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Affine interval exchange transformation without an isometric model

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Abstract. Contrary to the case of interval exchange transformation, we show that generalized affine interval exchange transformation (affine GIET), with or without flips and admitting dense orbits, may not be conjugated to an isometric GIET. This result is proved by constructing explicitly one such affine GIET.

Keywords: Generalized affine interval exchange transformation, no topological conjugation, isometric interval exchange transformation, recurrent orbits.

Mathematical subject classification: 37E05, 37A05, 37C15.

1 Introduction

Let C denote the circle S^1 and let us consider the usual topology on $C \times \{-1, 1\}$. A continuous injective map $E : C \times \{-1, 1\} \to C \times \{-1, 1\}$ whose domain ($\mathrm{Dom}(E)$) and range ($\mathrm{Im}(E)$) are open and dense subsets of $C \times \{-1, 1\}$ is said to be a *generalized interval exchange transformation*, or shortly a GIET, if E takes homeomorphically each connected component of $\mathrm{Dom}(E)$ onto a connected component of $\mathrm{Im}(E)$. If in particular E is affine (resp. isometric) in each such connect component, then E is said to be an *affine* GIET (resp. *isometric* GIET). If the restriction of E to some component of $\mathrm{Dom}(E)$ reverses the orientation, then we say that E admits a *flip*. An interval exchange transformation (IET) is an injective continuous transformation $T : C \to C$ such that $C \setminus \mathrm{Dom}(T)$ is a finite set and T is isometric in each connect component of $\mathrm{Dom}(T)$. An IET T is a particular case of a GIET. This can be seen by considering $E(x, 1) = (T(x), 1)$ and $E(x, -1) = (T^{-1}(x), -1)$. We remark also that identify the endpoints of interval $[0, 1]$, we can define a GIET to $C = [0, 1]$ (this follows because a GIET

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is not defined in a compact and totally disconnected set of points). Here and subsequently C will denote the circle S^1 but it would change nothing if we take $C = [0, 1]$.

Let $E : C \times \{-1, 1\} \to C \times \{-1, 1\}$ be a GIET and $\delta \in \{-1, 1\}$. For each point $p \in C$ such that $(p, \delta) \in \mathrm{Dom}(E)$ we will define the $E(\cdot, \delta)$-orbit of p as being

$$\{(p, \delta)\} \cup \{E^n(p, \delta); n \in \mathbb{Z}^+ \setminus \{0\} \quad \text{and} \quad E^{n-1}(p, \delta) \in \mathrm{Dom}(E)\}$$

where we use the notation $E^0(p, \delta) = (p, \delta)$ and $E^n(p, \delta) = E(E^{n-1}(p, \delta))$. The E*-orbit* of p is defined as the union of its $E(\cdot, 1)$- and $E(\cdot, -1)$-orbits. A point $p \in C$ is said to be $E(\cdot, \delta)$*-recurrent* if it is an accumulation point of its $E(\cdot, \delta)$-orbit. A point $p \in C$ is *recurrent* if it is $E(\cdot, 1)$- and $E(\cdot, -1)$-recurrent. An $E(\cdot, \delta)$-orbit is $E(\cdot, \delta)$*-recurrent* if it is the $E(\cdot, \delta)$-orbit of a $E(\cdot, \delta)$-recurrent point. A non-trivial recurrent point is a recurrent one whose orbit is not periodic. It is easy to check that if p is a non-trivial E- (resp. $E(\cdot, \delta)$-) recurrent point then the topological closure of its E- (resp. $E(\cdot, \delta)$-) orbit will be a perfect set. If in particular it is a Cantor set, then the E- (resp. $E(\cdot, \delta)$-) orbit of p will be said to be an exceptional orbit (resp. an exceptional $E(\cdot, \delta)$-orbit). An open interval $I \subset C$ is said to be a wandering interval to E if any E-orbit intersects I at most once.

Briefly, GIET's and foliations on two-manifolds are related as follows. Let $E : C \times \{-1, 1\} \to C \times \{-1, 1\}$ be a GIET and let $\sigma : C \times \{-1, 1\} \to C \times \{-1, 1\}$ be the map defined by $\sigma(x, \delta) = (x, -\delta)$. If $\sigma \circ E \circ \sigma \circ E$ is the identity map then through a process of suspension it is possible to obtain a foliation on a two-manifold containing C as a section, in such a way that E will be the return map on $C \times \{-1, 1\}$ induced by the foliation. We observe that if $E(x, \delta) = (y, \theta)$ then the condition on E imply that necessarily $E(y, -\theta) = (x, -\delta)$ (this allows the existence of an arc of leaf $L(x, y)$ starting at x with the orientation δ and ending at y with the orientation θ, such that $L(x, y) \cap C = \{x, y\}$). In [5] Gutierrez introduces a condition (called of *condition* S) to obtain a partial structure theorem connecting nonorientable foliations with GIET's. We remark that the referred conditions on E are necessary for the "connection" of E with nonorientable foliations. For details on the suspension of a GIET defined on $C \times \{-1, 1\}$, except possibly at finitely many points, see [13]. For when the GIET is not defined in an infinite many points see [5]. The suspension of IET's is described in [1], [9], [10]. The obtention of the "inverse process", on the other hand, is less immediate. In this direction, the most general result for flows (orientable foliations) on compact 2-manifolds was obtained by C. Gutierrez in his structure theorem ([6]). Essentially, Gutierrez shows that the first return map induced by a recurrent non-periodic orbit (recurrent non-compact leaf) on an adequate transversal circle is either "topologically conjugate" or "topologically semi-conjugate" to an IET which admits recurrent non-periodic orbits. On the other hand, the higher freedom that a flow can reach on a non-compact two-manifold implies that such theorem is not immediate in the non-compact case (in [8] non-compact two-manifolds admitting a dense subset of exceptional leaves are shown). Nevertheless, in the same way as Gutierrez's structure theorem, a structure theorem for flows on non-compact two-manifolds was obtained in [11]. In that case, however, the conjugation obtained is realizable with an affine GIET and not with an isometric GIET.

