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The normal subgroup structure of the extended Hecke groups

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Abstract. We consider the extended Hecke groups $\overline{H}(\lambda)$ generated by $T(z) = -1/z$, $S(z) = -1/(z + \lambda)$ and $R(z) = 1/\overline{z}$ with $\lambda \geq 2$. In this paper, firstly, we study the fundamental region of the extended Hecke groups $\overline{H}(\lambda)$. Then, we determine the abstract group structure of the commutator subgroups $\overline{H}'(\lambda)$, the even subgroup $\overline{H}_e(\lambda)$, and the power subgroups $\overline{H}^m(\lambda)$ of the extended Hecke groups $\overline{H}(\lambda)$. Also, finally, we give some relations between them.

Keywords: extended Hecke group, fundamental region, commutator subgroup, even subgroup, power subgroup.

Mathematical subject classification: 11F06, 20H05, 20H10.

1 Introduction

The Hecke groups are the set of linear fractional transformations $H(\lambda)$ generated by

$$T(z) = -\frac{1}{z} \qquad \text{and} \qquad W(z) = z + \lambda,$$

which take the upper half of the complex plane into itself. Let $S = T.W$, i.e.

$$S(z) = -\frac{1}{z+\lambda}.$$

Hecke [6], proved that the H(λ) are discrete only when $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, $q \in \mathbb{N}$, $q \geq 3$, or $\lambda \geq 2$. These groups have come to be known as the *Hecke groups*, and we will denote them by $H(\lambda_q)$, $H(\lambda)$ for $q \geq 3$, $\lambda \geq 2$, respectively.

The Hecke groups $H(\lambda_q)$ and $H(\lambda)$ and their normal subgroups have been extensively studied for many aspects in the literature, (see, [2], [3], [4], [9], [19],

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[20] and [21]). The Hecke group $H(\lambda_3) = \Gamma$, the modular group $PSL(2,\mathbb{Z})$, and its normal subgroups have especially been of great interest in many fields of Mathematics, for example number theory, automorphic function theory, group theory and graph theory. In [21], Schmidt and Sheingorn described the connection between the axis of a hyperbolic element of the Hecke triangle group $H(\lambda_q)$, and the simple closed geodesics on the Riemann surfaces obtained as the quotient of the upper half plane by $H(\lambda_q)$ and $H'(\lambda_q)$ where $H'(\lambda_q)$ is the first commutator subgroup of $H(\lambda_q)$. Also in [22] and [24], they studied the simple closed geodesics on the Riemann surfaces obtained as the quotient of the upper half plane by Γ^2, Γ^3 and Γ' where Γ^2 and Γ^3 are the second and third power subgroups of the modular group Γ and Γ' is the first commutator subgroup of Γ and analyzed geodesic arcs passing through an elliptic fixed point.

In this paper, we are going to be interested in the case $\lambda \geq 2$. In this case, $H(\lambda)$ is the Fuchsian group of the first kind and the element S is parabolic when $\lambda = 2$, or $H(\lambda)$ is the Fuchsian group of the second kind and hyperbolic (boundary) when $\lambda > 2$. It is known that when $\lambda \geq 2$, $H(\lambda)$ is a free product of a cyclic group of order 2 and infinity, (see [19], [26]), so all such $H(\lambda)$ have the same algebraic structure, i.e.

$$H(\lambda) = \langle T,\ S \mid T^2 = S^\infty = (TS)^\infty = I\rangle \cong C_2 * \mathbb{Z}.$$

Also, the signature of $H(\lambda)$ is $(0; +; [2, \infty; 1]; \{-\})$ when $\lambda > 2$, i.e. a sphere with one puncture, one elliptic fixed point of order 2 or $(0; +; [2, \infty, \infty]; \{-\}) \cong (0; +; [2, \infty^{(2)}]; \{-\})$ when $\lambda = 2$, i.e., a sphere with two punctures and one elliptic fixed point of order 2. Therefore all Hecke groups $H(\lambda)$, $\lambda \geq 2$, can be considered as a triangle group. The fundamental region of the Hecke groups $H(\lambda)$ have infinite volume, [19] and Hecke surface, $H(\lambda)\backslash\mathcal{U}$, is a Riemann surface.

The extended modular group $\overline{H}(\lambda_3)$ has been defined in (see [8], [10], [25]) by adding the reflection $R(z) = 1/\overline{z}$ to the generators of the modular group $H(\lambda_3)$. Then the extended Hecke groups $\overline{H}(\lambda_q)$ have been defined similar to the extended modular group case $\overline{H}(\lambda_3)$ in [13] and [14]. They were studied commutator subgroups $\overline{H}'(\lambda_q)$, $\overline{H}''(\lambda_q)$, even subgroups $\overline{H}_e(\lambda_q)$ and principal subgroups $\overline{H}_p(\lambda_q)$ of the extended Hecke groups $\overline{H}(\lambda_q)$. Also, in [15], [16] and [18], we were investigated the power and free subgroups of $\overline{H}(\lambda_3)$, $\overline{H}(\lambda_5)$ and $\overline{H}(\lambda_p)$ and the relations between power subgroups and commutator subgroups.

