)| \leq 2M_f h^\alpha$.

The integral I_2 and I_3 can be evaluated in the same way. We will write down only the estimation for I_2. We have

$$I_2 = \int_a^b R_f(x) dL_2 g(x) = \int_a^b R_f(x) L_2' g(x) dx$$

where

$$L_2 g(x) \;=\; g(x_1)\frac{x - x_2}{x_1 - x_2} \;+\; g(x_2)\frac{x - x_1}{x_2 - x_1}.$$

One gets

$$\begin{aligned} |I_2| \;&\leq\; \int_a^b |R_f(x) L_2' g(x)| dx \leq \int_a^b 2M_f h^\alpha \cdot \frac{1}{h} |g(x_2) - g(x_1)| dx \\ &=\; 2M_f h^\alpha |g(x_2) - g(x_1)| \leq 2M_f h^\alpha M_g |x_2 - x_1|^\beta = 2M_f M_g h^{\alpha+\beta}\;. \end{aligned}$$

Similarly $|I_3| \leq 2M_f M_g h^{\alpha+\beta}$.

The main point of this subsection is to estimate I_4. By using the definition of the Riemann Stieltjes integral one has:

$$\int_a^b R_f dR_g = (-1)^\gamma \int_a^b D_{a+}^\gamma R_{f_{a+}}(x) D_{b-}^{1-\gamma} R_{g_{b-}}(x) dx$$

where

$$\begin{aligned} D_{a+}^\gamma R_{f_{a+}}(x) \;&=\; \frac{1}{\Gamma(1-\gamma)} \left(\frac{R_{f_{a+}}(x)}{(x-a)^\gamma} + \gamma \int_a^x \frac{R_{f_{a+}}(x) - R_{f_{a+}}(y)}{(x-y)^{\gamma+1}}\, dy \right) \\ &:=\; \frac{1}{\Gamma(1-\gamma)}\, F \end{aligned}$$

and

$$\begin{aligned} D_{b-}^{1-\gamma} R_{g_{b-}}(x) \;&=\; \frac{(-1)^{1-\gamma}}{\Gamma(\gamma)} \left(\frac{R_{g_{b-}}(x)}{(b-x)^{1-\gamma}} + (1-\gamma) \int_x^b \frac{R_{g_{b-}} - R_{g_{b-}}(y)}{(y-x)^{2-\gamma}}\, dy \right) \\ &:=\; \frac{(-1)^\gamma}{\Gamma(\gamma)}\, G. \end{aligned}$$

With these notations $|I_4| \leq \dfrac{1}{\Gamma(1-\gamma)\Gamma(\gamma)} \displaystyle\int_a^b |F| \cdot |G| dx$.

Since $R_{f_{a+}} := (R_f(x) - R_f(a+))1_{(a,b)}(x) = R_f(x)1_{(a,b)}(x)$ and $R_f(a) = 0$ we deduce

$$\frac{|R_{f_{a+}}(x)|}{(x-a)^\gamma} = \frac{|R_f(x)|}{(x-a)^\gamma} = \frac{|R_f(x) - R_f(a)|}{(x-a)^\gamma}\,.$$

In our purpose we establish the following inequality

$$\begin{aligned} |R_f(x) - R_f(y)| \;&=\; \left| f(x) - f(y) + \frac{f(x_2) - f(x_1)}{x_2 - x_1} \cdot (x-y) \right| \\ &\le\; |f(x) - f(y)| + \left| \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right| |x-y| \\ &\le M_f |x-y|^\alpha + M_f h^{\alpha-1} |x-y|\,. \end{aligned}$$

By straightforward computations we have $|R_g(x) - R_g(y)| \le M_g |x-y|^\beta + M_g h^{\beta-1}|x-y|$ and consequently

$$\begin{aligned} \frac{|R_{f_{a+}}(x)|}{(x-a)^\gamma} \;&\le \big(M_f|x-a|^\alpha + M_f h^{\alpha-1}|x-a|\big)\frac{1}{(x-a)^\gamma} \\ &= M_f|x-a|^{\alpha-\gamma} + M_f h^{\alpha-1}|x-a|^{1-\gamma}. \end{aligned}$$

Due to the arbitrariness of γ we set $\gamma < \alpha$. It follows

$$\frac{|R_{f_{a+}}(x)|}{|x-a|^\gamma} \le 2M_f h^{\alpha-\gamma}.$$

So, we can write

$$|F| \le \frac{|R_{f_{a+}}(x)|}{(x-a)^\gamma} + \gamma \int_a^x \frac{|R_{f_{a+}}(x) - R_{f_{a+}}(y)|}{(x-y)^{\gamma+1}}\,dy\,.$$