The structure theorem of [6] and [11] are respectively a consequence of the following results

Lemma ([6]). *Let $E: S^1 \to S^1$ be a continuous injective map defined everywhere except at finitely many points. If E has a dense positive semi-orbit, then E is topologically conjugate to an interval exchange transformation.*

Proposition ([11]). *Let $E: S^1 \to S^1$ be a continuous injective map defined everywhere except in a compact, totally disconnected set of points. If E admits a dense subset of non-trivial recurrent points, then E is topologically conjugate to an affine GIET.*

In the present paper, is constructed explicitly an affine GIET (with or without flips and admitting dense orbits) which is not topologically conjugate to an isometric GIET. Thus, the structure theorem in [11] cannot be improved; that is, the conjugation of a continuous injective map (admitting a dense orbit) to an isometric continuous injective map obtained by Gutierrez is not valid, in general, when the set where the map is not defined is infinite. On the other hand, Arnoux-Ornstein-Weiss ([2]) show that generalized isometric interval exchange transformations can model any aperiodic measure-preserving transformation.

Briefly, we will start with a Denjoy map T, we will perturb T in a family of wandering subintervals (using Rosenberg's Labyrinths to obtain a transformation with flips, respectively an irrational rotation to obtain a transformation without flips) in such a way that we will obtain a transformation that admits dense orbits but does not admit any invariant probability measure with full support. Since, by construction, this transformation will permit a suspension, then we have that the associated relation between recurrent orbits (recurrent leaves) and the isometric IET's obtained in [6] (for flows and orientable foliations on compact two-manifolds) fails when flows on non-compact two-manifolds and nonorientable foliations on compact ones are considered.

We remark that the understanding of the structure of flows and foliations on two-manifolds has shown interesting consequences (see [6], [7], [12]).

Let us start by fixing some notations. Fix an orientation on C (remembering that C is the circle S^1). To differentiate the notation between an open subinterval of C and an ordered pair of $C \times \{-1, 1\}$, we will use the symbol $\langle x, y\rangle$ to denote the open subinterval of C of endpoints x and y with $x \prec y$. Here $\prec$ denote the linear order induced by the fixed orientation on C. Similarly to the construction of Denjoy C^1-diffeomorphisms associated to an irrational rotation of the circle C, we can obtain (see Lema 2.1 in [8]) a homeomorphism T defined at C which admit an wandering open interval, say $\langle a_0, b_0\rangle$, whose T-orbit is dense at C. Thus, let $\{I_{n_k} = \langle a_{n_k}, b_{n_k}\rangle\}_{k=-\infty}^{\infty}$ be a subfamily of open intervals of $\{\langle a_k, b_k\rangle = T^k(\langle a_0, b_0\rangle);\ k \in \mathbb{Z}\}$ so that the following properties are satisfied

(i) $n_0 = 0$, $\{n_k\}_{k\in\mathbb{Z}^+}$ is an increasing sequence of positive integers and $\{n_{-k}\}_{k\in\mathbb{Z}^+}$ is a decreasing sequence of negative integers;

(ii) The endpoints of I_{n_k} converge monotonely to b_0 as $k \to +\infty$ and to b_{-1} as $k \to -\infty$; and

(iii) $m_k = n_{k+1} - 1 - n_k$; $k = 0, 1, 2 \ldots$ is an increasing sequence of positive integers such that
$$\sum_{k=0}^{+\infty} \frac{m_k}{2^k} = \infty\,.$$

2 An Affine GIET with flips without an isometric model

Let $1/2 < \alpha < 1$ be an irrational number. Consider half-discs D_1, D_2, and D_3 (of diameter α, 1, and $1-\alpha$ respectively) foliated by half-circles. As in example 1 of [14], let $D = D_1 \cup D_2 \cup D_3$ (see Fig. 1) in such a way that we have a Rosenberg's labyrinths on D. Denote by R_i the holonomy map in each D_i; $i = 1, 2, 3$ and denote by $R_\alpha : [0, 1] \to [0, 1]$ the second return map: $R_1 \circ R_2$, $R_3 \circ R_2$. Notice that each R_i preserves the Lebesgue measure and that the leaf starting at α is dense in D. This follows because after identifying 0 and 1 we see that R_α is the irrational rotation by α (see [14]).

