BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 38(1), 81-99

Compact embedded rotation hypersurfaces of S^{n+1}

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Abstract. In this paper, we prove that $S^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times S^1\left(\sqrt{\frac{m}{n}}\right)$ and round geodesic spheres are the only n-dimensional compact embedded rotation hypersurfaces with $H_m = 0$ $(1 \leq m \leq n-1)$ in a unit sphere $S^{n+1}(1)$. When $m = 1$, our result reduces to the result of T. Otsuki [O1], [O2], Brito and Leite [BL].

Keywords: Rotation hypersurfaces, embedded hypersurfaces, Riemannian product.

Mathematical subject classification: Primary: 53C42; Secondary: 53C20.

1 Introduction

Do Carmo and Dajczer [DD] defined the rotation hypersurfaces in space forms and studied the rotation hypersurfaces with constant mean curvature in space forms. Some years later, Leite [LE] classified the complete rotation hypersurfaces with constant scalar curvature in space forms. In [P], Palmas studied the rotation hypersurfaces with constant H_m in space forms, where H_m is the normalized m-th symmetric function of the principal curvatures.

In this paper, we consider the n-dimensional rotation hypersurfaces in a unit sphere $S^{n+1}(1)$ of dimension n+1. The question whether compact rotation hypersurfaces which satisfy some special conditions are embedded is quite interesting. The embeddability of rotation minimal hypersurfaces of $S^{n+1}(1)$ has been treated by Otsuki in a long series of papers that started with [O1]. From a different point of view, Brito and Leite [BL] also considered the embeddability of compact minimal rotation hypersurfaces of $S^{n+1}(1)$. They proved the following important result:

Theorem 1.1 ([O1], [O2], [BL]). *There are no compact minimal embedded rotation hypersurfaces of S^{n+1} other than Clifford tori and round geodesic spheres.*

Received 2 January 2006.

The project is supported by the grant No. 10531090 of NSFC.

From [P], we know that there also exist many compact immersed rotation hypersurfaces of S^{n+1} with $H_m = 0$. From Theorem 1.1, we know that the following problem is interesting:

Problem. *Does there exist any n-dimensional compact embedded rotation hypersurface with $H_m = 0$ $(1 \leq m \leq n-1)$ in S^{n+1} other than Riemannian product$S^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times S^1\left(\sqrt{\frac{m}{n}}\right)$ and round geodesic spheres?*

In this paper, we solve this problem completely. In fact, we prove:

Theorem 1.2. *There are no compact embedded rotation hypersurfaces with $H_m = 0$ $(1 \leq m \leq n-1)$ of S^{n+1} other than the Riemannian product $S^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times S^1\left(\sqrt{\frac{m}{n}}\right)$ and round geodesic spheres,where H_m is the normalized m-th symmetric function of the principal curvatures, $S^{n-1}(a)$ denotes the $(n-1)$-dimensional sphere of radius a.*

Remark 1.1. When $m = 1$, our Theorem 1.2 reduces to Theorem 1.1.

Remark 1.2. Some interesting results for hypersurfaces with $H_m = constant$ $(1 \leq m \leq n-1)$ in space forms can be found in [ADE], [ADS], [BC], [BD], [CY], [DE], [HL], [L1], [L2].

2 Preliminaries

Let M be a rotation hypersurface of S^{n+1}, that is, invariant by the orthogonal group $O(n)$ considered as a subgroup of isometries of $S^{n+1}(1)$. Let us parametrize the profile curve α in $S^2(1)$ by $y_1 = y_1(s) \geq 0$, $y_{n+1} = y_{n+1}(s)$ and $y_{n+2} = y_{n+2}(s)$. We take $\varphi(t_1, \cdots, t_{n-1}) = (\varphi_1, \cdots, \varphi_n)$ as an orthogonal parametrization of the unit sphere $S^{n-1}(1)$. It follows that the rotation hypersurface (see [DD])

$$x : M^n \hookrightarrow S^{n+1}(1) \subset R^{n+2},$$

$$(s, t_1, \cdots, t_{n-1}) \mapsto (y_1(s)\varphi_1, \cdots, y_1(s)\varphi_n, y_{n+1}(s), y_{n+2}(s)). \tag{2.1}$$

$$\varphi_i = \varphi_i(t_1, \cdots, t_{n-1}), \quad \varphi_1^2 + \cdots + \varphi_n^2 = 1 \tag{2.2}$$

is a parametrization of a rotation hypersurface generated by a curve $y_1(s)$, $y_{n+1}(s)$ and $y_{n+2}(s)$. Since the curve $\{y_1(s), y_{n+1}(s), y_{n+2}(s)\}$ belongs to $S^2(1)$ and the parameter s can be chosen as its arc length, we have

$$y_1^2(s) + y_{n+1}^2(s) + y_{n+2}^2(s) = 1, \quad \dot{y}_1^2(s) + \dot{y}_{n+1}^2(s) + \dot{y}_{n+2}^2(s) = 1 \tag{2.3}$$

where the dot denotes the derivative with respect to s and from (2.3) we can obtain $y_{n+1}(s)$ and $y_{n+2}(s)$ as functions of $y_1(s)$. In fact, we can write

$$\begin{aligned} y_1(s) &= \cos r(s), \\ y_{n+1}(s) &= \sin r(s)\cos\theta(s), \\ y_{n+2}(s) &= \sin r(s)\sin\theta(s). \end{aligned} \tag{2.4}$$

