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Continuation methods in Banach manifolds

J.M. Soriano

Abstract. Sufficient conditions are given to assert that a perturbed mapping has a zero in a Banach manifold modelled over $\mathbb{R}^n$. The zero is estimated by means of sequences of Newton's iterations. The proof of the result is constructive and is based upon continuation methods.

Keywords: regular value, continuation methods, continuous dependence theorem, sequence of Newton's iterations.

Mathematical subject classification: Primary 58C30; Secondary 65H20.

1 Preliminaires

Since a process of measurements is used to describe scientific phenomena, the concept of Banach manifold is one of the most important in mathematical physics. At close range a manifold looks like a Banach-space. Phenomena are locally described by parameters. Since different Banach-space or parameter-space or coordinate-space are allowed for a phenomena local description, it is important to have a transformation rule for these different coordinates, which exist in a Banach manifold. It is clear that manifold properties, which are independent of the choice of local coordinates, are the most important from a mathematical or physical point of view.

Let X, Y be two Banach Spaces. Let $u\colon U \subset X \to Y$ be a continuous mapping. One way of solving the equation

$$u(x) = y \tag{1}$$

for any fixed $y \in Y$, is to embed (1) in a continuum of problems

$$H(x, t) = y, \, (0 \le t \le 1), \tag{2}$$

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which is solved when $t = 0$. When $t = 1$, problem (2) becomes (1). If it is possible to continue the solution for all $t \in [0, 1]$, then (1) is solved. This is the continuation method with respect to a parameter [1-21]. We introduce a continuation method for $u\colon M \to \mathbb{R}^n$, where M is a connected C^1-Banach manifold modelled on $\mathbb{R}^n$.

In this paper, sufficient conditions are given to prove that a C^1-mapping has a zero in a Banach manifold by using continuation methods. Other conditions, sufficient to guarantee the existence of zero points, have been given by the author in several other papers [7-21]. This zero is also estimated. The proof supplies the existence of a curve which leads to the solution. This can be approximated to as precise an error as desired. The key is the use of the Continuous Dependence Theorem [24], together with a consequence of the properties of Banach algebra of the linear continuous mappings from a Banach space into itself [22].

We briefly recall some Theorems and notation to be used.

Definitions and Notation [25]. Let M be a topological space. A *chart* (U, φ) in M is a pair where the set U is open in M and $\varphi\colon U \to U_\varphi$ is a homeomorphism onto an open subset U_φ of a Banach-space X_φ. We call φ a *chart map*, X_φ is called *chart space,* and U_φ *chart image.* For $x \in U$ we call $x_\varphi = \varphi(x)$ the *representative* of x in the chart (U, φ) or the *local coordinate* of x in the local coordinate system φ. The point $x \in M$ may have different local coordinates $x_\varphi = \varphi(x)$ and $x_\psi = \psi(x)$ for two different charts (U, φ) and (V, ψ). The transformation rules between them are $x_\varphi = \varphi(\psi^{-1}(x_\psi))$ and $x_\psi = \psi(\varphi^{-1}(x_\varphi))$.

Let M be a topological space, two charts (U, φ) and (V, ψ) are called C^k-*compatible* if and only if $U \cap V = \emptyset$ or $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$are C^k-mappings, $k \geq 0$, where $\varphi \circ \psi^{-1}\colon \psi(U \cap V) \to \varphi(U \cap V)$ and $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$. A C^k*-atlas* for M, $0 \leq k \leq \infty$ is a collection of charts (U_i, φ_i) (i ranging in some index set) which satisfies the following conditions: (i) The U_i cover M, (ii) any two charts are C^k-compatible, (iii) All chart spaces X_i are Banach-spaces over $\mathbb{K}$. M is said to be a C^k*-Banach manifold* if and only if there exists a C^k-atlas for M.

We call a chart in M, which is C^k-compatible with all atlas charts, *an admissible chart.* In particular, all atlas charts are admissible. If all chart spaces are equal to a fixed Banach-space X, then M is called a C^k*-Banach manifold modelled* on X. In this paper manifolds *without boundaries* will be considered, such as the surface of a ball in $\mathbb{R}^n$, an open set in a Banach-space X, etc.

Let M and N be C^k-Banach manifolds with chart spaces over $\mathbb{K}$, $k \geq 1$. Then $f\colon M \to N$ is called C^r-mapping, $r \leq k$ if and only if f is C^r at each point $x \in M$ in fixed admissible charts. Let (U, φ) and (V, ψ) be charts in M and N, respectively, with $x \in U$ and $f(x) \in V$, the mapping $\overline{f} = \psi \circ f \circ \varphi^{-1}$ which is well defined in a sufficient small neighbourhood of x_φ. This map is assumed to be C^r in the usual sense and we will call it a *representative* of f.