Let $\{p_k\}_{k=0}$ be an increasing sequence of rational numbers converging to α, and let $\{r_k\}_{k=0}$ and $\{q_k\}_{k=0}$be decreasing sequences of rational numbers converging to 0 and α respectively, so that for each $k \geq 0$ we have that

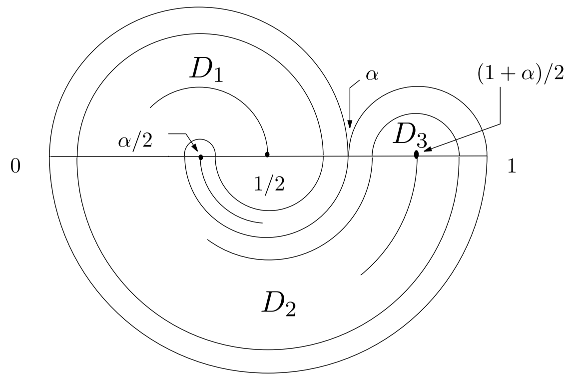

Figure 1:

(2.1) $0 < r_k < \alpha/2,\ \ 1/2 < p_k < \alpha < q_k < (1+\alpha)/2$, and $\lambda(\langle p_{k+1}, q_{k+1}\rangle) = \lambda(\langle p_k, q_k\rangle)/2$, where λ denotes the Lebesgue measure.

Before initiating the construction of our special map, let us illustrate our basic idea. Using the wandering intervals $\{T^k\langle a_0, b_0\rangle;\ k \in \mathbb{Z}\}$, we will build by suspensions an infinite vertical rectangle (strip) with vertical flow which admits $I_k = T^k\langle a_0, b_0\rangle;\ k \in \mathbb{Z}$ as a countable family of sections intervals. We will take the sub-family I_{n_k} (which by condition (ii) will converge to the endpoints b_0 of I_{n_0} and b_{-1} of $T^{-1}\langle a_0, b_0\rangle$ as $k \to \infty$ and $-\infty$ respectively) and will modify the vertical flow on I_{n_k} by figure 3(a) for positive k, and 3(b) for negative k, in such a way that the return map to the interval I_{n_0} will be the return one defined by the labyrinth on D (see Fig. 1). Geometrically speaking (see Fig. 2), if we follow each leaf starting at a point $x \in I_{n_0}$ with the orientation $+1$, then will we go up along the strip till we get caught in the labyrinth at some level. This always happens (except if $x = g_0(\alpha)$ which goes upward at every step) since after a long time we will be close to some discontinuity, that is, after time n_k we are out of the interval $[g_k \circ R_1(p_k), g_k(p_k)]$ or $[g_k(q_k), g_k \circ R_3(q_k)] \subset I_{n_k}$ (g_k will be an adequate gluing map of $\langle 0, 1\rangle$ onto I_{n_k}). After finding the labyrinth in the refereed level, we will go downward back (through the leaf) to the initial interval I_{n_0}, and then down again till we get caught in the labyrinth forcing the return to I_{n_0}, starting thereafter the whole dynamic again. All this will be done in such a way that the return map on the section I_{n_0} be uniquely ergodic, being the Lebesgue measure the unique invariant measure. As this map will be defined at $\langle g_0(p_k), g_0(q_k)\rangle \subset I_{n_0}$ and in its n_k iterates, and as $\langle g_0(p_k), g_0(q_k)\rangle;\ k = 0, 1, 2, \ldots$ is a decreasing sequence of intervals with $m_k = n_{k+1} - 1 - n_k$ having a strong growth, then the obtained map could not be conjugate to an isometric GIET.

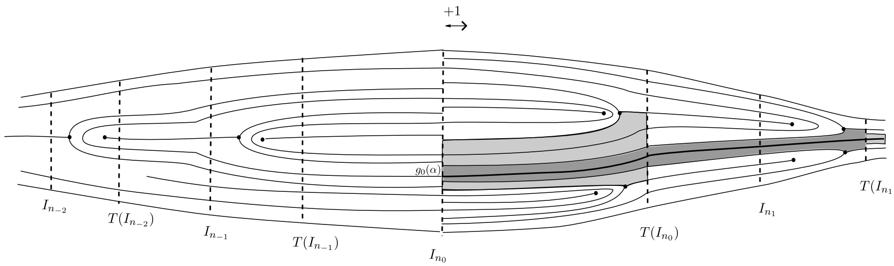

Figure 2: $\langle g_0(p_0), g_0(q_0)\rangle \supset \langle g_0(p_1), g_0(q_1)\rangle$ are the transversal sections, contained in I_{n_0}, of the shaded quadrilaterals. The first return map on I_{n_0} will be the return one defined by the labyrinth on D.

Now we will formalize all this. To simplify the writing let us use the following notation. For each open subinterval $\Sigma \subset C$, $\phi_\Sigma \colon \langle 0, 1\rangle \to \Sigma$ will denote the linear oriented homeomorphism between $\langle 0, 1\rangle$ and Σ.