We can deduce from (2.3) that

$$\dot{r}^2 + \dot{\theta}^2\sin^2 r = 1. \tag{2.5}$$

It follows from equation (2.5) that $\dot{r}^2 \leq 1$. Combining these with $\dot{r}^2 = \frac{\dot{y}_1^2}{1-y_1^2}$, we have

$$\dot{y}_1^2 + y_1^2 \leq 1. \tag{2.6}$$

We can get the plane curve γ from α by projection of $S^2_+ = \{(y_1, y_{n+1}, y_{n+2}) \mid y_1 \geq 0, y_1^2 + y_{n+1}^2 + y_{n+2}^2 = 1\}$ onto the unit disk $E = \{(y_{n+1}, y_{n+2}) \mid y_{n+1}^2 + y_{n+2}^2 \leq 1\}$. Then the plane curve γ can be written as

$$y_{n+1}(s) = \sin r(s)\cos\theta(s), \quad y_{n+2}(s) = \sin r(s)\sin\theta(s). \tag{2.7}$$

The parameter $\widetilde{s}$ can be chosen as γ's arc length. Let $h(\widetilde{s})$ be the supporting function of γ, $\widetilde{\theta}$ is the oriented angle between two vectors $y_{n+1}-$axis and tangent direction of γ (see Figure 2), then we can obtain by (2.3)

$$(d\widetilde{s})^2 = (dy_{n+1})^2 + (dy_{n+2})^2 = (ds)^2 - (dy_1)^2, \tag{2.8}$$

$$\tan\widetilde{\theta} = \frac{\dot{y}_{n+2}(s)}{\dot{y}_{n+1}(s)}. \tag{2.9}$$

Writing $f(s) = y_1(s)$, do Carmo and Dajczer proved the following

Lemma 2.1 ([DD]). *Let M^n be a rotation hypersurface of $S^{n+1}(1)$. Then the principal curvatures λ_i of M^n are*

$$\lambda_i = \lambda = -\frac{\sqrt{1 - f^2 - \dot{f}^2}}{f} \tag{2.10}$$

for $i = 1, \cdots, n-1$, and

$$\lambda_n = \mu = \frac{\ddot{f} + f}{\sqrt{1 - f^2 - \dot{f}^2}}. \tag{2.11}$$

Let H_m be the normalized m-th symmetric function of the principal curvatures of an hypersurface M:

$$C_n^m H_m = \sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_m} \tag{2.12}$$

where $C_n^m = \frac{n(n-1)\cdots(n-m+1)}{m(m-1)\cdots 1}$, λ_i are the principal curvatures of M.

If M is a rotation hypersurface with $H_m = 0$ $(m < n)$ in $S^{n+1}(1)$, then we can deduce that

$$0 = C_n^m H_m = C_{n-1}^{m-1} \lambda^{m-1} \mu + C_{n-1}^m \lambda^m$$

That is,

$$\lambda^{m-1}\{(n-m)\lambda + m\mu\} = 0. \tag{2.13}$$

By putting (2.10) and (2.11) into (2.13), we get the following result of Oscar Palmas [P]:

Lemma 2.2 ([P]). *The rotation hypersurface M^n in $S^{n+1}(1)$ has $H_m = 0$ $(m < n)$ if and only if f satisfies the following differential equation:*

$$(n-m)(1 - f^2 - \dot{f}^2)^{\frac{m}{2}} - m(1 - f^2 - \dot{f}^2)^{\frac{m-2}{2}}(\ddot{f} + f) f = 0. \tag{2.14}$$

Equation (2.14) is equivalent to its first order integral

$$f^{n-m}(1 - f^2 - \dot{f}^2)^{\frac{m}{2}} = K, \tag{2.15}$$

where K is a constant.

For a constant solution $f = f_0$ in (2.14), one has that

$$f_0^2 = \frac{n-m}{n}, \quad K_0 = \left(\frac{m}{n}\right)^{\frac{m}{2}} \left(\frac{n-m}{n}\right)^{\frac{m(n-m)}{2n}}. \tag{2.16}$$

Moreover, the constant solutions of equation (2.14) correspond to the Riemannian product $S^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times S^1\left(\sqrt{\frac{m}{n}}\right)$.

Now we follow the techniques in the paper of Leite [LE] and Palmas [P] to study (2.15). Equation (2.15) tells us that a local solution f of (2.14) paired with its first derivative is a subset, denoted by $(f, \dot{f})$, of a level curve for the function G_m defined by

$$G_m(u, v) = u^{n-m}(1 - u^2 - v^2)^{\frac{m}{2}}, \tag{2.17}$$

with $u > 0$ and $u^2 + v^2 \le 1$.

Let us map the open half plane $\{(u, v) \mid u > 0\}$ by level curve $G_m = K$ (see Figure 1). Each curve is a smooth union of two graphs

$$v = \pm\sqrt{1 - u^2 - \left(\frac{K}{u^{n-m}}\right)^{2/m}}, \tag{2.18}$$

except for the level K_0 given by (2.16). The level curve $G_m = K_0$ consists of the unique critical point of G_m, which is on the horizontal axis, as it can be seen from

$$\nabla G_m(u, v) = u^{n-m-1}\left(1-u^2-v^2\right)^{\frac{m-2}{2}}\left((n-m)(1-v^2)-nu^2, -muv\right). \tag{2.19}$$

For $K = 0$, the level curve $u^2 + v^2 = 1$ is a semi-circle. For $K \neq 0$, we can get easily that the level curve is closed in the open half plane (in fact, in the semicircular region, see Figure 1).