Now we can define the extended Hecke group $\overline{H}(\lambda)$ by adding the reflection $R(z) = 1/\overline{z}$ to the generators of $H(\lambda)$ similar to the extended Hecke groups $\overline{H}(\lambda_q)$. Then $\overline{H}(\lambda)$ has a presentation

$$\overline{H}(\lambda) = \langle T, S, R \mid T^2 = S^\infty = R^2 = I,\ RT = TR,\ RS = S^{-1}R\rangle$$

or

$$\overline{H}(\lambda) = \langle T, S, R \mid T^2 = R^2 = (TR)^2 = (RS)^2 = I \rangle. \tag{1.1}$$

Let

$$G_1 = \langle T, R \mid T^2 = R^2 = (TR)^2 = I \rangle \cong D_2,$$

and let

$$G_2 = \langle S, R \mid S^\infty = R^2 = (RS)^2 = I \rangle \cong D_\infty.$$

Then $\overline{H}(\lambda)$ is $G_1 * G_2$ with the identification $R = R$. In G_1, the subgroup generated by R is $\mathbb{Z}_2$,this is also true in G_2. Therefore the identification induces an isomorphism and $\overline{H}(\lambda)$ is a generalized free product with the subgroup $\mathbb{Z}_2$ amalgamated, i.e.,

$$\overline{H}(\lambda) \cong D_2 *_{\mathbb{Z}_2} D_\infty.$$

It is clear that Hecke group $H(\lambda)$ is a subgroup of index 2 in $\overline{H}(\lambda)$. Also if we put $R = R_1$, $T = R_2R_1 = R_1R_2$, and $S = R_3R_1$, then the extended Hecke groups $\overline{H}(\lambda)$ has the presentation

$$\overline{H}(\lambda) = \langle R_1, R_2, R_3 \mid R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^2 = I \rangle, \tag{1.2}$$

where

$$R_1(z) = \frac{1}{\overline{z}}, \qquad R_2(z) = -\overline{z}, \qquad R_3(z) = \frac{-\overline{z}}{\lambda\overline{z}+1}.$$

The signature of the extended Hecke group $\overline{H}(\lambda)$ is $(0; +; [-]; \{2, \infty, \infty\})$ or $(0; +; [-]; \{2, \infty; 1\})$ and the quotient space $\overline{H}(\lambda)\backslash\mathcal{U}$ is a sphere with one branch point and two cusps when $\lambda = 2$ or a sphere with one branch point, one cusp and one hole, respectively. Since the extended Hecke groups $\overline{H}(\lambda)$ contain a reflection, they are NEC groups. Thus quotient space $\overline{H}(\lambda)\backslash\mathcal{U}$ is a Klein surface. Also $H(\lambda)\backslash\mathcal{U}$ is the canonical double cover of $\overline{H}(\lambda)\backslash\mathcal{U}$. In our recent paper [17], we showed that there is a relation between the extended Hecke groups $\overline{H}(\lambda_q)$ and the automorphism groups of compact bordered Klein surfaces of algebraic genus $p \geq 2$.

In this work, we will consider extended Hecke groups $\overline{H}(\lambda)$ and their some subgroups. Firstly, we will study the fundamental region of the extended Hecke group $\overline{H}(\lambda)$. Then, we will discuss the commutator subgroups $\overline{H}'(\lambda)$ and even subgroups $\overline{H}_e(\lambda)$ of $\overline{H}(\lambda)$ and their fundamental regions. Finally, we will determine the abstract group structure of the power subgroups $\overline{H}^m(\lambda)$ of $\overline{H}(\lambda)$ and the relations between the commutator subgroups and the power subgroups of $\overline{H}(\lambda)$.

2 Fundamental Region of $\overline{H}(\lambda_q)$

E. Hecke showed that, when $\lambda \geq 2$ and real, or when $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, $q \in \mathbb{N}$, $q \geq 3$, the set

$$F_\lambda = \left\{ z \in U : |Re\ z| < \frac{\lambda}{2}\ \ |z| > 1 \right\}$$

is a fundamental region for the group $H(\lambda)$, and also F_λ fails to be a fundamental region for all other $\lambda > 0$ [6]. When $\lambda > 2$, F_λ has free sides and thus infinite area and when $\lambda = 2$, the area is finite, the two real intervals shrink to single points and the removed disc shrinks to a point.