Under this consideration, let $g_0 \colon \langle 0, 1\rangle \to I_{n_0}$ be the map defined by $g_0 = \phi_{I_{n_0}}$ and let $h_k \colon \mathrm{Dom}(h_k) \to T(I_{n_k})$; $k = 0, 1, 2\ldots$ be a sequence of orientation preserving piecewise affine homeomorphisms recursively characterized as follows

$$\mathrm{Dom}(h_k) = I_{n_k} \setminus \{[g_k \circ R_1(p_k), g_k(p_k)], [g_k(q_k), g_k \circ R_3(q_k)]\};$$

$$\mathrm{Im}(h_k) = T(I_{n_k}) \setminus \{\phi_{T(I_{n_k})}(\alpha/2), \phi_{T(I_{n_k})}((1+\alpha)/2)\}, \text{ and}$$

$$h_k(g_k(\alpha)) = \phi_{T(I_{n_k})}(\alpha)$$

where for each $k \geq 1$, $g_k \colon \mathrm{Dom}(g_k) \to I_{n_k}$ denotes the map

$$g_k = T^{n_k - n_{k-1} - 1} \circ h_{k-1} \circ g_{k-1}$$

with $\mathrm{Dom}(g_k) = \langle 0, 1\rangle \setminus \{[R_1(p_k), p_k], [q_k, R_3(q_k)]\}$.

Similarly, define the map $g_{-1} \colon \langle 0, 1\rangle \to T(I_{n_{-1}})$ by $g_{-1} = T^{n_{-1}+1} \circ g_0$, and take a sequence of orientation preserving piecewise affine homeomorphisms $h_{-k} \colon \mathrm{Dom}(h_{-k}) \to T(I_{n_{-k}})$; $k = 1, 2, \ldots$ recursively characterized as follows

$$\mathrm{Dom}(h_{-k}) = I_{n_{-k}} \setminus \{\phi_{I_{n_{-k}}}(1/2)\} \text{ and,}$$

$$\mathrm{Im}(h_{-k}) = T(I_{n_{-k}}) \setminus [g_{-k}(r_k), g_{-k} \circ R_2(r_k)]$$

where for each integer $k \geq 2$, $g_{-k} \colon \mathrm{Dom}(g_{-k}) \to T(I_{n_{-k}})$ denotes the map

$$g_{-k} = T^{n_{-k}-n_{-k+1}+1} \circ h_{-k+1}^{-1} \circ g_{-k+1}$$

with $\mathrm{Dom}(g_{-k}) = \langle 0, 1\rangle \setminus [r_k, R_2(r_k)]$.

Now let us consider the following definition. Let $\sigma \colon C \times \{-1, 1\} \to C \times \{-1, 1\}$ be the map defined by $\sigma(x, \delta) = (x, -\delta)$. Let that $E \colon C \times \{-1, 1\} \to C \times \{-1, 1\}$ be an injective continuous map whose $\mathrm{Dom}(E)$ is an open set. Let $I \subset C \times \{-1, 1\}$. We shall say that $E|_I$ is an involution of I if $E(\mathrm{Dom}(E|_I)) = \mathrm{Dom}(E|_I)$ and, where it is defined, $E|_I \circ E|_I$ is the identity. The connection between involutions maps, nonorientable foliations and first return map can be found in works of Danthony and Nogueira (see [3]), and of Gutierrez in [5].

Under these notations and considerations, we are now in position to define our special continuous injective map $E \colon C \times \{-1, 1\} \to C \times \{-1, 1\}$ as follows:

- $E|_{I_{n_0} \times \{1\}}(x, 1) =$

$$\begin{cases} (g_0 \circ R_1 \circ g_0^{-1}(x), -1), & \text{if; } x \in \langle g_0 \circ R_1(p_0), g_0(p_0)\rangle \setminus \{g_0(\alpha/2)\} \\ (g_0 \circ R_3 \circ g_0^{-1}(x), -1), & \text{if; } x \in \langle g_0(q_0), g_0 \circ R_3(q_0)\rangle \setminus \{g_0((1+\alpha)/2)\} \\ (h_0(x), 1), & \text{if. } x \in I_{n_0} \setminus \{[g_0 \circ R_1(p_0), g_0(p_0)] \cup [g_0(q_0), g_0 \circ R_3(q_0)]\} \end{cases}$$

notice that

$$\sigma \circ E|_{\langle g_0 \circ R_1(p_0), g_0(p_0)\rangle \times \{1\}}, \quad \text{and} \quad \sigma \circ E|_{\langle g_0(q_0), g_0 \circ R_3(q_0)\rangle \times \{1\}}$$

are involutions of

$$\langle g_0 \circ R_1(p_0), g_0(p_0)\rangle \times \{1\} \quad \text{and} \quad \langle g_0(q_0), g_0 \circ R_3(q_0)\rangle \times \{1\}$$

respectively (see Fig. 3(a))