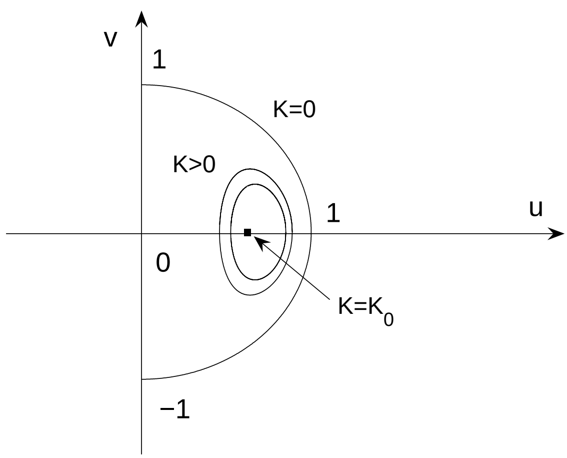

Figure 1: Level curves for $K \geq 0$.

We consider the foliation of the open half plane by level curves $G_m = K$. Since G_m has a maximum at K_0, $K \in [0, K_0]$. Clearly any curve at an intermediate level K is compact and the associated solutions $r(s)$ attains a unique minimum $r_1 > 0$.

Now we have to consider three cases.

Case 1: $K = K_0$.

The value $K = K_0$ implies $\lambda_1 = \cdots = \lambda_{n-1} = -\sqrt{\frac{m}{n-m}}$, $\lambda_n = \sqrt{\frac{n-m}{m}}$ by (2.10), (2.11) and (2.16), corresponding to the Riemannian product $S^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times S^1\left(\sqrt{\frac{m}{n}}\right)$.

Case 2: $K = 0$.

$K = 0$ gives us a totally geodesic n-sphere. In fact, from $K = 0$ and equation (2.15), we get $f^2 + \dot{f}^2 = 1$. Integration of $f^2 + \dot{f}^2 = 1$ with $f(0) = 0$, we obtain $f = \sin s$ and θ =constant, so the profile curve is a great circle which generates a totally geodesic n-sphere.

Case 3: $K \in (0, K_0)$.

If $K \in (0, K_0)$, then we have

$$f^2 + \dot{f}^2 < 1, \quad 0 < f < 1, \quad f \neq \text{constant}. \tag{2.20}$$

It follows from $f(s) = y_1(s) = \cos r(s)$ that

$$0 < \cos r < 1, \quad 0 < \sin r < 1. \tag{2.21}$$

Using $f(s) = \cos r(s)$, (2.5) can be written as

$$\dot{\theta}^2 = \frac{1 - \dot{r}^2}{\sin^2 r} = \frac{1 - f^2 - \dot{f}^2}{(1 - f^2)^2}, \tag{2.22}$$

we can deduce from (2.20) and (2.22) that

$$\dot{\theta} \neq 0, \quad \dot{r} < 1. \tag{2.23}$$

We see from (2.10) and (2.20) that $\lambda = -\frac{\sqrt{1-f^2-\dot{f}^2}}{f} \neq 0$, then it follows from (2.13) that

$$(n - m)\lambda + m\mu = 0. \tag{2.24}$$

From (2.10), (2.11), (2.24) and $f(s) = y_1(s) = \cos r(s)$, we can deduce that

$$\ddot{r} = (1 - \dot{r}^2)\left(\cot r - \frac{n - m}{m}\tan r\right). \tag{2.25}$$

Without loss of generality, from (2.8), (2.21) and (2.23), we have

$$\frac{d\widetilde{s}}{ds} = \sqrt{1 - \sin^2 r(\theta)\dot{r}^2} > 0. \tag{2.26}$$

From (2.9), we get

$$\frac{d\widetilde{\theta}}{ds} = \frac{\ddot{y}_{n+2}\dot{y}_{n+1} - \dot{y}_{n+2}\ddot{y}_{n+1}}{\dot{y}_{n+1}^2 + \dot{y}_{n+2}^2}.$$

Hence
$$\frac{d\widetilde{\theta}}{d\widetilde{s}} = \frac{d\widetilde{\theta}}{ds}\frac{ds}{d\widetilde{s}} = \frac{B}{\sqrt{1-\dot{r}^2\sin^2 r}\,(\dot{y}_{n+1}^2+\dot{y}_{n+2}^2)}, \tag{2.27}$$
where $B := \ddot{y}_{n+2}\dot{y}_{n+1} - \dot{y}_{n+2}\ddot{y}_{n+1}$.

Next we will prove $\frac{d\widetilde{\theta}}{d\widetilde{s}} \neq 0$ in order to show that $\widetilde{\theta}$ can be chosen as a parameter of the plane curve γ. From (2.27), we know that we only need to prove $B \neq 0$.

By a direct calculation, we obtain from (2.4)

$$\dot{y}_{n+1} = \dot{r}\cos r\cos\theta - \dot{\theta}\sin r\sin\theta,\ \ \dot{y}_{n+2} = \dot{r}\cos r\sin\theta + \dot{\theta}\sin r\cos\theta, \tag{2.28}$$

$$\begin{aligned} B =& \ddot{y}_{n+2}\dot{y}_{n+1} - \dot{y}_{n+2}\ddot{y}_{n+1} = \dot{r}^2\dot{\theta} + \dot{r}^2\dot{\theta}\cos^2 r \\ &+ \dot{r}\ddot{\theta}\sin r\cos r - \ddot{r}\dot{\theta}\sin r\cos r + \dot{\theta}^3\sin^2 r. \end{aligned} \tag{2.29}$$

Combining (2.29) with (2.5), we have

$$B = \dot{\theta} + \dot{r}^2\dot{\theta}\cos^2 r + \dot{r}\ddot{\theta}\sin r\cos r - \ddot{r}\dot{\theta}\sin r\cos r. \tag{2.30}$$

Taking derivative of (2.5), we get

$$\dot{r}\ddot{r} + \dot{r}\dot{\theta}^2\sin r\cos r + \dot{\theta}\ddot{\theta}\sin^2 r = 0. \tag{2.31}$$

Next, we have to consider two subcases.

Subcase 3.1: $\dot{r} = 0$.