Let M be a C^k-Banach manifold, $k \geq 1$, and $x \in M$. Two C^1-curves in M, which pass through the point x, are called *equivalent* at point x if and only if the representatives have the same tangent vector at x in some admissible chart. A *tangent vector* v to M at x consists of all C^1-curves which are equivalent at x for a fixed C^1-curve. The tangent space TM_x to M at point x is the set of all tangent vectors at x.

Let $f : M \to N$ be a C^k-mapping, $k \geq 1$, where M and N are C^k-Banach manifolds with chart space over $\mathbb{K}$, f is called a *submersion* at x if and only if $f'(x)$ is surjective and the null space $N(f'(x))$ splits TM_x (which is automatic when M and N are finite-dimensional). A point $x \in M$ is called a *regular point of* f if and only if f is a submersion at x. A point $y \in N$ is called a regular value of f if and only if the set $f^{-1}(y)$ is empty or consists only of regular points.

If X, Y are Banach spaces, let $\mathcal{L}(X, Y)$ denote the set of all linear continuous mappings $L : X \to Y$. Let $\mathrm{Isom}(X, Y)$ denote the set of all the isomorphisms $L : X \to Y$. Let $B(x_0, \rho)$ be the open ball of centre x_0 and radius ρ. If $u : X \to Y$ is a linear continuous bijective operator, then the inverse linear continuous operator will be denoted by u^{-1}. If U is a set in X, let ∂U denote the boundary of the set U, and $\overline{U}$ its closure. $d(x, U)$ denotes the distance between point x and set U. $d(U, V)$ denotes the distance between the sets U and V.

$H_x(x, t)$ denotes the partial F-derivative of H with respect to X at the point (x, t), where $H : X \times [0, 1] \to Y$.

Theorem 1. Continuous Dependence Theorem ([24], pp. 188). *Let the following conditions be satisfied:*

(i) *P is a metric space, called the parameter space.*

(ii) *For each parameter $p \in P$, mapping T_p satisfies the following hypotheses:*

  (1) *$T_p : M \to M$, i.e. M is mapped into itself by T_p.*
  (2) *M is a closed non-empty set in a complete metric space (X, d).*
  (3) *T_p is k-contractive, for any fixed $k \in [0, 1)$.*

(iii) *For each $p_0 \in P$, and any $x \in M$, $\lim_{p \to p_0} T_p(x) = T_{p_0}(x)$.*

Thus for each $p \in P$, the equation $x_p = T_p x_p$ has exactly one solution x_p, where $x_p \in M$ and $\lim_{p \to p_0} x_p = x_{p_0}$.

Theorem 2 ([22] pp. 23-24). *If X, Y are Banach spaces then*

a) $\mathrm{Isom}(X, Y)$ *is open in* $\mathcal{L}(X, Y)$.

b) *the mapping* $\beta\colon \mathrm{Isom}(X, Y) \to \mathcal{L}(X, Y)$, $\beta(u) := u^{-1}$ *is continuous.*

Theorem 3 ([23], p. 45). *Let $L, L' \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and assume that L is invertible, with $\| L^{-1} \| \leq C$. If $\| L - L' \| \leq \varepsilon$, and $C \times \varepsilon < 1$, therefore L' is also invertible, and $\| L'^{-1} \| \leq \frac{C}{1 - C \times \varepsilon}$.*

2 Continuation Methods in a Banach Manifold

Theorem 4. *Let $f, g\colon M \to \mathbb{R}^n$ be two C^1-mappings,where $M \subset \mathbb{R}^p$, $p > n$ is a compact connected set, which is a C^1-Banach manifold modelled on $\mathbb{R}^n$ and a C^1-atlas for M is $(U_i, \varphi_i)_{i \in I}$, $I = 1, 2, ..., N$. Suppose that the following conditions are fulfilled:*

(i) *Mapping f has only one zero $x^\star$ in M, with $x^* \in U_j$, and*

(ii) *Zero is a regular value of the mappings $f, g, H(\cdot, t)$ for each $t \in [0, 1]$ of the parameter t, where $H\colon M \times [0, 1] \to \mathbb{R}^n$, $H(x, t) := f(x) - tg(x)$.*

(iii) *Any chart (U_i, φ_i), $i \in I$ can be extended an admissible chart (U_i^*, φ_i), $i \in I$, with $\overline{U}_i \subset U_i^*$ and $R' > d(\partial U_{\varphi_i}, \partial U^*_{\varphi_i}) > R$.*