We therefore take a fundamental region for $H(\lambda)$ as

$$F_\lambda = \left\{ z \in U : |Re\ z| < \frac{\lambda}{2}\ \ |z| > 1 \right\}$$

that is, bounded by the unit circle and two vertical lines through $\pm\lambda/2$. It is well-known that fundamental region of a group is not unique. We have already seen that $F_\lambda = F_1 \cup F_2$ in Figure 1 is a fundamental region for $H(\lambda)$. Actually a shaded region together with an unshaded one form a fundamental region for $H(\lambda)$. Therefore sometimes, for convenience, we shall take it as

$$F'_\lambda = \left\{ z \in U : -\frac{\lambda}{2} < Re\ z < 0,\ \ \left| z + \frac{1}{\lambda} \right| > \frac{1}{\lambda} \right\}.$$

which is $F_1 \cup T(F_2)$.

Now, we can find fundamental region of extended Hecke groups $\overline{H}(\lambda)$ using the above results.

Theorem 2.1. *The set*

$$\overline{F}_\lambda = \left\{ z \in U : |z| > 1, -\frac{\lambda}{2} < Re\ z < 0 \right\}$$

is a fundamental region of extended Hecke groups $\overline{H}(\lambda)$.

Proof. We know that the set

$$F_\lambda = \left\{ z \in U : |Re\ z| < \frac{\lambda}{2},\ \ |z| > 1 \right\}$$

is a fundamental region of the Hecke groups $H(\lambda)$ and Hecke groups $H(\lambda)$ is a subgroup of index 2 in the extended Hecke groups $\overline{H}(\lambda)$. If we use the Riemann-Hurwitz formula

$$\left[\overline{H}(\lambda) : H(\lambda)\right] = \frac{\mu(H(\lambda))}{\mu(\overline{H}(\lambda))}$$

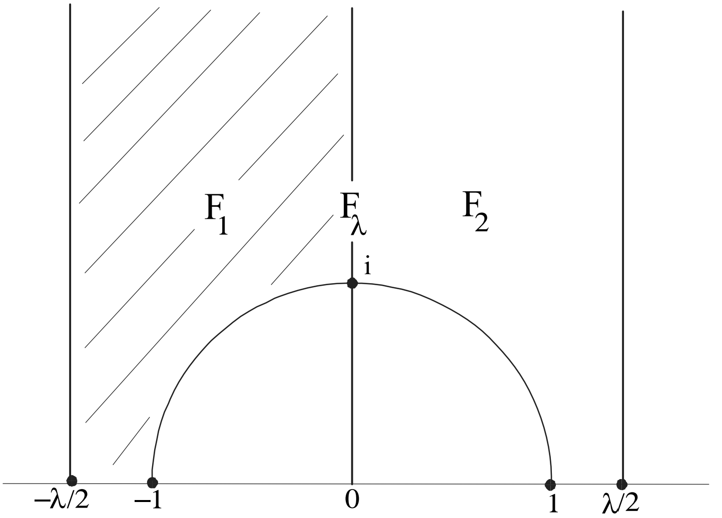

Figure 1

where $\mu(H(\lambda))$ denotes the hyperbolic area of a fundamental region for $H(\lambda)$. Then we find the fundamental region of extended Hecke groups $\overline{H}(\lambda)$ as half of a fundamental region of Hecke groups $H(\lambda)$ in Figure 2, since every points in F_2 equivalent to points in F_1 under $TR = R_2$, is the reflection on the line $x = 0$, i.e. $R_2(F_2) = F_1$.

Notice that the extended Hecke group $\overline{H}(\lambda)$ is a discrete group generated by the reflections in the edges of $\overline{F}_\lambda$.

3 Commutator Subgroups of $\overline{H}(\lambda)$

The commutator subgroup of $\overline{H}(\lambda)$ is denoted by $\overline{H}'(\lambda)$ and defined by

$$\left\langle [A, B] \mid A, B \in \overline{H}(\lambda) \right\rangle$$

where $[A, B]=ABA^{-1}B^{-1}$. The commutator subgroup $\overline{H}'(\lambda)$ is a normal subgroup of $\overline{H}(\lambda)$, and therefore we can form the quotient group $\overline{H}(\lambda)/\overline{H}'(\lambda)$. The quotient group $\overline{H}(\lambda)/\overline{H}'(\lambda)$ is the group obtained by adding the relation of abelianizing to the presentation of $\overline{H}(\lambda)$ (see [5]).

Now we study the commutator subgroups of the extended Hecke groups $\overline{H}(\lambda)$.