Combining $\dot{r} = 0$ with (2.25), we see that

$$\begin{aligned} B &= \dot{\theta} + \dot{r}^2\dot{\theta}\cos^2 r + \dot{r}\ddot{\theta}\sin r\cos r - \ddot{r}\dot{\theta}\sin r\cos r \\ &= \dot{\theta} - \ddot{r}\dot{\theta}\sin r\cos r \\ &= \dot{\theta}[1 - \sin r\cos r(\cot r - \frac{n-m}{m}\tan r)] \\ &= \frac{n}{m}\dot{\theta}\sin^2 r. \end{aligned}$$

From (2.21) and (2.23), we obtain

$$B \neq 0. \tag{2.32}$$

Subcase 3.2: $\dot{r} \neq 0$.

If $\dot{r} \neq 0$, we can deduce from (2.31) that

$$\ddot{r} = -\frac{\sin^2 r}{\dot{r}}\dot{\theta}\ddot{\theta} - \dot{\theta}^2 \sin r \cos r. \tag{2.33}$$

Then we see from (2.5), (2.30) and (2.33) that

$$\begin{aligned}
B &= \dot{\theta} + \dot{r}^2\dot{\theta}\cos^2 r + \dot{r}\ddot{\theta}\sin r\cos r + \dot{\theta}\sin r\cos r\left(\frac{\sin^2 r}{\dot{r}}\dot{\theta}\ddot{\theta} + \dot{\theta}^2\sin r\cos r\right)\\
&= \dot{\theta} + \dot{\theta}\cos^2 r + \ddot{\theta}\left(\dot{r}\sin r\cos r + \frac{\sin^3 r\cos r}{\dot{r}}\dot{\theta}^2\right)\\
&= \dot{\theta}(1+\cos^2 r) + \ddot{\theta}\left(\dot{r}\sin r\cos r + \frac{\sin r\cos r}{\dot{r}}(1-\dot{r}^2)\right)\\
&= \dot{\theta}(1+\cos^2 r) + \ddot{\theta}\frac{\sin r\cos r}{\dot{r}}.
\end{aligned}$$

Combining (2.25) and (2.33), we have

$$\ddot{\theta} = -\frac{\dot{r}\dot{\theta}}{\sin r\cos r}\left(2\cos^2 r - \frac{n-m}{m}\sin^2 r\right). \tag{2.34}$$

Substituting (2.34) into above equation, we can deduce that

$$\begin{aligned}
B &= \dot{\theta}(1+\cos^2 r) + \ddot{\theta}\frac{\sin r\cos r}{\dot{r}}\\
&= \dot{\theta}(1+\cos^2 r) - \dot{\theta}(2\cos^2 r - \frac{n-m}{m}\sin^2 r)\\
&= \frac{n}{m}\dot{\theta}\sin^2 r\\
&\neq 0.
\end{aligned}$$

Therefore we can obtain $\frac{d\widetilde{\theta}}{d\widetilde{s}} \neq 0$ in Case 3.

We only need to consider the question in Case 3 whether compact rotation hypersurfaces with $H_m = 0$ are nontrivial embedded (that is, except Riemannian product and round geodesic spheres).

3 The Rotation Hypersurfaces with $H_m = 0$ in Case 3

In this section, we will consider our problem in Case 3.

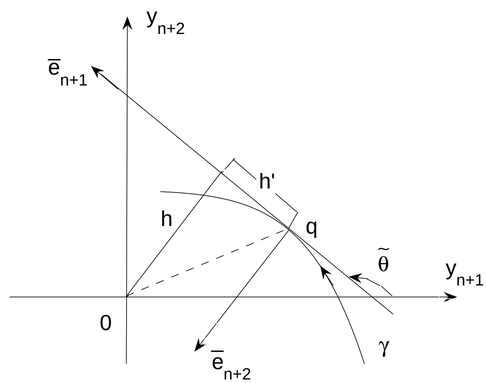

Figure 2: Plane curve γ.

Since $\frac{d\widetilde{\theta}}{d\widetilde{s}} \neq 0$, the plane curve γ can be written in the form $h = h(\widetilde{\theta})$ (see Figure 2). Let $h^{'}$ and $h^{''}$ denote $\frac{dh}{d\widetilde{\theta}}$ and $\frac{d^2h}{d\widetilde{\theta}^2}$ respectively.

From the definition of h, we have (see Figure 2)

$$h = (y_{n+1} - \frac{1}{\tan\widetilde{\theta}} y_{n+2}) \sin\widetilde{\theta} = y_{n+1} \sin\widetilde{\theta} - y_{n+2} \cos\widetilde{\theta},$$

it follows from the above equation and (2.9) that $h^{'} = y_{n+1} \cos\widetilde{\theta} + y_{n+2} \sin\widetilde{\theta}$, then $y_{n+1} = h \sin\widetilde{\theta} + h^{'} \cos\widetilde{\theta}$, $\;y_{n+2} = -h\cos\widetilde{\theta} + h^{'} \sin\widetilde{\theta}$ and the generic point $q(\widetilde{\theta})$ of γ is given by

$$\begin{aligned} q(\widetilde{\theta}) &= (0, \cdots, 0, y_{n+1}(\widetilde{s}), y_{n+2}(\widetilde{s})) \\ &= (0, \cdots, 0, h \sin\widetilde{\theta} + h^{'} \cos\widetilde{\theta}, h^{'} \sin\widetilde{\theta} - h \cos\widetilde{\theta}). \end{aligned} \tag{3.1}$$

It follows from (3.1) that

$$y^2_{n+1}(\widetilde{s}) + y^2_{n+2}(\widetilde{s}) = h^2 + (h^{'})^2. \tag{3.2}$$