Hence the following statement holds true:

(a) *$f - g$ has a zero $x^{**} \in M$, and there is a continuous mapping $\alpha^*\colon [0, 1] \to M$, with $H(\alpha^*(t), t) = 0, \forall t \in [0, 1], \alpha^*(0) = x^*, \alpha^*(1) = x^{**}$.*

(b) *Furthermore if $H(x, t) \neq 0, \forall (x, t) \in \partial U_j \times [0, 1]$, $j \in I$ then there is a partition $0 = t_0 < t_1 < t_2 < ... < t_N = 1$ of $[0, 1]$, an integer m, and a sequence $\{x^{i,k}_{\varphi_j}\}$ of Newton's iterations defined by:*

$$x^{i,k+1}_{\varphi_j} = x^{i,k}_{\varphi_j} - H_x(x^{i,k}_{\varphi_j}, t_i)^{-1} H(x^{i,k}_{\varphi_j}, t_i), \, x^{1,0}_{\varphi_j} = x^*_{\varphi_j}, x^{i+1,0}_{\varphi_j} = x^{i,m}_{\varphi_j},$$

where

$$k = 0, 1, ..., m - 1 \;\; \text{if} \;\; i \in 1, ..., N - 1 \, , \text{and} \;\; k = 0, 1, ... \, , \text{if} \;\; i = N,$$

which converges towards $x^{**}_{\varphi_j}$.

Proof. $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is provided by the topology given by its operator norm and $\mathbb{R}^n \times \mathbb{R}$ is provided by a product topology.

x^* belongs to U_j, $j \in I$, and $x^*_{\varphi_j} = \varphi_j(x^*)$ is its representative in the chart (U_j, φ_j). The representative mapping of

$$H\colon M \times [0,1] \to \mathbb{R}^n,\, H(x,t) = f(x) - tg(x)$$

in this chart is the C^1-mapping

$$\overline{H}\colon U_{j_{\varphi j}} \times [0,1] \subset \mathbb{R}^n \times [0,1] \to \mathbb{R}^n,\, \overline{H}(x_{\varphi_j}, t) := (H \circ (\varphi_j^{-1}, I))(x_{\varphi_j}, t),$$

with $I(t) = t$. This verifies that $\overline{H}(x^*_{\varphi_j}, 0) = 0$.

(a) For simplicity we will call $\overline{H}$ the representative of H in any chart. We fix any chart of the atlas $(U_i, \varphi_i)_{i\in I}$, for example (U_j, φ_j), which we will call (U, φ). Anologously we call (U^*, φ) the corresponding chart in the atlas $(U_i^*, \varphi_i)_{i\in I}$. It is supposed that (x_a, t_a) is a zero to H and $x_{a_\varphi} \in U_\varphi$, therefore the representative of H,

$$\overline{H}\colon U_\varphi \times [0,1] \subset \mathbb{R}^n \times [0,1] \to \mathbb{R}^n,$$

has the zero (x_{a_φ}, t_a). We will prove there is a neighbourhood A' of t_a and a continuous mapping $\alpha(\cdot)\colon A' \subset [0,1] \to U_\varphi \subset \mathbb{R}^n$ with $\overline{H}(\alpha(t), t) = 0, \forall t \in A'$:

(a1) By hypothesis (ii), $\overline{H}_x(x_\varphi, t)(\cdot) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ maps onto $\mathbb{R}^n$ for any $(x_\varphi, t) \in U^*_\varphi \times [0,1]$, therefore $\operatorname{rank}(\overline{H}_x(x_\varphi, t)) = n$, and hence $\overline{H}_x(x_\varphi, t)$ belongs to $\operatorname{Isom}(\mathbb{R}^n, \mathbb{R}^n)$. We will prove here that there is a real number $C > 0$, such that if $(x_{\varphi_i}, t), i = 1, \ldots, N$ belongs to $(\overline{H}^{-1}(0)) \cap (U^*_{\varphi_i} \times [0,1])$ then $\| \overline{H}_x(x_{\varphi_i}, t)^{-1} \| \leq C$.

Since H is a C^1-mapping, the mapping

$$\overline{H}_x\colon U^*_\varphi \times [0,1] \subset \mathbb{R}^n \times [0,1] \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n),\, (x_\varphi, t) \mapsto \overline{H}_x(x_\varphi, t)$$

is continuous.

From Theorem 2, the inverse formation

$$\beta\colon \operatorname{Isom}(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n),\, \beta(u) = u^{-1}$$

is a continuous mapping.