- for all $k \geq 1$, $E|_{I_{n_k} \times \{1\}}(x, 1) =$

$$\begin{cases} (g_k \circ R_1 \circ g_k^{-1}(x), -1), & \text{if; } x \in \langle g_k \circ R_1(p_k), g_k(p_k)\rangle \setminus \\ & \qquad \{T^{n_k - n_{k-1} - 1} \circ \phi_{T(I_{n_{k-1}})}(\alpha/2)\} \\ (g_k \circ R_3 \circ g_k^{-1}(x), -1), & \text{if; } x \in \langle g_k(q_k), g_k \circ R_3(q_k)\rangle \setminus \\ & \qquad \{T^{n_k - n_{k-1} - 1} \circ \phi_{T(I_{n_{k-1}})}((1+\alpha)/2)\} \\ (h_k(x), 1), & \text{if. } x \in I_{n_k} \setminus \{[g_k \circ R_1(p_k), g_k(p_k)] \cup [g_k(q_k), g_k \circ R_3(q_k)]\} \end{cases}$$

notice that

$$\sigma \circ E|_{\langle g_k \circ R_1(p_k), g_k(p_k)\rangle \times \{1\}}, \quad \text{and} \quad \sigma \circ E|_{\langle g_k(q_k), g_k \circ R_3(q_k)\rangle \times \{1\}}$$

are involutions of

$$\langle g_k \circ R_1(p_k), g_k(p_k)\rangle \times \{1\} \quad \text{and} \quad \langle g_k(q_k), g_k \circ R_3(q_k)\rangle \times \{1\}$$

respectively (see Fig. 3(a))

- for all $k \geq 1$, $E|_{T(I_{n_{-k}})\times\{-1\}}(x,-1) =$

$$\begin{cases} (\, g_{-k} \circ R_2 \circ g_{-k}^{-1}(x), 1\,), & \text{if; } x \in \langle g_{-k}(r_k), g_{-k} \circ R_2(r_k)\rangle \setminus \\ & \qquad \{T^{n_{-k}-n_{-(k-1)}+1} \circ \phi_{I_{n_{-(k-1)}}}(1/2)\} \\ (\, h_{-k}^{-1}(x), -1\,), & \text{if. } x \in T(I_{n_{-k}}) \setminus [g_{-k}(r_k), g_{-k} \circ R_2(r_k)] \end{cases}$$

notice that

$$\sigma \circ E|_{\langle g_{-k}(r_k), g_{-k}\circ R_2(r_k)\rangle \times\{-1\}}$$

is an involutions of

$$\langle g_{-k}(r_k), g_{-k} \circ R_2(r_k)\rangle \times \{-1\}$$

(see Fig. 3(b)).

- If Σ is a connected component of $C \setminus \bigcup_{k\in\mathbb{Z}} I_{n_k}$, then (see Fig 3(c))

$$E|_{\Sigma\times\{1\}}(x,1) = (\,T(x), 1\,), \text{ for all } x \in \Sigma.$$

Deleting from C the points b_0, b_{-1}, $T(b_0)$ and $T(b_{-1})$ it follows from the definition of E that

(2.2) $E|_{C\times\{1\}}$ is not defined at b_0, b_{-1}, $\phi_{I_{n_0}} \circ R_1(p_0)$, $\phi_{I_{n_0}}(p_0)$, $\phi_{I_{n_0}} \circ R_3(q_0)$, $\phi_{I_{n_0}}(q_0)$, $\phi_{I_{n_0}}(\alpha/2)$, $\phi_{I_{n_0}}((\alpha+1)/2)$, and at $\phi_{I_{n_{-k}}}(1/2)$, $T^{n_k-n_{k-1}-1} \circ \phi_{T(I_{n_{k-1}})}(\alpha/2)$, $T^{n_k-n_{k-1}-1} \circ \phi_{T(I_{n_{k-1}})}((\alpha+1)/2)$, $g_k \circ R_1(p_k)$, $g_k(p_k)$, $g_k \circ R_3(q_k))$ and $g_k(q_k)$ for all $k \geq 1$;

(2.3) The map $E|_{C\times\{-1\}}$ is not defined at $T(b_0)$, $T(b_{-1})$, and at $\phi_{T(I_{n_k})}(\alpha/2)$, $\phi_{T(I_{n_k})}((\alpha+1)/2)$, $T^{n_{-k}+n_{-(k-1)}+1} \circ \phi_{I_{n_{-(k-1)}}}(1/2)$, $g_{-k} \circ R_2(r_k)$ and $g_{-k}(r_k)$, for all $k \geq 1$;

(2.4) $C \setminus \mathrm{Dom}(E)$ is a totally disconnected compact set whose accumulation set is $\{b_0, b_{-1}, T(b_0), T(b_{-1})\}$

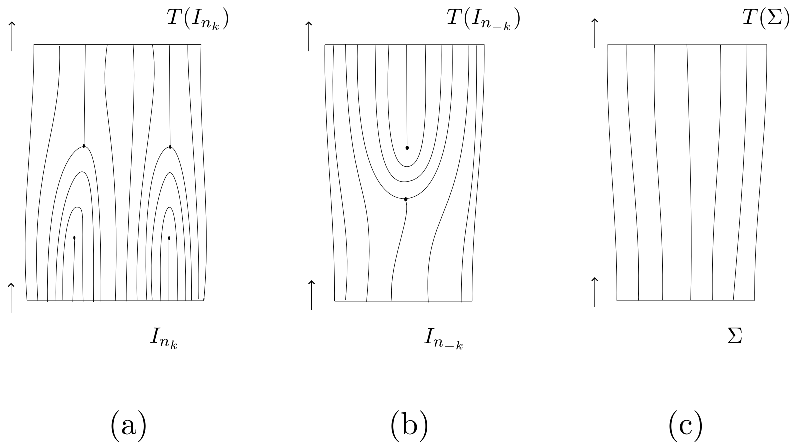

Figure 3:

The following Lemma is an immediate consequence from the definition of E.