From (2.3) and (3.2), we have $f = y_1 = \sqrt{1 - h^2 - (h^{'})^2}$.
By (2.20) and (3.2), we can obtain

$$0 < h^2 + (h^{'})^2 < 1. \tag{3.3}$$

Let $(\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_{n+2})$ be the moving orthonormal frame of R^{n+2} with the following conditions (c.f. [O1]):

$$\overline{e}_n = (\varphi_1, \cdots, \varphi_n, 0, 0), \quad \varphi_1^2 + \cdots + \varphi_n^2 = 1, \tag{3.4}$$

$$\overline{e}_{n+1} = (0, \cdots, 0, \cos\widetilde{\theta}, \sin\widetilde{\theta}), \quad \overline{e}_{n+2} = (0, \cdots, 0, -\sin\widetilde{\theta}, \cos\widetilde{\theta}), \tag{3.5}$$

where $(\varphi_1, \cdots, \varphi_n)$ is an orthogonal parametrization of the unit sphere.

We put

$$d\overline{e}_i = \sum_{j=1}^{n} \overline{\omega}_{ij}\overline{e}_j, \quad \overline{\omega}_{ij} + \overline{\omega}_{ji} = 0. \tag{3.6}$$

Then from (2.1), (3.1), (3.4) and (3.5), we know that the position vector p of the rotation hypersurface M^n in $S^{n+1}(1)$ can be written as

$$p = f\overline{e}_n + h'\overline{e}_{n+1} - h\overline{e}_{n+2} = q + f\overline{e}_n. \tag{3.7}$$

The arc length $\widetilde{s}$ of γ is given by

$$d\widetilde{s} = (h + h^{''})d\widetilde{\theta}. \tag{3.8}$$

Using $\overline{e}_{n+1}$ and $\overline{e}_{n+2}$, we have

$$q = h'\overline{e}_{n+1} - h\overline{e}_{n+2}, \quad dq = \overline{e}_{n+1}d\widetilde{s}. \tag{3.9}$$

By means of (3.5), (3.6) and (3.9), we have

$$dp = f\sum_{a=1}^{n-1} \overline{\omega}_{na}\overline{e}_a + df\overline{e}_n + (h + h^{''})d\widetilde{\theta}\overline{e}_{n+1}.$$

Putting

$$\begin{aligned} &e_a = \overline{e}_a, \quad e_n = \frac{f^{'}\overline{e}_n + (h + h^{''})\overline{e}_{n+1}}{\sqrt{(f^{'})^2 + (h + h^{''})^2}}, \\ &\omega_a = f\overline{\omega}_{na}, \quad \omega_n = \sqrt{(f^{'})^2 + (h + h^{''})^2}d\widetilde{\theta}, \end{aligned} \tag{3.10}$$

where $f^{'} = \frac{df}{d\widetilde{\theta}}$ and $1 \le a \le n-1$.

The above equation can be written as

$$dp = \sum_{i=1}^{n} \omega_i e_i. \tag{3.11}$$

From (3.2), by a direct calculation, we get

$$h'(h + h^{''}) + ff^{'} = 0, \tag{3.12}$$

it follows from (3.12) that

$$(f^{'})^2 + (h + h^{''})^2 = \frac{1 - h^2}{f^2}(h + h^{''})^2.$$

Therefore ω_n and e_n can be written as

$$\omega_n = \frac{\sqrt{1-h^2}}{f}(h+h^{''})d\widetilde{\theta} = \frac{\sqrt{1-h^2}}{f}d\widetilde{s}, \tag{3.13}$$

$$e_n = \frac{1}{\sqrt{1-h^2}}(-h'\overline{e}_n + f\overline{e}_{n+1}). \tag{3.14}$$

By a simple computation, we can choose the normal unit vector of M in S^{n+1} by

$$e_{n+1} = -\frac{h}{\sqrt{1-h^2}}(f\overline{e}_n + h'\overline{e}_{n+1}) - \sqrt{1-h^2}\overline{e}_{n+2}. \tag{3.15}$$

If we take $e_{n+2} = -p$ as normal unit vector of S^{n+1} in R^{n+2}, then $\{e_1, \cdots, e_{n+2}\}$ make a basis in R^{n+2} with the same orientation of $\{\overline{e}_1, \cdots, \overline{e}_{n+2}\}$.

Now, from (3.10) and (3.14) we obtain

$$\begin{aligned}\omega_{an+1} &= -\langle e_a, De_{n+1}\rangle = -\langle e_a, de_{n+1}\rangle \\ &= \frac{hf}{\sqrt{1-h^2}}\langle e_a, d\overline{e}_n\rangle = \frac{hf}{\sqrt{1-h^2}}\overline{\omega}_{na} = \frac{h}{\sqrt{1-h^2}}\omega_a,\end{aligned}$$

that is

$$\omega_{an+1} = \lambda\omega_a, \quad \lambda = \frac{h}{\sqrt{1-h^2}}, \tag{3.16}$$

where $1 \leq a \leq n-1$, D denotes the covariant differentiation on $S^{n+1}(1)$. Then we get

$$\begin{aligned}\omega_{nn+1} &= -\langle e_n, De_{n+1}\rangle = -\langle e_n, de_{n+1}\rangle \\ &= \frac{1}{\sqrt{1-h^2}}\langle -h'\overline{e}_n + f\overline{e}_{n+1}, d(-e_{n+1})\rangle \\ &= \left\{\frac{h}{1-h^2}(fh^{''} - f^{'}h') - f\right\} d\widetilde{\theta}.\end{aligned}$$

Using (3.12), we obtain

$$\omega_{nn+1} = \left\{\frac{h(h+h^{''})}{f} - \frac{f}{1-h^2}\right\} d\widetilde{\theta} = \mu\omega_n. \tag{3.17}$$