Since

$$\| \cdot \| \circ\beta \circ \overline{H}_x \colon (\overline{H}^{-1}(0)) \cap (U_\varphi^* \times [0,1]) \to \mathbb{R}, \overline{H}_x(x_\varphi, t)^{-1} \mapsto \|H_x(x_\varphi, t)^{-1}\|$$

is continuous due to being a composition of continuous mappings, and given that $(\overline{H}^{-1}(0)) \cap (U_\varphi^* \times [0,1])$ is a compact set, the Weierstrass Theorem implies that

$$\| \overline{H}_x(x_\varphi, t)^{-1} \| \leq C, \forall (x_\varphi, t) \in (\overline{H}^{-1}(0)) \cap (U_\varphi^* \times [0,1]).$$

The number C has hence been defined to a particulart chart, and since the atlas $(U_i^*, \varphi_i)_{i \in I}$, has N charts, then there are N positive numbers C_i, $i \in I$. We select $C = \min\{C_i : i \in I\}$.

(a2) In this Section we will find two numbers r, r_0, to be used in a later section. Let us define the mapping

$$h \colon \overline{U}_\varphi^* \times [0,1] \times \overline{U}_\varphi^* \times [0,1] \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n,$$
$$h(x_{a_\varphi}, t_a; x_\varphi, t) := \overline{H}_x(x_{a_\varphi}, t_a)(x_\varphi) - \overline{H}(x_{a_\varphi} + x_\varphi, t).$$

Since $\overline{H}$ is a C^1-mapping, h is a composition of continuous mappings, and since $\overline{U}_\varphi^* \times [0,1] \times \overline{U}_\varphi^* \times [0,1]$ is a compact set of the topological space $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$, therefore for any $r > 0$ and for C given in Section (a1), there is a

$$\delta\left(\frac{r}{2C}\right) > 0, \tag{3}$$

such that, if $(x_{a_\varphi}, t_a; x_\varphi, t)$, $(x_{a'_\varphi}, t_{a'_\varphi}; x'_\varphi, t') \in \overline{U}_\varphi^* \times [0,1] \times \overline{U}_\varphi^* \times [0,1]$ with

$$\| (x_{a_\varphi}, t_a; x_\varphi, t) - (x_{a'_\varphi}, t_{a'}; x'_\varphi, t') \| < \delta\left(\frac{r}{2C}\right) \text{ then}$$

$$\| h(x_{a_\varphi}, t_a; x_\varphi, t) - h(x_{a'_\varphi}, t_{a'}; x'_\varphi, t') \| < \frac{r}{2C}. \tag{4}$$

On the other hand, the mapping $h_x \colon \overline{U}_\varphi^* \times [0,1] \times \overline{U}_\varphi^* \times [0,1] \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$,

$$h_x(x_{a_\varphi}, t_a; x_\varphi, t) := \overline{H}_x(x_{a_\varphi}, t_a) - \overline{H}_x(x_{a_\varphi} + x_\varphi, t)$$

is also uniformly continuous in the compact set $\overline{U}_\varphi \times [0,1] \times \overline{U}_\varphi \times [0,1]$, and therefore there is an

$$r := \delta\left(\frac{1}{2C}\right) > 0 \tag{5}$$

such that, if $(x_{a_\varphi}, t_a; x_\varphi, t), (x_{a'_\varphi}, t_{a'_\varphi}; x', t') \in \overline{U}^*_\varphi \times [0,1] \times \overline{U}^*_\varphi \times [0,1]$ with

$$\| (x_{a_\varphi}, t_a; x_\varphi, t) - (x_{a'_\varphi}, t_{a'}; x'_\varphi, t') \| < \delta\left(\frac{1}{2C}\right), \text{ then}$$

$$\| h_x(x_{a_\varphi}, t_a; x_\varphi, t) - h_x(x_{a'_\varphi}, t_{a'_\varphi}; x'_\varphi, t') \| < \frac{1}{2C}. \tag{6}$$

We take r given by Equation (5) and fix $r'_0 := \delta\left(\frac{r}{2C}\right)$ given by Equation (3), and we define the number $r_0 := \min\{r, r'_0\}$. The numbers r, and r_0 have hence been defined in a particular chart, and since the atlas $(U^*_i, \varphi_i)_{i\in I}$ has N charts there are $r_i, r_{0_i}, i \in I$ positive numbers. We select $r = \min\{r_i, R\colon i \in I\}$ and $r_0 = \min\{r_{0_i} : i \in I\}$.