Theorem 3.1. *The commutator subgroup $\overline{H}'(\lambda)$ is a normal subgroup of index 8 of $\overline{H}(\lambda)$. Also*

$$\overline{H}'(\lambda) = \langle S^2 \rangle * \langle TS^2T \rangle * \langle TSTS^{-1} \rangle.$$

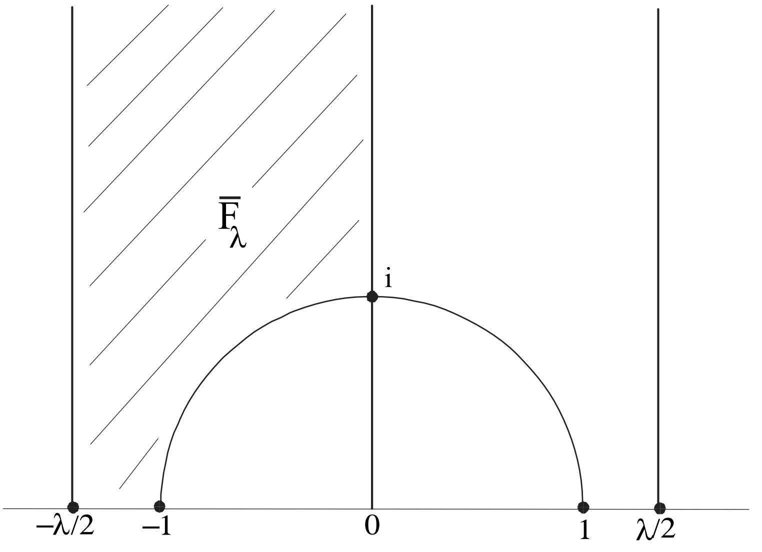

Figure 2

Proof. The quotient group $\overline{H}(\lambda)/\overline{H}'(\lambda)$ is the group obtained by adding the relations $TS = ST$, $RT = TR$ and $RS = SR$ to the relations of $\overline{H}(\lambda)$. Then $\overline{H}(\lambda)/\overline{H}'(\lambda)$ has a presentation

$$\frac{\overline{H}(\lambda)}{\overline{H}'(\lambda)} = \Big\langle T, S, R \mid T^2 = R^2 = I,\ RT = TR,\ TS = ST,$$
$$RS = SR,\ RS = S^{-1}R\Big\rangle.$$

We obtain $S^2 = I$ as $RS = S^{-1}R$ and $RS = SR$. Then we find $(TS)^2 = I$ as $T^2 = S^2 = I$. Therefore

$$\frac{\overline{H}(\lambda)}{\overline{H}'(\lambda)} = \langle T, S, R \mid T^2 = S^2 = R^2 = (RT)^2 = (RS)^2 = (TS)^2 = I\rangle$$

and so

$$\begin{aligned} \overline{H}(\lambda)/\overline{H}'(\lambda) &\cong C_2 \times C_2 \times C_2. \\ \left|\overline{H}(\lambda) : \overline{H}'(\lambda)\right| &= 8 \end{aligned}$$

Now we choose a Schreier transversal for $\overline{H}'(\lambda)$ as

$$I, T, R, S, TR, SR, TS, TSR.$$

Hence, all possible products are

$$
\begin{array}{ll}
I.T.(T)^{-1} = I, & TR.T.(R)^{-1} = TRTR, \\
T.T.(I)^{-1} = I, & SR.T.(TSR)^{-1} = SRTRS^{-1}T, \\
R.T.(TR)^{-1} = RTRT, & TS.T.(S)^{-1} = TSTS^{-1}, \\
S.T.(TS)^{-1} = STS^{-1}T, & TSR.T.(SR)^{-1} = TSRTRS^{-1}, \\
I.S.(S)^{-1} = I, & TR.S.(TSR)^{-1} = TRSRS^{-1}T, \\
T.S.(TS)^{-1} = I, & SR.S.(R)^{-1} = SRSR, \\
R.S.(SR)^{-1} = RSRS^{-1}, & TS.S.(T)^{-1} = TS^2T, \\
S.S.(I)^{-1} = S^2, & TSR.S.(TR)^{-1} = TSRSRT, \\
I.R.(R)^{-1} = I, & TR.R.(T)^{-1} = I, \\
T.R.(TR)^{-1} = I, & SR.R.(S)^{-1} = I, \\
R.R.(I)^{-1} = I, & TS.R.(TSR)^{-1} = I, \\
S.R.(SR)^{-1} = I, & TSR.R.(TS)^{-1} = I.
\end{array}
$$