By (3.13) and (3.17), we have

$$\mu = \frac{h}{\sqrt{1-h^2}} - \frac{1-h^2-(h')^2}{(h+h'')\sqrt{(1-h^2)^3}}. \tag{3.18}$$

Using (3.16) and (3.18), the condition (2.24) can be given by

$$(n-m)\frac{h}{\sqrt{1-h^2}}+m\left\{\frac{h}{\sqrt{1-h^2}}-\frac{1-h^2-(h')^2}{(h+h'')\sqrt{(1-h^2)^3}}\right\}=0.$$

That is

$$\frac{n}{m}h(1-h^2)h^{''}+(h')^2+h^2-1+\frac{n}{m}h^2(1-h^2)=0. \tag{3.19}$$

Conversely, if a function $h(\widetilde{\theta})$ satisfying (3.19) gives a plane curve in R^{n+2} by (3.1), then by (3.7) we get a rotation hypersurface $M^n \hookrightarrow S^{n+1}(1)$ with $H_m=0$. The properties of this hypersurface M^n completely depend on the properties of $h(\widetilde{\theta})$.

In the following, we will investigate the properties of the ordinary differential equation (3.19) of second order.

Writing $F=h^2+(h^{'})^2$. From (3.3), we obtain $0<F<1$.

By a direct calculation, we have

$$\begin{aligned}\frac{1}{2}\frac{dF}{d\widetilde{\theta}}&=hh^{'}+h^{'}h^{''}\\&=\frac{mh^{'}}{nh(1-h^2)}(1-F),\end{aligned}$$

hence

$$\frac{dF}{1-F}=\frac{2m}{n}\left\{\frac{1}{h}+\frac{1}{2(1-h)}-\frac{1}{2(1+h)}\right\}dh. \tag{3.20}$$

Integrating (3.20), we get

$$1-F=C\left(\frac{h^2}{1-h^2}\right)^{-\frac{m}{n}}, \qquad C=\text{constant}>0,$$

that is

$$\left(\frac{dh}{d\widetilde{\theta}}\right)^2=1-h^2-C\left(\frac{1}{h^2}-1\right)^{\frac{m}{n}}. \tag{3.21}$$

In this case, we can deduce from $\lambda=\frac{h}{\sqrt{1-h^2}}\neq 0$ that $h>0$. From (3.3), we have $h<1$.

Hence we will only consider the solutions of (3.19) such that

$$0<h(\widetilde{\theta})<1, \quad 0<h^2+(h^{'})^2<1 \tag{3.22}$$

From (2.20), we have $h \neq$ constant, otherwise, $f =$ constant. This is a contradiction. For non-constant solution $h(\widetilde{\theta})$ of (3.21) with (3.22), its range is given by

$$1 - h^2 - C(\frac{1}{h^2} - 1)^{\frac{m}{n}} \geq 0,$$

which is a closed interval $a_0 \leq h \leq a_1$, $0 < a_0 < a_1 < 1$, where a_0, a_1 are the two solutions of the equation

$$1 - h^2 - C\left(\frac{1}{h^2} - 1\right)^{\frac{m}{n}} = 0.$$

It follows that

$$a_0 < \sqrt{\frac{m}{n}} < a_1.$$

Lemma 3.1. *$h(\widetilde{\theta})$ is periodic with respect to $\widetilde{\theta}$.*

Proof. Any solution $h(\theta)$ of (3.19) such that $0 < h < 1$ will be obtained by integrating the following equation

$$\left(\frac{dh}{d\widetilde{\theta}}\right)^2 = 1 - h^2 - C\left(\frac{1}{h^2} - 1\right)^{\frac{m}{n}},$$

where C is a positive constant. Since the range of x such that

$$1 - x - C\left(\frac{1}{x} - 1\right)^{\frac{m}{n}} \geq 0, \quad 0 < x < 1 \tag{3.23}$$

is given by the set of points of the curve $y = \left(\frac{1}{x} - 1\right)^{\frac{m}{n}}$ $(0 < x < 1)$ beneath the line $y = \frac{1}{C}(1 - x)$. As easily seen, this curve intersects at two points with this line through (1,0) when

$$0 < C < (1 - \alpha)^{1-\alpha}\alpha^{\alpha}, \tag{3.24}$$

where $\alpha = \frac{m}{n}$. And let the x-coordinates of the above two points x_0, x_1, then

$$0 < x_0 < \alpha < x_1 < 1. \tag{3.25}$$

Therefore, we get

$$a_0 = \sqrt{x_0} \leq h(\widetilde{\theta}) \leq a_1 = \sqrt{x_1}.$$

The minimum and maximum of $h(\widetilde{\theta})$ must be a_0 and a_1 because at such points $h^{'}(\widetilde{\theta}) = 0$. Furthermore, $y = h(\widetilde{\theta})$ is symmetric with respect to $\widetilde{\theta} = \widetilde{\theta}_0, \widetilde{\theta}_1$, where $a_0 = h(\widetilde{\theta}_0)$, $a_1 = h(\widetilde{\theta}_1)$. From (3.21), we can easily see that $h(\widetilde{\theta})$ is periodic and its minimal positive period is given by

$$T(C) = 2\int_{a_0}^{a_1} \frac{dh}{\sqrt{1 - h^2 - C(\frac{1}{h^2} - 1)^{\frac{m}{n}}}}. \tag{3.26}$$

we complete the proof of Lemma 3.1.

Denote the solution of (3.21) by $h(\widetilde{\theta}, C)$ and the hypersurface immersed in S^{n+1} corresponding to $h(\widetilde{\theta}, C)$ by $M^n(C)$.