The next step is to evaluate the expression $\dfrac{|R_{f_{a+}}(x) - R_{f_{a+}}(y)|}{(x-y)^{\gamma+1}}$. We may write

$$\begin{aligned} \frac{|R_{f_{a+}}(x) - R_{f_{a+}}(y)|}{(x-y)^{\gamma+1}} \;&=\; \frac{|R_f(x) - R_f(y)|}{(x-y)^{\gamma+1}} \\ &\le\; \big(M_f|x-y|^\alpha + M_f h^{\alpha-1}|x-y|\big)\cdot \frac{1}{(x-y)^{\gamma+1}} \\ &=\; M_f \frac{1}{|x-y|^{1+\gamma-\alpha}} + M_f h^{\alpha-1}\frac{1}{|x-y|^\gamma} \end{aligned}$$

Thus $|F| \le \dfrac{|R_{f_{a+}}(x)|}{|x-a|^\gamma} + \gamma \displaystyle\int_a^x \left(M_f \frac{1}{|x-y|^{1+\gamma-\alpha}} + M_f h^{\alpha-1} \frac{1}{|x-y|^\gamma} \right) dy.$

Let us remark that the condition imposed $\gamma < \alpha$ assures the convergence of the (improper) integrals which appear in the above formula. At this point we can conclude

$$
\begin{aligned}
|F| \;&\leq\; 2M_f h^{\alpha-\gamma} + \gamma\left(M_f \frac{(x-a)^{\alpha-\gamma}}{\alpha-\gamma} + M_f h^{\alpha-1}\frac{(x-a)^{1-\gamma}}{1-\gamma}\right) \\
&\leq M_f\left(2+\gamma\left(\frac{1}{\alpha-\gamma}+\frac{1}{1-\gamma}\right)\right)h^{\alpha-\gamma}.
\end{aligned}
$$

In the following we will do an analogue calculus for G. Choosing $\gamma > 1-\beta$ (recall that γ is arbitrary) we obtain

$$
\begin{aligned}
|G| \;&\leq\; 2M_g h^{\beta+\gamma-1} + (1-\gamma)M_g\Bigg(\int_x^b \frac{1}{(y-x)^{2-\beta-\gamma}}\,dy \\
&\quad + \; h^{\beta-1}\int_x^b \frac{1}{(y-x)^{1-\gamma}}\,dy\Bigg) \\
&\leq\; M_g\left(2+(1-\gamma)\left(\frac{1}{\beta+\gamma-1}+\frac{1}{\gamma}\right)\right)h^{\beta+\gamma-1}
\end{aligned}
$$

(the condition $\gamma > 1-\beta$ assures the convergence of the integrals).

At this moment we fix $\gamma \in (1-\beta, \alpha)$ and consequently

$$
|I_4| \leq \frac{1}{\Gamma(1-\gamma)\Gamma(\gamma)} M_f M_g K h^{\alpha+\beta},
$$

where K is a constant depending on α and β.

Since

$$
\begin{aligned}
\Gamma(\gamma)\Gamma(1-\gamma) \;&=\; \int_0^\infty \left(x^{\frac{\gamma-1}{2}} e^{-\frac{x}{2}}\right)^2 dx \cdot \int_0^\infty \left(x^{\frac{\gamma}{2}} e^{-\frac{x}{2}}\right)^2 dx \\
&\geq \left(\int_0^\infty x^{-\frac{1}{2}} e^{-x} dx\right)^2 = \Gamma\left(\frac{1}{2}\right)^2
\end{aligned}
$$

we have

$$
\frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} \leq \frac{1}{\Gamma(\frac{1}{2})^2} \quad \text{and} \quad |I_4| \leq \frac{K M_f M_g}{\Gamma(\frac{1}{2})^2} \cdot h^{\alpha+\beta}\ .
$$

At the end we can conclude the estimation of the rest

$$|R| \leq |I_2| + |I_3| + |I_4| \leq M_f M_g \left(4 + \frac{K}{\Gamma(\frac{1}{2})^2}\right) \cdot h^{\alpha+\beta} . \tag{10}$$