Since $(STS^{-1}T)^{-1} = TSTS^{-1}$, $(TRTR)^{-1} = RTRT = I$, $(RSRS^{-1}) = (S^2)^{-1}$, $SRSR = I$, $(SRTRS^{-1}T)^{-1} = TSRTRS^{-1} = TSTS^{-1}$, $TRSRS^{-1}T = (TS^2T)^{-1}$, $TSRSRT = I$, the generators of $\overline{H}'(\lambda)$ are S^2, TS^2T, $TSTS^{-1}$. Thus $\overline{H}'(\lambda)$ has a presentation

$$\overline{H}'(\lambda) = \langle S^2 \rangle * \langle TS^2T \rangle * \langle TSTS^{-1} \rangle. \qquad \square$$

It is clear that $\overline{H}'(\lambda)$ is a normal subgroup of index 4 in the Hecke group $H(\lambda)$. Since F_λ is the fundamental region for $H(\lambda)$, we find the fundamental region of $\overline{H}'(\lambda)$ as $F_\lambda \cup W(F_\lambda) \cup W^2(F_\lambda) \cup W^3 F_\lambda$ where $W : z \to z+\lambda$ (defined above).(see Figure 3). The action of W constitutes a conformal isometry of $\overline{H}'(\lambda)\backslash\mathcal{U}$. Also $TR : z \to -\bar{z}$ is an anti-conformal isometry of $\overline{H}'(\lambda)\backslash\mathcal{U}$.

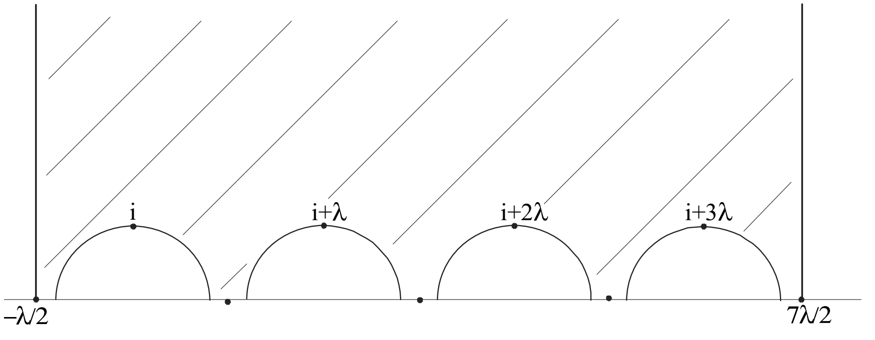

Figure 3

Using the theorem 3.1, we obtain the following result.

Corollary 3.2. *$\overline{H}'(\lambda)$ is a free group of rank 3.*

Let us now investigate the group theoretical structure of $\overline{H}''(\lambda)$. We have seen above that $\overline{H}'(\lambda)$ is a free normal subgroup of finite index, 8, and of rank 3, of $\overline{H}(\lambda)$. Therefore the second commutator subgroup $\overline{H}''(\lambda)$, as the commutator subgroup of a free group, $\overline{H}'(\lambda)$, is of infinite index in $\overline{H}'(\lambda)$ and hence in $\overline{H}(\lambda)$. Hence

Corollary 3.3. *$\overline{H}''(\lambda)$ is a free normal subgroup of infinite index in $\overline{H}(\lambda)$.* □

We also have

Theorem 3.4. *$\dfrac{\overline{H}'(\lambda)}{\overline{H}''(\lambda)}$ is a free abelian group with free generators*

$$S^2\overline{H}''(\lambda),\ TS^2T\overline{H}''(\lambda),\ TSTS^{-1}\overline{H}''(\lambda)$$

and

$$r\left(\frac{\overline{H}'(\lambda)}{\overline{H}''(\lambda)}\right) = r(\overline{H}'(\lambda)) = 3.$$ □

4 The Even Subgroup of $\overline{H}(\lambda)$

We now investigate the structure of another important normal subgroup of $\overline{H}(\lambda)$ namely the even subgroup. In fact it contains infinitely many other normal subgroups. We know that all elements of the extended Hecke groups $\overline{H}(\lambda)$ form two classes (see [12]):

$$\text{(i)} \quad \begin{pmatrix} a & b\lambda \\ c\lambda & d \end{pmatrix}, \qquad ad - bc\lambda^2 = \mp 1,$$

$$\text{(ii)} \quad \begin{pmatrix} a\lambda & b \\ c & d\lambda \end{pmatrix}, \qquad ad\lambda^2 - bc = \mp 1$$

where a, b, c, d are all polynomials of λ^2. But the converse is not true. That is, all elements of type (i) or (ii) need not belong to $\overline{H}(\lambda)$. Those of type (i) are called even while those of type (ii) are called odd. Note that if we consider the muptiplication of these elements, the situation is similar to the multiplication of negative and positive numbers. Here we have

$$\text{odd} \cdot \text{odd} = \text{even} \cdot \text{even} = \text{even}\,,$$
$$\text{even} \cdot \text{odd} = \text{odd} \cdot \text{even} = \text{odd}\,.$$

and the sets of all even elements forms a subgroup of $\overline{H}(\lambda)$ of index 2 called the even subgroup, denoted by $\overline{H}_e(\lambda)$. Therefore

$$\overline{H}_e(\lambda) = \left\{ A = \begin{pmatrix} a & b\lambda \\ c\lambda & d \end{pmatrix} : A \in \overline{H}(\lambda) \right\}.$$