$M^n(C)$ is the compact embedded rotation hypersurface in $S^{n+1}(1)$ if and only if the minimum positive period $T(C)$ of the solution $h(\widetilde{\theta}, C)$ is $\frac{2\pi}{k}$ $(k = 1, 2, \ldots)$.

4 Proof of Theorem

The proof of Theorem 1.2. It is sufficient to prove $\pi < T(C) < 2\pi$.

Lemma 4.1. $T(C) > \pi$.

Proof. Putting $h^2 = x,\ (a_0)^2 = x_0,\ (a_1)^2 = x_1,\ \frac{m}{n} = \alpha < 1$, we get

$$T(C) = \int_{x_0}^{x_1} \frac{dx}{\sqrt{x(1-x) - Cx^{1-\alpha}(1-x)^{\alpha}}}, \tag{4.1}$$

Putting

$$g(x) = x(1-x) - Cx^{1-\alpha}(1-x)^{\alpha},$$

we have

$$g^{'}(x) = 1 - 2x - \frac{C(1-\alpha-x)}{x^{\alpha}(1-x)^{1-\alpha}}, \tag{4.2}$$

$$g^{''}(x) = -2 + \frac{C\alpha(1-\alpha)}{x^{1+\alpha}(1-x)^{2-\alpha}} > -2 \tag{4.3}$$

and $\{x \mid g(x) \geq 0, 0 < x < 1\} = [x_0, x_1]$. Clearly, by (4.2), $g(x)$ takes its maximum on $[x_0, x_1]$ at one point $x = x_2$, $x_0 < x_2 < x_1$. Let $g(x_2) = b > 0$. Now putting

$$L(x) = (g^{'}(x))^2 + 4g(x),$$

we obtain

$$L^{'}(x) = 2(g^{''}(x) + 2)g^{'}(x).$$

Hence by (4.2) and (4.3), $L(x)$ is increasing on $[x_0, x_2]$ and decreasing on $[x_2, x_1]$ and its maximum is $L(x_2) = 4g(x_2) = 4b$, therefore $(g')^2 < 4(b-g)$ on (x_0, x_2) and (x_2, x_1). Thus we get

$$\begin{aligned} T(C) &= \int_{x_0}^{x_2} \frac{dx}{\sqrt{g(x)}} + \int_{x_2}^{x_1} \frac{dx}{\sqrt{g(x)}} \\ &= \int_{x_0}^{x_2} \frac{g'(x)dx}{\sqrt{g(x)(g'(x))^2}} - \int_{x_2}^{x_1} \frac{g'(x)dx}{\sqrt{g(x)(g'(x))^2}} \\ &> \int_0^b \frac{dg}{\sqrt{g(b-g)}} = [\sin^{-1} \frac{2g-b}{b}]_0^b = \pi. \end{aligned}$$

Lemma 4.2. $T(C) < 2\pi$.

Proof. Putting $h^2 = x,\ (a_0)^2 = x_0,\ (a_1)^2 = x_1,\ \frac{m}{n} = \alpha < 1$, we obtain

$$T(C) = \int_{x_0}^{x_1} \frac{dx}{\sqrt{x(1-x) - C\psi(1-x)}}, \tag{4.4}$$

where

$$\psi(x) = x^\alpha (1-x)^{1-\alpha}, \quad \text{on } 0 < x < 1 \tag{4.5}$$

and

$$C = \psi(x_0) = \psi(x_1), \quad 0 < x_0 < \alpha < x_1 < 1, \tag{4.6}$$

It is clear that

$$\psi(x)\psi(1-x) = x(1-x), \tag{4.7}$$

$$\frac{d\psi(x)}{dx} = \frac{\alpha - x}{x(1-x)}\psi(x), \tag{4.8}$$

$$\frac{d\psi(1-x)}{dx} = \frac{1-\alpha-x}{x(1-x)}\psi(1-x). \tag{4.9}$$

Since $\psi(x)$ is monotone increasing on $0 < x < \alpha$ and monotone decreasing on $\alpha < x < 1$. Let $X_0(u)$ and $X_1(u)$ be the inverse functions of $u = \psi(x)$ on $0 < x < \alpha$ and $\alpha < x < 1$ respectively. Thus

$$\begin{aligned} T(C) &= \int_{x_0}^{\alpha} \frac{dx}{\sqrt{x(1-x) - C\psi(1-x)}} + \int_{\alpha}^{x_1} \frac{dx}{\sqrt{x(1-x) - C\psi(1-x)}} \\ &= \int_{x_0}^{\alpha} \frac{\sqrt{x^{1-\alpha}(1-x)^\alpha}}{\sqrt{x^\alpha(1-x)^{1-\alpha} - C}} x^{\alpha-1}(1-x)^{-\alpha} dx \end{aligned}$$

$$+\int_{\alpha}^{x_1}\frac{\sqrt{x^{1-\alpha}(1-x)^{\alpha}}}{\sqrt{x^{\alpha}(1-x)^{1-\alpha}-C}}x^{\alpha-1}(1-x)^{-\alpha}dx$$

$$=\int_{x_0}^{\alpha}\frac{\sqrt{x(1-x)}}{(\alpha-x)\sqrt{x^{\alpha}(1-x)^{1-\alpha}[x^{\alpha}(1-x)^{1-\alpha}-C]}}(\alpha-x)x^{\alpha-1}(1-x)^{-\alpha}dx$$

$$+\int_{\alpha}^{x_1}\frac{\sqrt{x(1-x)}}{(\alpha-x)\sqrt{x^{\alpha}(1-x)^{1-\alpha}[x^{\alpha}(1-x)^{1-\alpha}-C]}}(\alpha-x)x^{\alpha-1}(1-x)^{-\alpha}dx$$

$$=\int_{C}^{A}\frac{\sqrt{X_0(u)(1-X_0(u))}}{(\alpha-X_0(u))\sqrt{u(u-C)}}du+\int_{A}^{C}\frac{\sqrt{X_1(u)(1-X_1(u))}}{(\alpha-X_1(u))\sqrt{u(u-C)}}du$$

$$=\int_{C}^{A}\frac{\sqrt{X_0(u)(1-X_0(u))(A-u)}}{(\alpha-X_0(u))\sqrt{u}}\frac{du}{\sqrt{(A-u)(u-C)}}$$

$$+\int_{C}^{A}\frac{\sqrt{X_1(u)(1-X_1(u))(A-u)}}{(X_1(u)-\alpha)\sqrt{u}}\frac{du}{\sqrt{(A-u)(u-C)}}.$$