Lemma 2.1. $E : C \times \{-1, 1\} \to C \times \{-1, 1\}$ *is an injective continuous map with exceptional and dense orbits whose domain is an open and dense subset. Besides, it satisfies the following properties:*

(a) R_1, R_2 *and* R_3 *are respectively the first return map of* $E(\cdot, 1)$ *to* $\langle a_0, g_0(\alpha)\rangle$, *of* $E(\cdot, -1)$ *to* $I_{n_0} = \langle a_0, b_0\rangle$, *and of* $E(\cdot, 1)$ *to* $\langle g_0(\alpha), b_0\rangle$. *Here* R_i *denotes the holonomy map induced by Rosenberg's labyrinths on* D_i *(see beginning of section 2);*

(b) *The* $E(\cdot, -1)$ *and* $E(\cdot, 1)$*-orbit of* $g_0(\alpha)$ *are infinite. More precisely, its* $E(\cdot, -1)$*-orbit is* $E(\cdot, -1)$*-recurrent (it is dense in* C*) but its* $E(\cdot, 1)$*-orbit is not* $E(\cdot, 1)$*-recurrent;*

(c) *The* E*-orbit of* a_0 *is an exceptional one contained in the closure of* $E(\cdot, -1)$*-orbit of* $g_0(\alpha)$*;*

(d) $C \setminus Dom(E)$ *is a totally disconnected compact set whose accumulation set is* $\{b_0, b_{-1}, T(b_0), T(b_{-1})\}$.

Now, we are in condition to show our first result

Proposition 2.2. *There exists a continuous injective map from circle* S^1 *to* S^1 *which admit flips, dense orbits, and it is topologically conjugate to an affine GIET, but it is not topologically conjugate to an isometric GIET.*

The domain of the refereed map is an open and dense subset of S^1 and (necessarily) it is not defined on an infinite set.

We remark that the foliation obtained by suspension of E on the two-torus necessarily must have infinite singularities.

Proof of Proposition 2.2. We will prove that the map E defined above satisfies the required properties. From the previous Lemma, we need only to prove that E is topologically conjugate to an affine GIET and it is not topologically conjugate to an isometric GIET.

By construction of E it follows that for each $k \in \mathbb{Z}^+ \cup \{0\}$ and x belonging to the open interval $\langle g_0(p_k), g_0(q_k)\rangle$, we have that $E^i(x, 1)$ is defined for all $i \in \{0, 1, 2, \ldots, n_{k+1}\}$. Moreover, for each $k \in \mathbb{Z}^+ \cup \{0\}$, $i \in \{n_k, n_k + 1, \ldots, n_{k+1} - 1\}$, and $j \in \{n_{k+1}, n_{k+1} + 1, \ldots, n_{k+2} - 1\}$

$$E^i(\langle g_0(p_k), g_0(q_k)\rangle, 1) \quad \text{and} \quad E^j(\langle g_0(p_{k+1}), g_0(q_{k+1})\rangle, 1)$$

are disjoint open subintervals of C, i.e., $\{E^i(\langle g_0(p_k), g_0(q_k)\rangle, 1);\ k \in \mathbb{Z}^+ \cup \{0\},\ n_k \le i \le n_{k+1} - 1\}$ is a family of pairwise disjoint open intervals. Notice that this is possible because p_k, q_k satisfy (2.1), and because the $E(\cdot, 1)$-orbit of $g_0(\alpha)$ is not $E(\cdot, 1)$-recurrent. Consequently, we can affirm that

$$\sum_{k=0}^{+\infty} \sum_{i=n_k}^{n_{k+1}-1} \mu(E^i(\langle g_0(p_k), g_0(q_k)\rangle, 1)) < +\infty \tag{2.5}$$

where $\mu = \lambda \times \delta_{\{-1,1\}}$ is the usual product measure in $C \times \{-1, 1\}$. Here λ denotes the usual Lebesgue measure at C and $\delta_{\{-1,1\}}$ is the usual measure in $\{-1, 1\}$. On the other hand, from the previous Lemma and Proposition ([11]), we have that the map E is topologically conjugate to an *affine* GIET. Now, suppose that there exists a homeomorphism h which conjugates E with an isometric GIET, say $\tilde{E}$. Since $\bigcup_k \langle a_k, b_k\rangle$ is an E-invariant set, then under the assumption that such an h exists, it follows that $E|_{\cup_k \langle a_k, b_k\rangle}$ and $\tilde{E}|_{\cup_k h(\langle a_k, b_k\rangle)}$ are topologically conjugate. As the first return map induced by E on the interval $\langle a_0, b_0\rangle$ (see Lemma 2.1) is uniquely ergodic (its second return maps is an irrational rotations), then the first return map induced by $\tilde{E}|_{\cup_k h(\langle a_k, b_k\rangle)}$ to $h(\langle a_0, b_0\rangle)$ will also be uniquely ergodic. Therefore, we can conclude that if $\tilde{\mu}$ denote the $\tilde{E}|_{\cup_k h(\langle a_k, b_k\rangle) \times \{-1,1\}}$-invariant measure, then necessarily

$$\tilde{\mu}|_{\cup_k h(\langle a_k, b_k\rangle) \times \{-1,1\}} = (\lambda \circ h^{-1}) \times \delta_{\{-1,1\}},$$

and consequently (2.1) will be valid under conjugation, more precisely, for all $k \geq 0$

$$\tilde{\mu}(\langle h \circ g_0(p_{k+1}), h \circ g_0(q_{k+1})\rangle, \cdot) = \frac{1}{2}\tilde{\mu}(\langle h \circ g_0(p_k), h \circ g_0(q_k)\rangle, \cdot) \quad (2.6)$$