The set of odd elements

$$\overline{H}_o(\lambda) = \left\{ B = \begin{pmatrix} a\lambda & b \\ c & d\lambda \end{pmatrix} : B \in \overline{H}(\lambda) \right\},$$

forms the other coset of $\overline{H}_e(\lambda)$ in $\overline{H}(\lambda)$.

Theorem 4.1. *The even subgroup $\overline{H}_e(\lambda)$ is a normal subgroup of index two of $\overline{H}(\lambda)$. Also*

$$\begin{aligned} \overline{H}(\lambda) &= \overline{H}_e(\lambda) \cup T.\overline{H}_e(\lambda), \\ \overline{H}_e(\lambda) &\cong \langle TS, ST, RT \mid (TR)^2 = (STRT)^2 = I \rangle. \end{aligned} \tag{4.1}$$

Proof. Having index two, $\overline{H}_e(\lambda)$ is normal subgroup of $\overline{H}(\lambda)$. Let us now choose I, T as a Schreier transversal for the even subgroup. According to the Reidemeister-Schreier method, we should form all possible products :

$$\begin{aligned} &I.T.(T)^{-1} = I, \quad I.S.(T)^{-1} = ST, \quad I.R.(T)^{-1} = RT, \\ &T.T.(I)^{-1} = I, \quad T.S.(I)^{-1} = TS, \quad T.R.(I)^{-1} = TR. \end{aligned}$$

Since $TR = RT$, the generators are TS, ST and RT. We have

$$\overline{H}_e(\lambda) \cong \langle TS, ST, RT \mid (TR)^2 = (STRT)^2 = I \rangle.$$

Finally as $T \notin \overline{H}_e(\lambda)$, (2.1) follows. □

$\overline{H}_e(\lambda)$ is very important amongst the normal subgroups of $\overline{H}(\lambda)$. It contains infinitely many normal subgroups of $\overline{H}(\lambda)$.The fundamental region of $\overline{H}_e(\lambda)$ is the following (Figure 4).

The following result connects the even subgroup and the commutator subgroup of $\overline{H}(\lambda)$:

Theorem 4.2. *The commutator subgroup $\overline{H}'(\lambda)$ is a normal subgroup of the even subgroup $\overline{H}_e(\lambda)$ with index 4.*

Proof. We have just seen that $\overline{H}'(\lambda)$ is a normal subgroup of $\overline{H}(\lambda)$ with index 8. The even subgroup $\overline{H}_e(\lambda)$, having index 2, is also normal in $\overline{H}(\lambda)$. Then the required index is 4.

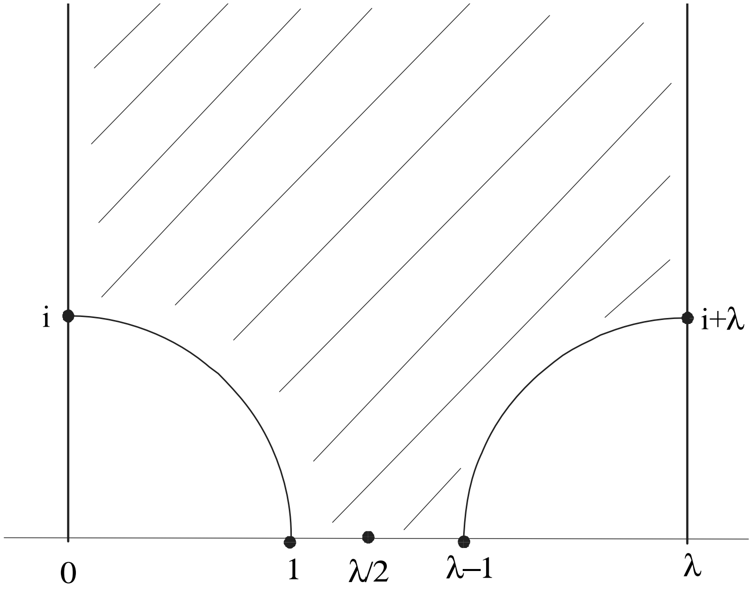

Figure 4

Let now two elements A, B of $\overline{H}(\lambda)$ be given. Then whatever A and B are, their commutator is always even. Hence for every pair of elements we have

$$[A, B] \in \overline{H}_e(\lambda).$$

That is

$$\overline{H}'(\lambda) \lhd \overline{H}_e(\lambda). \qquad \square$$

Then we have

Corollary 4.3. *A subgroup of $\overline{H}'(\lambda)$ consists of only even elements.*