(a3) Let us suppose that the representative (x_{a_φ}, t_a) of (x_a, t_a) in the chart $(U_\varphi \times [0,1], (\varphi, I))$ verifies $(x_{a_\varphi}, t_a) \in \overline{H}^{-1}(0) \cap (U_\varphi \times [0,1])$. Such a point (x_{a_φ}, t_a) will be called a "starting point" in $U^*_\varphi \times [0,1]$.

The sets

$$A' := \{t \in [0,1]\colon \mid t - t_a \mid \leq r_0\}, A := \{x_\varphi \in \mathbb{R}^n\colon \| x_\varphi \| \leq r, \},$$

with r_0, r found in Section (a2), will be associated to the "starting point" (x_{a_φ}, t_a). Since $\| x_\varphi \| \leq r, \forall x_\varphi \in A$, therefore

$$x_\varphi + x_{a_\varphi} \in U^*_\varphi, \forall x_\varphi \in A.$$

Given a "starting point" (x_{a_φ}, t_a), we will prove here the existence of a key continuous mapping

$$\alpha(\cdot)\colon A' \subset \mathbb{R} \to A + x_{a_\varphi} \subset U^*_\varphi \subset \mathbb{R}^n, \text{ such that } \overline{H}(\alpha(t), t) = 0, \forall t \in A'.$$

Let us solve the equation

$$\overline{H}(x_{a_\varphi} + x_\varphi, t) = 0, \tag{7}$$

for fixed $t \in A'$ when x_φ is in A. Obviously $\overline{H}(x_{a_\varphi}, t_a) = 0$. Equation (7) can be transformed into the following equivalent equation

$$\overline{H}_x(x_{a_\varphi}, t_a)^{-1}[\overline{H}_x(x_{a_\varphi}, t_a)(x_\varphi) - \overline{H}(x_{a_\varphi} + x_\varphi, t)] = x_\varphi, \tag{8}$$

which leads us to define the mappings

$$h\colon A \times t \to \mathbb{R}^n, \text{ for fixed } t \in A', h(x_\varphi, t) := \overline{H}_x(x_{a_\varphi}, t_a)(x_\varphi) - \overline{H}(x_{a_\varphi} + x_\varphi, t),$$

and

$$T_t\colon A \to \mathbb{R}^n,\ T_t(x_\varphi) := \overline{H}_x(x_{a_\varphi}, t_a)^{-1} h(x_\varphi, t).$$

Let us observe that $h(x_\varphi, t)$ is defined in Section (a2) as $h(x_{a_\varphi}, t_a; x_\varphi, t)$ when (x_{a_φ}, t_a) is a fixed "starting point", and when t is fixed and belongs to A', and the variable x_φ belongs to A. Let us also observe that t in the definition of T_t is an index and not a partial derivative as is usually written.

Evidently

$$h(0, t_a) = 0, \tag{9}$$

and

$$h_x(0, t_a) = 0. \tag{10}$$

Equation (7) is equivalent to the following key Fixed Point Equation

$$T_t(x_\varphi) = x_\varphi, \tag{11}$$

which will be studied later in the paper.

Let $x_\varphi, x'_\varphi \in A, t \in A'$, and hence the Taylor Theorem together with Equations (6) and (10) imply that

$$\begin{aligned} &\| h(x_\varphi, t) - h(x'_\varphi, t) \| \\ &\leq \sup\{\| h_x(x'_\varphi + \theta(x_\varphi - x'_\varphi), t) \| : \theta \in [0,1]\}. \| x_\varphi - x'_\varphi \| \leq \frac{1}{C} r. \end{aligned} \tag{12}$$

Equations (4) and (9) imply that

$$\| h(x_\varphi, t) \| \leq \| h(x_\varphi, t) - h(0, t_a) \| + \| h(0, t_a) \| \leq \frac{r}{C},$$

hence

$$\| T_t(x_\varphi) \| \leq \| \overline{H}_x(x_{a_\varphi}, t_a)^{-1} \| \| h(x_\varphi, t) \| \leq r. \tag{13}$$

We are already able to prove that the hypotheses of Theorem 1 are verified by the spaces and mappings which have just been defined. The Metric space $(A', |\cdot|)$ is the parameter space of hypothesis (i) of Theorem 1. A is considered as the closed and non-empty set and $\mathbb{R}^n$ as the complete metric space of hypothesis (ii), which is verified below:

From Equation (13), for any fixed $t \in A'$, and for all $x_\varphi \in A$, we obtain $\| T_t x_\varphi \| \leq r$, therefore $T_t x_\varphi \in A$, and hence $T_t\colon A \to A$, i.e. T_t maps the closed and non-empty set of the Banach space $\mathbb{R}^n$ into itself.