In order to obtain a better approximation, the quadrature formula must be iterate. With this end in view we will consider a division of the interval $[a, b]$ of the form $a = \tau_1 < \tau_2 < \cdots < \tau_m = b$. The fact that g is (Hölder) continuous function, allows as to write

$$\int_a^b f(x)dg(x) = \sum_{j=1}^{m-1} \int_{\tau_j}^{\tau_{j+1}} f(x)dg(x).$$

Consider $m = 2^N + 1$, i.e. the interval $[a, b]$ was divided into 2^N subintervals. Suppose that $\tau_{i+1} - \tau_i = \frac{1}{2^N}$ and denote by $R_{2,i}$ the rest of the quadrature formula for the interval $[\tau_i, \tau_{i+1}]$. The total rest is

$$\begin{aligned} |R_2| &\leq \sum_{i=1}^{2^N} |R_{2,i}| \leq \sum_{i=1}^{2^N} M_f M_g \left(4 + \frac{K}{\Gamma(\frac{1}{2})^2}\right) h_i^{\alpha+\beta} \\ &= M_f M_g \left(4 + \frac{K}{\Gamma(\frac{1}{2})^2}\right) 2^{(1-\alpha-\beta)\cdot N} \\ &= M_f M_g \left(4 + \frac{K}{\Gamma(\frac{1}{2})^2}\right) \frac{1}{2^{(\alpha+\beta-1)N}} \\ &= M_f M_g \left(4 + \frac{K}{\Gamma(\frac{1}{2})^2}\right) h^{\alpha+\beta-1}. \end{aligned}$$

The condition $\alpha + \beta > 1$ yields to $\lim_{N\to\infty} R_2 = 0$.

3.2 Case n=3 (Simpson formula)

The coefficients of the quadrature formula are:

$$\begin{aligned} &I_1^{11} = -\frac{1}{2}, \quad I_1^{12} = \frac{2}{3}, \; I_1^{13} = -\frac{1}{6}, \\ &I_1^{23} = -\frac{2}{3}, \quad I_1^{22} = 0, \; I_1^{23} = \frac{2}{3}, \\ &I_1^{31} = \frac{1}{6}, \quad I_1^{32} = -\frac{2}{3}, \; I_1^{33} = \frac{1}{2}. \end{aligned}$$

In this case, the quadrature formula is:

$$\begin{aligned} I_1 \;=\; & \frac{1}{6}\Big\{[\,f(x_1)+4\,f(x_2)+\,f(x_3)]\,[g(x_3)-g(x_1)] \\ & -\;[g(x_1)+4g(x_2)+g(x_3)]\,[\,f(x_3)-\,f(x_1)]\Big\} \\ & +\;\frac{1}{2}\,[g(x_3)+g(x_1)]\,[\,f(x_3)-\,f(x_1)]\,. \end{aligned} \tag{11}$$

We will work as in previous case. After a straightforward computation we obtain

$$\begin{aligned} & |R_f(x)| \le 6M_f h^{\alpha}\ ,\ |R_g(x)| \le 6M_g h^{\beta}\ , \\ & |L_3 f'(x)| \le 2M_f h^{\alpha-1}\ ,\ |L_3 g'(x)| \le 2M_g^{\beta-1}. \end{aligned}$$

Hence

$$|I_2| \le \int_a^b |R_f|\cdot|L_3' g(x)|dx \le 24 M_f M_g h^{\alpha+\beta} \quad \text{and} \quad |I_3| \le 24 M_f M_g h^{\alpha+\beta}.$$

To finalize we have to evaluate I_4. The following estimations hold:

$$\begin{aligned} |R_f(x)-R_f(y)| \;&\le\; M_f|x-y|^{\alpha} + 2M_f h^{\alpha-1}|x-y|, \\ \frac{|R_f(x)-R_f(y)|}{|x-y|^{\gamma+1}} \;&\le\; M_f(|x-y|^{\alpha-\gamma-1} + 2h^{\alpha-1}|x-y|^{-\gamma}) \\ |D_{a+}^{\gamma} R_{fa+}(x)| \;&\le\; \frac{1}{\Gamma(1-\gamma)}\cdot\Bigg\{\frac{M_f(|x-a|^{\alpha}+2h^{\alpha-1}|x-a|)}{|x-a|^{\gamma}} \\ &\qquad + \gamma\int_a^x M_f|x-y|^{\alpha-\gamma-1} + 2h^{\alpha-1}|x-y|^{-\gamma}dy\Bigg\} \\ &\le\; \frac{M_f}{\Gamma(1-\gamma)}\,(2^{\alpha-\gamma}+2^{2-\gamma}+\frac{\gamma}{\alpha-\gamma}+\frac{2\gamma}{1-\gamma})h^{\alpha-\gamma}. \end{aligned}$$

The integrals are convergent if we take $\gamma < \alpha$. Similarly,

$$\begin{aligned} |D_{b-}^{1-\gamma} R_{gb-}(x)| \;\le\; & \frac{M_g}{\Gamma(\gamma)}\Bigg\{(2^{\beta-1+\gamma}+2^{1+\gamma})h^{\beta-1+\gamma} \\ & +\;(1-\gamma)\int_x^b\left(\frac{1}{(y-x)^{2-\beta-\gamma}}+\frac{2h^{\beta-1}}{(y-x)^{1-\gamma}}\right)dy\Bigg\} \end{aligned}$$