5 Power subgroups of $\overline{H}(\lambda)$

Let m be a positive integer. Let us define $\overline{H}^m(\lambda)$ to be the subgroup generated by the m^{th} powers of all elements of $\overline{H}(\lambda)$. The subgroup $\overline{H}^m(\lambda)$ is called the m^{th} power subgroup of $\overline{H}(\lambda)$. As fully invariant subgroups, they are normal in $\overline{H}(\lambda_q)$.

From the definition one can easily deduce that

$$\overline{H}^m(\lambda) > \overline{H}^{mk}(\lambda)$$

and that

$$(\overline{H}^m(\lambda))^k > \overline{H}^{mk}(\lambda).$$

Also, it is easy to deduce that

$$\overline{H}^m(\lambda).\overline{H}^k(\lambda) = \overline{H}^{(m,k)}(\lambda),$$

where (m, k) denotes the greatest common divisor of m and k.

The power subgroups of the Hecke groups $H(\lambda_q)$, $q \geq 3$ integer, are studied in [2], [7] and [11]. Also the power subgroups of the extended modular group are investigated in [15].

We now discuss the group theoretical structure of these subgroups. Let us consider the presentation of the extended Hecke group $\overline{H}(\lambda)$ given in (1.1):

$$\overline{H}(\lambda) = \langle T, S, R \mid T^2 = R^2 = (TR)^2 = (SR)^2 = I \rangle.$$

We find a presentation for the quotient $\overline{H}(\lambda)/\overline{H}^m(\lambda)$ by adding the relation $X^m = I$ to the presentation of $\overline{H}(\lambda)$. The order of $\overline{H}(\lambda)/\ \overline{H}^m(\lambda)$ gives us the index. We have

$$\begin{aligned} \frac{\overline{H}(\lambda)}{\overline{H}^m(\lambda)} \cong \langle T, S, R \mid T^2 = R^2 = (TR)^2 = (SR)^2 = I, \\ T^m = S^m = R^m = (TS)^m = (TR)^m = (SR)^m = \cdots = I \rangle. \end{aligned} \tag{5.1}$$

Thus we use Reidemeister-Schreier process to find the presentation of the power subgroups $\overline{H}^m(\lambda)$. First we have

Theorem 5.1. *The normal subgroup $\overline{H}^2(\lambda)$ is the free product of three infinite cyclic groups. Also*

$$\begin{aligned} \frac{\overline{H}(\lambda)}{\overline{H}^2(\lambda)} &= \mathbb{C}_2 \times C_2 \times C_2, \\ \overline{H}^2(\lambda) &= \langle S^2 \rangle * \langle TS^2T \rangle * \langle TSTS^{-1} \rangle, \\ \overline{H}(\lambda) &= \overline{H}^2(\lambda) \cup T\overline{H}^2(\lambda) \cup R\overline{H}^2(\lambda) \cup S\overline{H}^2(\lambda) \cup TR \\ &\quad \overline{H}^2(\lambda) \cup RS\overline{H}^2(\lambda) \cup TS\overline{H}^2(\lambda) \cup TSR\overline{H}^2(\lambda). \end{aligned}$$

The elements of $\overline{H}^2(\lambda)$ can be characterized by the requirement that the sums of the exponents of R and S are both even.

Proof. By (5.1), we obtain

$$\frac{\overline{H}(\lambda)}{\overline{H}^2(\lambda)} \cong \left\langle T, S, R \mid T^2 = S^2 = R^2 = (TR)^2 = (SR)^2 = (TS)^2 = I \right\rangle.$$

and therefore we get

$$\begin{aligned} \frac{\overline{H}(\lambda)}{\overline{H}^2(\lambda)} &\cong C_2 \times C_2 \times C_2 \\ \left|\overline{H}(\lambda) : \overline{H}^2(\lambda)\right| &= 8. \end{aligned}$$