On the other hand, it is clear that, in a natural way, the assertions the Lemma 2.1 remain valid to the isometric GIET $\tilde{E}$. Therefore, (2.5) will also be valid with E, μ, g_0 replaced by $\tilde{E}$, $\tilde{\mu}$ and $h \circ g_0$, respectively. Thus, if there exists such a map h, this clearly forces

$$\sum_{k=0}^{+\infty} \sum_{i=n_k}^{n_{k+1}-1} \tilde{\mu}(\tilde{E}^i(\langle h \circ g_0(p_k), h \circ g_0(q_k)\rangle, 1)) =$$

$$\sum_{k=0}^{+\infty}(n_{k+1} - 1 - n_k)\tilde{\mu}(\langle h \circ g_0(p_k), h \circ g_0(q_k)\rangle, 1))$$

but from (2.6)

$$\tilde{\mu}(\langle h \circ g_0(p_k), h \circ g_0(q_k)\rangle, 1) = \frac{1}{2^k}\tilde{\mu}(\langle h \circ g_0(p_0), h \circ g_0(q_0)\rangle, 1), \ \forall k \geq 0$$

Combining this with the conditions on $\{m_k\}_{k=0}$ gives

$$\sum_{k=0}^{+\infty} \sum_{i=n_k}^{n_{k+1}-1} \tilde{\mu}(\tilde{E}^i(\langle h \circ g_0(p_k), h \circ g_0(q_k)\rangle, 1)) =$$

$$\tilde{\mu}(\langle h \circ g_0(p_0), h \circ g_0(q_0)\rangle, 1) \sum_{k=0}^{+\infty} \frac{m_k}{2^k} = +\infty$$

which leads to a contradiction with its equivalent version of (2.5). Therefore, such homeomorphism h does not exist and the proof is complete. □

3 An oriented Affine GIET without an isometric model

We can user the same argument, with the obvious change, given in the section 2. To obtain the desired oriented map, we "will use" the infinite strip (with orientable foliation) shown in the Figure 4. This will be formalized as follows:

Let $F : [0, 1] \to [0, 1]$ be the IET defined by

$$F(x) = \begin{cases} x+\alpha, & \text{if; } 0 \le x < 1-\alpha \\ x+\alpha-1, & \text{if. } 1-\alpha < x \le 1 \end{cases}$$

where $0 < \alpha < 1/2$ is an irrational number. As in the previous section, consider in $[0,1]$ increasing sequences $\{q_k\}_{k=0}$, $\{s_k\}_{k=0}$ and decreasing sequences $\{p_k\}_{k=0}$, $\{r_k\}_{k=0}$ of rational numbers such that for each $k \ge 0$

(3.1) $0 < p_k < q_k < 1-\alpha$, p_k converges to 0, and q_k converges to $1-\alpha$;

(3.2) $1-\alpha < r_k < s_k < 1$, r_k converges to $1-\alpha$, and s_k converges to 1;

(3.3) $\lambda(\langle q_{k+1}, r_{k+1}\rangle) = \lambda(\langle q_k, r_k\rangle)/2$, where λ denotes the Lebesque measure.

Let $T\colon [0,1] \to [0,1]$ and $\langle a_0, b_0\rangle$ be as in the beginning but with the following additional properties (see proof of Lemma 2.1 in [8]). The subsequence of wandering intervals $\{I_{n_k}\}_{k\in\mathbb{Z}}$ satisfies $(i)-(iii)$ of Section 1 and for each $k > 0$

- $\lambda(I_{n_k}) = \lambda(I_{n_{-k}})$;
- $\lambda(I_{n_k}) = \lambda(I_{n_0} \setminus \{\langle \phi(p_{k-1}), \phi(q_{k-1})\rangle \cup \langle \phi(r_{k-1}), \phi(s_{k-1})\rangle\})$; and,
- $\lambda(I_{n_{-k}}) = \lambda(I_{n_0} \setminus \{\langle \psi(p_{k-1}), \psi(q_{k-1})\rangle \cup \langle \psi(r_{k-1}), \psi(s_{k-1})\rangle\})$.

where $\phi\colon \langle 0,1\rangle \to I_{n_0}$ denotes the linear oriented homeomorphism of $\langle 0,1\rangle$ onto I_{n_0}, and $\psi\colon \langle 0,1\rangle \to I_{n_0}$ is defined by $\psi = \phi \circ F$.