Now, we assume that

$$\frac{\sqrt{X_i(u)(1-X_i(u))(A-u)}}{\mid\alpha-X_i(u)\mid\sqrt{u}}<\lambda_i \tag{4.10}$$

for $C<u<A$, $i=0,1$. Then, we obtain

$$T(C)<(\lambda_0+\lambda_1)\int_{C}^{A}\frac{du}{\sqrt{(A-u)(u-C)}}=(\lambda_0+\lambda_1)\pi. \tag{4.11}$$

In the following, we will prove that we can take the values of λ_0 and λ_1 as $\lambda_0=\lambda_1=1$. The inequalities (4.10) are equivalent to

$$\sqrt{x(1-x)(A-\psi(x))}<\lambda_i\mid\alpha-x\mid\sqrt{\psi(x)} \tag{4.12}$$

for $x_0<x<\alpha$ and $\alpha<x<x_1$ respectively. Setting $\lambda=\lambda_i$, (4.12) can be read

$$x(1-x)(A-\psi(x))<\lambda^2(\alpha-x)^2\psi(x),$$

that is

$$x(1-x)A<\psi(x)[\lambda^2(\alpha-x)^2+x(1-x)]. \tag{4.13}$$

Putting $f_\lambda(x)=\frac{\lambda^2(\alpha-x)^2+x(1-x)}{\psi(1-x)}$. By (4.7), the inequality (4.13) can be written as

$$A<f_\lambda(x). \tag{4.14}$$

For this positive valued function $f_\lambda(x)$ on $0 < x < 1$ for any $\lambda > 0$, we have

$$f_\lambda(\alpha) = A, \tag{4.15}$$

and

$$\begin{aligned}\frac{f_\lambda'}{f_\lambda} &= \frac{-2\lambda^2(\alpha - x) + 1 - 2x}{\lambda^2(\alpha - x)^2 + x(1-x)} - \frac{1-\alpha-x}{x(1-x)} \\ &= \frac{g_\lambda(x)}{x(1-x)[\lambda^2(\alpha-x)^2 + x(1-x)]},\end{aligned}$$

where

$$f_\lambda' = \frac{d(f_\lambda)}{dx}, \quad g_\lambda(x) = (\alpha - x)[-\lambda^2\alpha(1-\alpha) + (1-\lambda^2)x(1-x)]. \tag{4.16}$$

If $\lambda = 1$, then we get

$$g_\lambda(x) = (x - \alpha)\alpha(1-\alpha). \tag{4.17}$$

When $x \in (x_0, \alpha)$, we obtain $g_\lambda(x) < 0$, then $f_\lambda(x)$ is a strictly monotone decreasing function of x in (x_0, α). When $x \in (\alpha, x_1)$, we get $g_\lambda(x) > 0$, then $f_\lambda(x)$ is a strictly monotone increasing function of x in (α, x_1). Hence

$$f_\lambda(x) > f_\lambda(\alpha) = A, \quad \text{for } x \in (x_0, \alpha) \bigcup (\alpha, x_1).$$

Hence we have

$$\pi < T(C) < 2\pi.$$

From Case 1, Case 2 and Case 3, we can get our result. This completes the proof of Theorem 1.2.

Finally, we also mention the following fact. In [LE], Leite proved that there exists many complete immersed rotation hypersurfaces of S^{n+1} with constant scalar curvature $n(n-1)$. i.e. $H_2 = 0$. She also asked the following

Problem 4.1. *Are there embedded hypersurfaces of S^{n+1} with $H_2 = 0$ other than product of spheres?*

Our theorem 1.2 gives a partial answer to her problem. In fact, we prove the following

Corollary 4.1. *There are no compact embedded rotation hypersurfaces M with constant scalar curvature $n(n-1)$ of S^{n+1} other than the Riemannian product $S^{n-1}\left(\sqrt{\frac{n-2}{n}}\right) \times S^1\left(\sqrt{\frac{2}{n}}\right)$ and round geodesic spheres.*

The proof of Corollary 4.1. From Theorem 1.2, we can easily get our result. In fact, $H_2 = 0$ is equivalent to that M has constant scalar curvature $n(n-1)$.

Remark 4.1. The referee tells us that Dr. Fernando Espinosa in his Ph.D. Thesis at Universidade Federal do Ceará (paper not yet published) has proved some rigidity results for k-umbilical hypersurfaces in S^{n+1}, where a hypersurface is called k-umbilical if $AP_{k-1}(A) = \lambda I$, where A is the second fundamental form and P_{k-1} is the $(k-1)$-th Newton polynomial.

Acknowledgements. This work was started since H. Li visited University of Sao Paulo in July of 2004, he would like to express his thanks to FADESP for financial support and Professor Fabiano Brito for his hospitality. The authors express their thanks for Prof. M. Leite for her kindness to send us her reprints about this topics. We also would like to thank the referee for some helpful comments and suggestions.

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