Now we choose $I, T, R, S, TR, SR, TS, TSR$ for our transversal. Hence, all possible products are

$$\begin{array}{ll}
I.T.(T)^{-1} = I, & TR.T.(R)^{-1} = TRTR, \\
T.T.(I)^{-1} = I, & SR.T.(TSR)^{-1} = SRTRS^{-1}T, \\
R.T.(TR)^{-1} = RTRT, & TS.T.(S)^{-1} = TSTS^{-1}, \\
S.T.(TS)^{-1} = STS^{-1}T, & TSR.T.(SR)^{-1} = TSRTRS^{-1}, \\
I.S.(S)^{-1} = I, & TR.S.(TSR)^{-1} = TRSRS^{-1}T, \\
T.S.(TS)^{-1} = I, & SR.S.(R)^{-1} = SRSR, \\
R.S.(SR)^{-1} = RSRS^{-1}, & TS.S.(T)^{-1} = TS^2T, \\
S.S.(I)^{-1} = S^2, & TSR.S.(TR)^{-1} = TSRSRT, \\
I.R.(R)^{-1} = I, & TR.R.(T)^{-1} = I, \\
T.R.(TR)^{-1} = I, & SR.R.(S)^{-1} = I, \\
R.R.(I)^{-1} = I, & TS.R.(TSR)^{-1} = I, \\
S.R.(SR)^{-1} = I, & TSR.R.(TS)^{-1} = I.
\end{array}$$

Since $(STS^{-1}T)^{-1} = TSTS^{-1}$, $(TRTR)^{-1} = RTRT = I$, $(RSRS^{-1}) = (S^2)^{-1}$, $SRSR = I$, $(SRTRS^{-1}T)^{-1} = TSRTRS^{-1} = TSTS^{-1}$, $TRSRS^{-1}T = (TS^2T)^{-1}$, $TSRSRT = I$, the generators of $\overline{H}^2(\lambda)$ are S^2, TS^2T, $TSTS^{-1}$. Thus $\overline{H}^2(\lambda)$ has a presentation

$$\overline{H}^2(\lambda) = \langle S^2 \rangle * \langle TS^2T \rangle * \langle TSTS^{-1} \rangle.$$

and we get

$$\begin{aligned} \overline{H}(\lambda) = {} & \overline{H}^2(\lambda) \cup T\overline{H}^2(\lambda) \cup R\overline{H}^2(\lambda) \cup S\overline{H}^2(\lambda) \cup TR\overline{H}^2(\lambda) \\ & \cup RS\overline{H}^2(\lambda) \cup TS\overline{H}^2(\lambda) \cup TSR\overline{H}^2(\lambda). \end{aligned}$$ □

Theorem 5.2. *Let m be an odd integer. Then* $\overline{H}^m(\lambda) = \overline{H}(\lambda)$.

Proof. By (5.1), we find $S = T = R = I$ from the relations

$$R^2 = R^m = I, \ (SR)^2 = (SR)^m = I, \ T^2 = T^m = I.$$

Thus we have

$$\left|\overline{H}(\lambda) : \overline{H}^m(\lambda)\right| = 1,$$

that is,

$$\overline{H}^m(\lambda) = \overline{H}(\lambda). \qquad \square$$

Finally, if $m > 2$ is even, then in the quotient group $\overline{H}(\lambda)/\overline{H}^m(\lambda)$ we have the relations $T^2 = S^m = R^2 = (TS)^m = (TR)^2 = (SR)^2$. Therefore the above techniques do not say much about $\overline{H}^m(\lambda)$ in this apart from the fact that they are all normal subgroups with torsion elements. To discuss $\overline{H}^m(\lambda)$ we use the commutator subgroup $\overline{H}'(\lambda)$.

Theorem 5.3. *The commutator subgroup* $\overline{H}'(\lambda)$ *of* $\overline{H}(\lambda)$ *satisfies*

$$\overline{H}'(\lambda) = \overline{H}^2(\lambda). \qquad \square$$

By means of this result, we are going to be able to investigate the subgroups $\overline{H}^m(\lambda)$. As $\overline{H}^2(\lambda) \supset \overline{H}^m(\lambda)$, and clear that

$$\overline{H}'(\lambda) \supset \overline{H}^m(\lambda).$$

As $\overline{H}'(\lambda)$ is a free group, we can conclude that $\overline{H}^m(\lambda)$ is also a free group. Therefore we have the following theorem:

Theorem 5.4. *Let* $m > 2$ *be an even integer. The groups* $\overline{H}^m(\lambda)$ *are free groups.* $\square$

Notice that since both $H(\lambda)$ (a free product) and $\overline{H}(\lambda)$ act on the upper half plane, one may construct a tree $\mathcal{T}$ such that both $H(\lambda)$ and $\overline{H}(\lambda)$ act on the tree. As a consequence, the above mentioned facts can be proved by investigating the action of $\overline{H}(\lambda)$ on $\mathcal{T}$ or the fundamental region of $H(\lambda)$. Construction of such trees and fundamental regions can be found in [10] and [23].

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