From Equations (6), (10), and the Taylor Theorem, for any $x_\varphi, x'_\varphi \in A$, and for any $t \in A'$

$$
\begin{aligned}
\| T_t(x_\varphi) - T_t(x'_\varphi) \| &\leq \| \overline{H}_x(x_{a_\varphi}, t_a)^{-1} \| \| h_x(x'_\varphi + b(x_\varphi - x'_\varphi), t) \| \| x_\varphi - x'_\varphi \| \\
&\leq C \| h_x(x'_{\varphi_\varphi} + b(x_\varphi - x'_\varphi), t) - h_x(0, t_a) \| \| x_\varphi - x'_\varphi \| \\
&\leq \frac{1}{2} \| x_\varphi - x'_\varphi \|, (b \in [0, 1]).
\end{aligned}
$$

Therefore T_t is half-contractive for any fixed $t \in A'$. Hence hypothesis (ii) of Theorem 1 is verified.

For any fixed $t_0 \in A'$ and for all $x_\varphi \in A$,

$$
\begin{aligned}
T_t(x_\varphi) &= \overline{H}_x(x_{a_\varphi}, t_a)^{-1}(\overline{H}_x(x_{a_\varphi}, t_a)(x_\varphi) - \overline{H}(x_{a_\varphi} + x_\varphi, t)) \\
&\quad \to \overline{H}_x(x_{a_\varphi}, t_a)^{-1}(\overline{H}_x(x_{a_\varphi}, t_a)(x_\varphi) - \overline{H}(x_{a_\varphi} + x_\varphi, t_0)) \\
&= T_{t_0}(x_\varphi) \quad \text{as} \quad t \to t_0, t \in A',
\end{aligned}
$$

therefore hypothesis (iii) of Theorem 1 is also verified. Hence Theorem 1 implies, for any $t \in A'$, that T_t has a unique fixed point $x_\varphi \in A$, $T_t(x_\varphi) = x_\varphi := x_\varphi(t)$, and $x_\varphi(t) \to x_\varphi(t_0)$ while $t \to t_0$; $t, t_0 \in A'$, i.e. $x_\varphi(\cdot)$ is a continuous mapping. Thus for any $t \in A'$ there is only one $x_\varphi(t) \in A$ such that

$$
\overline{H}(x_{a_\varphi} + x_\varphi(t), t) = 0, \tag{14}
$$

and furthermore $\overline{H}(x_{a_\varphi} + x_\varphi(t), t) \to \overline{H}(x_{a_\varphi} + x_\varphi(t_0), t_0) = 0$ while $t \to t_0, t, t_0 \in A'$. Let us observe that $T_{t_a}(0) = 0, x_\varphi(t_a) = 0$.

Equation (14) can be written as $\overline{H}(\alpha(t), t) = 0$, which is verified for all $t \in A'$, where α is the continuous mapping

$$
\alpha \colon A' \to U^*_\varphi \subset \mathbb{R}^n, \alpha(t) := x_{a_\varphi} + x_\varphi(t), \quad \text{where} \quad \alpha(t_a) = x_{a_\varphi}. \tag{15}
$$

(b) Conclusion (a) will be proved here. Since $(x^*_\varphi, 0) \in \overline{H}^{-1}(0) \cap (U_\varphi \times [0, 1])$, i.e. $(x^*_\varphi, 0)$ is a "starting point", therefore from Section (a3) there are A, A', α such that

$$
\alpha \colon A' \to A + x^*_\varphi, \overline{H}(\alpha(t), t) = 0, \forall t \in A'.
$$

Let suppose that $\alpha(t) \in U_\varphi, \forall t \in [0, r_0]$, then mapping α is extended to the right of r_0 by taking $(\alpha(r_0), r_0)$ which belongs to $\overline{H}^{-1}(0) \cap (U_\varphi \times [0, 1])$ as the following “starting point”. We also call α the continuous prolonged mapping and by supposing that $\alpha(t) \in U_\varphi, \forall t \in [0, 2r_0]$, we obtain

$$\alpha\colon [0, 2r_0] \to U_\varphi \subset \mathbb{R}^n, \quad \text{which verifies} \quad \overline{H}(\alpha(t), t) = 0, \forall t \in [0, 1].$$

Mapping α is successively extended to the right in the same way if $\alpha(t)$ does not reach ∂U_φ. One of these two cases occurs:

1) Due to $[0, 1]$ being a compact set, there is an

$$\alpha\colon [0, 1] \to U_\varphi^* \subset \mathbb{R}^n, \quad \text{which verifies} \quad \overline{H}(\alpha(t), t) = 0, \forall t \in [0, 1].$$