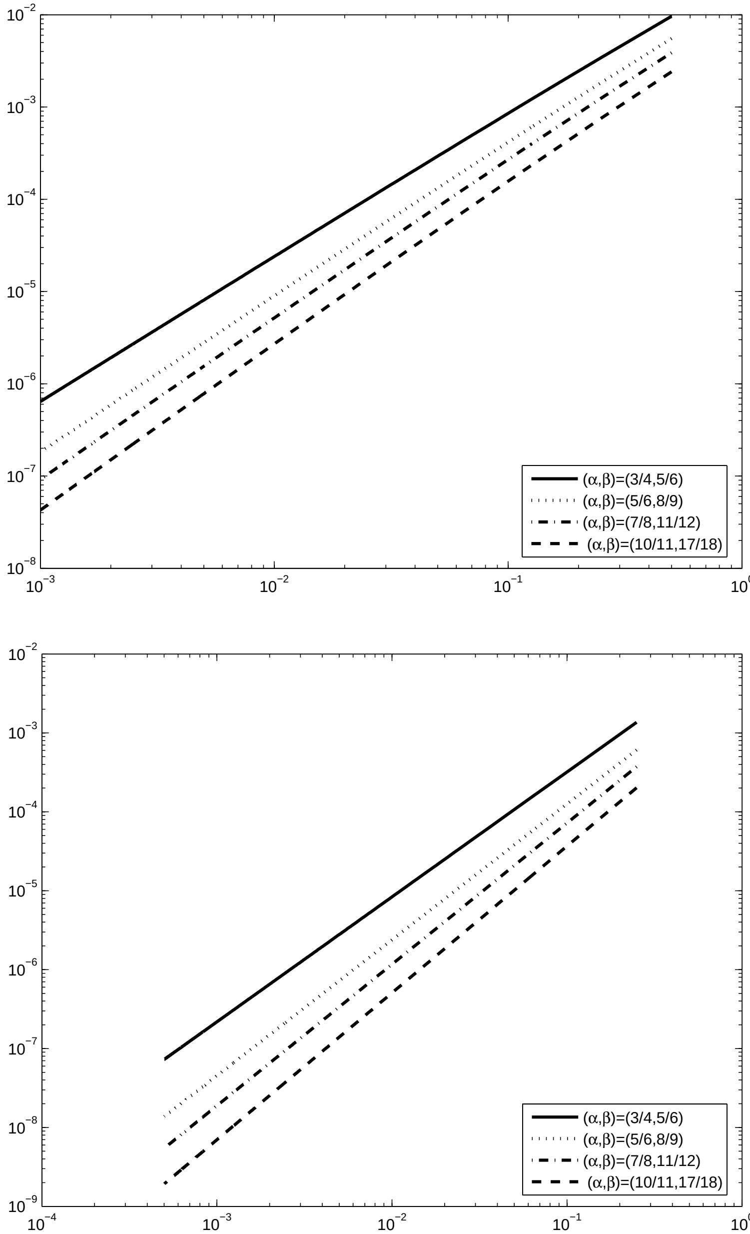

Figure 1: Logarithmic representation of the errors versus h for different values of α and β; the trapezoidal method on the left side and Simpson method on the right side.

$$\leq \frac{M_g}{\Gamma(\gamma)}\Big\{2^{\beta-1+\gamma} + 2^{1+\gamma} + \frac{1-\gamma}{\beta+\gamma-1}\cdot 2^{\beta+\gamma-1} + 2^{\gamma+1}\cdot\frac{1-\gamma}{\gamma}\Big\}h^{\beta+\gamma-1}$$

with $1-\beta<\gamma$.

Thus we have obtained an estimation for I_4

$$|I_4| \leq \frac{2M_f M_g c(\alpha,\beta,\gamma)}{\Gamma(\gamma)\Gamma(1-\gamma)}\, h^{\alpha+\beta}.$$

In the end, we may conclude that

$$|R_3| = |I_2+I_3+I_4| \leq |I_2|+|I_3|+|I_4| \leq cM_f M_g h^{\alpha+\beta}. \tag{12}$$

As in the case $n=2$, for the iterate formula we have $\lim_{n\to\infty} R_3 = 0$.

In order to illustrate the behavior of the quadratures formulas we considered $f(x)=x^\alpha,\ g(x)=x^\beta$ and we present some numerical results. (see Figure 1).

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