Under this consideration, for each $k \in \mathbb{Z}\setminus\{0\}$ let us denote by $\rho_k\colon \mathrm{Dom}(\rho_k) \to I_{n_k}$ the isometric oriented map such that for each $k > 0$,

$$\mathrm{Dom}(\rho_k) = I_{n_0} \setminus \{\langle \phi(p_{k-1}), \phi(q_{k-1})\rangle \cup \langle \phi(r_{k-1}), \phi(s_{k-1})\rangle\} \text{ and}$$

$$\mathrm{Dom}(\rho_{-k}) = I_{n_0} \setminus \{\langle \psi(p_{k-1}), \psi(q_{k-1})\rangle \cup \langle \psi(r_{k-1}), \psi(s_{k-1})\rangle\}.$$

For $k = 0$, consider ρ_0 as being the identidy map on I_{n_0}.

Finally, remembering that $n_0 = 0$, define the map $E\colon [0,1] \to [0,1]$, as follows. Let $k \in \mathbb{Z}^+ \cup \{0\}$

if; $x \in I_{n_k} \setminus \{[\rho_k \circ \phi(p_k), \rho_k \circ \phi(q_k)] \cup [\rho_k \circ \phi(r_k), \rho_k \circ \phi(s_k)]\}$ then $E(x) = T^{n_k - n_{k+1}+1} \circ \rho_{k+1} \circ \rho_k^{-1}(x)$

if; $x \in \langle \rho_k \circ \phi(p_k), \rho_k \circ \phi(q_k)\rangle \cup \langle \rho_k \circ \phi(r_k), \rho_k \circ \phi(s_k)\rangle$ then $E(x) = \rho_{-k} \circ \phi \circ F \circ \phi^{-1} \circ \rho_k^{-1}(x)$

if; $x \in T^{-1}(I_{n_{-k}}) \cap T^{-n_{-(k+1)}+n_{-k}-1}(\mathrm{Im}(\rho_{-(k+1)}))$ then $E(x) = \rho_{-k} \circ \rho_{-(k+1)}^{-1} \circ T^{n_{-(k+1)}-n_{-k}+1}(x)$

and, $E(x) = T(x)$ in all the other cases, except at $x \in \{\rho_k \circ \phi(p_k), \rho_k \circ \phi(q_k), \rho_k \circ \phi(r_k), \rho_k \circ \phi(s_k); k = 0, 1, 2 \ldots\} \cup \{I_{n_{-k}} \setminus \mathrm{Im}(\rho_{-k}); k = 1, 2, \ldots\}$.

Thus, $\mathrm{Dom}(E) = [0, 1] \setminus \cup_{k=0}\{\rho_k \circ \phi(p_k), \rho_k \circ \phi(q_k), \rho_k \circ \phi(r_k), \rho_k \circ \phi(s_k)\} \cup \{I_{n_{-k}} \setminus \mathrm{Im}(\rho_{-k}); k = 1, 2, \ldots\}$.

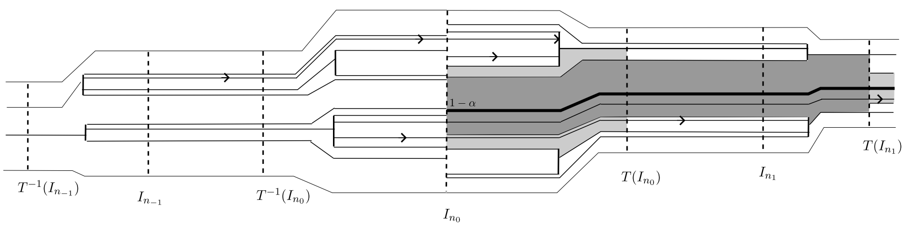

Figure 4: $\langle \phi(q_0), \phi(r_0) \rangle \supset \langle \phi(q_1), \phi(r_1) \rangle$ are the transversal sections, contained in I_{n_0}, of the shaded quadrilaterals. The first return map on I_{n_0} will be the map F.

It follows from definition of E, that, the accumulation sets of $[0, 1] \setminus \mathrm{Dom}(E)$ is $\{b_0, b_{-1}\}$. Therefore deleting from $[0, 1]$ the points b_0 and b_{-1} we see that the following lemma is valid.

Lemma 3.1. $E : [0, 1] \to [0, 1]$ *is an oriented injective map with the following properties. It is defined in an open and dense set, the first return map of E to* $I_{n_0} = \langle a_0, b_0 \rangle$ *is topologically conjugate to the map* $F|_{\langle 0,1 \rangle}$, *admits dense orbits, and the E-orbit through* $\phi(1-\alpha)$ *is not w-recurrent but the closure of its E-orbit contains an exceptional one.*

On the other hand, notice that condition (2.1), necessary in the proof of Proposition 2.2, is also required to the construction of the map E (see condition (3.3)). As after identifying 0 and 1 we see that F is an irrational rotation then, from definition of E is easy see that a similar argument used in the proof of Proposition 2.2 remains valid for the following proposition.

Proposition 3.2. *There exists a continuous injective map* $E : [0, 1] \to [0, 1]$ *preserving orientation, such that it is defined in an open and dense set, admit dense and exceptional orbits, and it is topologically conjugate to an affine GIET but it is not topologically conjugate to an isometric GIET.*

We remark that the foliations obtained by suspension of E necessarily will be obtained on a two-manifolds of infinite genus.

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