Therefore a point $(x_\varphi^{**}, 1) \in U_\varphi^* \times [0, 1]$ exists, which verifies

$$\overline{H}(x_\varphi^{**}, 1) = \overline{H}(\alpha(1), 1) = 0,$$

and hence

$$\overline{H}(x_\varphi^{**}, 1) = (H \circ (\varphi^{-1}, I))(x_\varphi^{**}, 1) = H(x^{**}, 1) = 0 \text{ or } f(x^{**}) - g(x^{**}) = 0,$$

and furthermore $\overline{H}(\alpha(t), t) = 0, \forall t \in [0, 1]$. Equivalently

$$(H \circ (\varphi^{-1}, I))(\alpha(t), t) = H(\alpha^*(t), t) = 0, \forall t \in [0, 1], \quad \text{with} \quad \alpha^*(t) := \varphi^{-1} \circ \alpha(t),$$

which is continuous since it is composed of continuous mappings.

2) α^* can only be extended in this way until $t_1 < 1$,where $\alpha(t_1) \in U_\varphi^* = U_{\varphi_j}^*$, and $(\alpha(t_1), t_1)$ is not a starting point in $U_{\varphi_j}^* \times [0, 1]$. Since U_i, U_i^* $i \in I$ are open covers for M, $x_1 = \varphi_j^{-1}(\alpha(t_1)) = \varphi_j^{-1}(x_{1_{\varphi j}})$ verifies $x_1 \in U_j^* \cap U_k$, $j, k \in I$. The continuous mapping $\alpha^*(\cdot)\colon [0, t_1] \to M, \alpha^*(t) = (\varphi_j^{-1} \circ \alpha)(t)$ will now be extended to the right of t_1 in the same way as before, but by using the chart (U_k^*, φ_k) with the point $(x_{1_{\varphi k}}, t_1)$ as the first “starting point” in $U_{\varphi_k}^* \times [0, 1]$, and by using the continuous mapping $\alpha\colon [t_1, t_2] \to U_{\varphi_k}^*$, which verifies $\overline{H}(\alpha(t), t) = 0, \forall t \in [t_1, t_2], \alpha(t_2) = x_{2_{\varphi k}}$. Equation (15) establishes that $t_2 - t_1 \geq r_0$, when the chart at x_2 is changed again, when necessary.

This situation can be repeated, however, since $[0, 1]$ is a compact set α^*, reaches $\alpha^*(1)$ in a finite number of repetitions. Therefore

$$H(\alpha^*(1), 1) = f(\alpha^*(1), 1) - g(\alpha^*(1), 1) = 0.$$

That is, there is an x^{**} such that $f(x^{**}) = g(x^{**})$.

(c) Section (a1) gives $C > 0$ such that if $(x_\varphi, t) \in ((\overline{H}^{-1}(0)) \cap (U_\varphi \times [0,1])$ then $\| \overline{H}_x(x_\varphi, t)^{-1} \| \leq C$. Section (b) and the hypothesis in (b) give a continuous mapping $\alpha^*\colon [0,1] \to \mathbb{R}^n$,with $H(\alpha^*(t), t) = 0, \alpha^*(t) \in M, \forall t \in [0,1]$.

Since H is a C^1-mapping, therefore $\overline{H}_x(x_\varphi, t)\colon U^*_\varphi \times [0, t_1] \to \mathbb{R}^n$ is uniformly continuous on the compact set $\overline{U}_\varphi \times [0,1]$ and hence, for any fixed $\varepsilon \in (0, \frac{1}{3C}), \varepsilon < 1$, the value $\delta > 0$ exists for which

$$\overline{B}(\alpha(t), \delta) \subset \overline{U}_{\varphi_j}, \forall t \in [0,1],$$

and furthermore

$$\begin{gathered} \| \overline{H}_x(x_\varphi, t) - \overline{H}_x(y_\varphi, t) \| < \varepsilon, \forall x_\varphi, y_\varphi \in \overline{U}_\varphi, \\ \text{with } \| x_\varphi - y_\varphi \| < \delta, t \in [0,1]. \end{gathered} \tag{16}$$

Theorem 3 implies that $\overline{H}_x(x_\varphi, t)^{-1}$ exists for any $x_\varphi \in \overline{B}(\alpha(t), \delta)$, and furthermore

$$\| \overline{H}_x(x_\varphi, t)^{-1} \| \leq \frac{C}{1 - C\varepsilon}. \tag{17}$$

The Newton process for any fixed $t \in [0, t_1]$ and $x^0_\varphi \in B(\alpha(t), \delta)$ is

$$x^{k+1}_\varphi = x^k_\varphi - \overline{H}_x(x^k_\varphi, t)^{-1} H(x^k_\varphi, t), k = 0, 1, \ldots \tag{18}$$

It will be proved by induction that

$$\| x^k_\varphi - \alpha(t) \| \leq \gamma^k \delta, k = 0, 1, \ldots, \text{with } \gamma = \frac{C}{1 - C\varepsilon} < 1 : \tag{19}$$

1) Equation (19) is true by assumption for $k = 0$.

2) If Equation (19) is true for any $k \in \mathbb{N}$, and therefore $x^k_\varphi \in \overline{B}(\alpha(t), \gamma^k\delta)$, then, from Equations (16), (17), (19) and Theorem 3 and the Mean-Value Theorem together with $H(\alpha(t), t) = 0, t \in [0,1]$, Equation (19) is true for $k+1$:

$$\begin{gathered} \| \alpha(t) - x^{k+1}_\varphi \| = \| \alpha(t) - x^k_\varphi + \overline{H}_x(x^k_\varphi, t)^{-1} \overline{H}(x^k_\varphi, t) \| \\ \leq \| \overline{H}_x(x^k_\varphi, t)^{-1} \| \, \| \overline{H}(\alpha(t), t) - \overline{H}(x^k_\varphi, t) - \overline{H}_x(x^k_\varphi, t)(\alpha(t) - x^k_\varphi) \| \\ = \| \overline{H}_x(x^k_\varphi, t)^{-1} \| \, \| \overline{H}_x(u, t)(\alpha(t) - x^k) - \overline{H}_x(x^k_\varphi, t)(\alpha(t) - x^k) \| \\ \leq \frac{C}{1 - C\varepsilon} \varepsilon \gamma^k \delta \leq \gamma^{k+1} \delta. \end{gathered} \tag{20}$$

Hence the Newton sequence (18) remains in $\overline{B}(\alpha(t), \delta)$ and converges towards $\alpha(t)$.

Since $\alpha(\cdot)$ is a continuous mapping, therefore there are $t_0, t_1, \ldots, t_L$, $0 = t_0 < t_1 < \ldots < t_L = 1$ for which $\| \alpha(t_{i+1}) - \alpha(t_i) \| \leq \delta' < \delta$, $i = 0, 1, \ldots, L-1$ is verified.

Let $m \in \mathbb{N}$ exists such that $\gamma^m \leq 1 - (\frac{\delta'}{\delta})$. We consider the sequence of Newton's iterations:

$$x_\varphi^{i,k+1} = x_\varphi^{i,k} - \overline{H}_x(x_\varphi^{i,k}, t_i)^{-1}\overline{H}(x_\varphi^{i,k}, t_i),$$

$$x_\varphi^{1,0} = x_\varphi^*, x_\varphi^{i+1,0} = x_\varphi^{i,m}, k = 0, 1, \ldots, m-1, i = 1, 2, \ldots, L-1, \tag{21}$$

$$x_\varphi^{L,k+1} = x_\varphi^{L,k} - H_x(x_\varphi^{L,k}, 1)^{-1}H(x_\varphi^{L,k}, 1), k = 0, 1, \ldots, \tag{22}$$

which verifies:

1) $x_\varphi^* \in B(\alpha(t_0), \delta - \delta')$,

2) If $x_\varphi^{i,0} \in B(\alpha(t_{i-1}), \delta - \delta')$ and since,

$$\| x_\varphi^{i,0} - \alpha(t_i) \| < \| x_\varphi^{i,0} - \alpha(t_{i-1}) \| + \| \alpha(t_{i-1}) - \alpha(t_i) \| < \delta,$$

therefore $x_\varphi^{i,0} \in B(\alpha(t_i), \delta) \subset U_\varphi$, $i = 1, \ldots, L$. Furthermore, from Equation (20),

$$\| x_\varphi^{i+1,0} - \alpha(t_{0,i}) \| = \| x_\varphi^{i,m} - \alpha(t_i) \| \leq \gamma^m \delta \leq \delta - \delta',$$

hence the process from (21) to (22) can be continued upwards where all $x_\varphi^{i,k}$ are in U_φ. Thus the final iteration (22) converges towards $\alpha(1)$.

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José M. Soriano Arbizu
Departamento de Análisis Matemático
Universidad de Sevilla, Aptdo. 1160
Sevilla 41080
SPAIN

E-mail: soriano@us.es