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Relative mean curvature configurations for surfaces in $\mathbb{R}^n$, $n \geq 5$

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Abstract. We define the relative mean curvature directions on surfaces immersed in $\mathbb{R}^n$, $n \geq 4$, generalizing the concept of mean curvature directions for surfaces in 4-space studied by Mello. We obtain their differential equations and study their corresponding generic configurations.

Keywords: immersed surfaces, curvature ellipse, mean curvature, mean curvature configurations, semiumbilics, pseudo-umbilics.

Mathematical subject classification: 53A07, 58K40, 58K50.

1 Introduction

The second fundamental form α determines the shape operators associated to the family of normal vector fields on a surface S immersed in $\mathbb{R}^n$, $n \geq 3$, and hence their corresponding principal configurations. The study of this dynamics goes back to the works of Monge [19] and Darboux [4], who described the behavior of the principal curvature lines in the neighborhood of umbilic points of analytic surfaces in Euclidean 3-space. A complete treatment of the subject in terms of the structural stability of the principal lines for surfaces of class C^r, $r \geq 4$, has been provided more recently (Gutierrez and Sotomayor [13], [14], Bruce and Fidal [1]). The generic behavior of principal configurations on surfaces in $\mathbb{R}^4$ has been studied along these lines by Ramirez Galarza and Sánchez Bringas in [24]. Besides the principal configurations, the extrinsic geometry of

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the surfaces determines other interesting foliations such as the mean curvature configurations of surfaces in $\mathbb{R}^3$ described by Garcia and Sotomayor in [9] and [8], the axiumbilic configurations ([12], [7]), the asymptotic configurations ([6], [2]) and the mean curvature direction configurations for surfaces in $\mathbb{R}^4$ described by L.F. Mello [18].

The definition of these configurations relies on the concept of curvature ellipse of a surface S immersed in n-space ([17], [26]). This is defined as the image through α of the unit tangent vectors circle into the normal space N_pS at each point $p \in S$. The vector $H(p) \in N_pS$ determined by the center of the curvature ellipse at p is known as the mean curvature vector. For a surface immersed into $\mathbb{R}^4$, the normal line defined by $H(p)$ cuts the ellipse in two opposite points (except at the special situations in which the ellipse degenerates into a radial segment, or if $H(p) = 0$). These two points determine a couple of orthogonal tangent directions known as the mean curvature direction at p. These are characterized by the fact that the curvature vector of the normal section of the surface along them is parallel to the mean curvature vector $H(p)$.

The generalization of this procedure to surfaces immersed in $\mathbb{R}^n$ with $n > 4$ embodies some problems due to the fact that the plane determined by the curvature ellipse does not pass through the origin of the normal space at a generic point p. This means that there are no tangent directions whose normal section's curvature vector is parallel to $H(p)$. In other words, there aren't mean curvature directions on surfaces immersed with codimension higher than 2. The way we use here to overcome this difficulty is based on the property described in [21] that, from a qualitative viewpoint, all the principal configurations on S arise from normal vector fields parallel to the subspace determined by the curvature ellipse at every point. In fact, any normal vector $v \in N_pS$ can be decomposed into a sum $v^\top + v^\perp$, with $v^\top$ and $v^\perp$ respectively parallel and orthogonal to the plane determined by the curvature ellipse. It can be shown that the shape operator associated to $v^\perp$ is a multiple of the identity ([20]) and thus the eigenvectors of the shape operator W_v coincide with those of $W_{v^\top}$. This induces us to eliminate the orthogonal part $H(p)^\perp$ of the mean curvature vector and apply the above setting to the direction $H(p)^\top$ contained in the plane defined by the curvature ellipse at p translated to the origin. Then we define the *relative mean curvature directions* at a point p of a surface immersed in $\mathbb{R}^n$ with $n \geq 4$ as those inducing normal sections whose curvature vector is parallel to $H(p)^\top$. We obtain in this way two orthogonal foliations globally defined on the surface whose critical points are the semiumbilics and the pseudo-umbilics (with inflection points and minimal points considered as non-generic particular cases).

In section 2 we introduce the basic geometrical concepts and notations for surfaces immersed in $\mathbb{R}^n$. In section 3 we analyze their generic behavior with respect to the relative positions of the vector $H(p)$ and the curvature ellipse at each point. We determine in section 4 the differential equations associated to the relative mean curvature configuration. Section 5 is devoted to the description of the generic behavior of the foliations in a neighborhood of their critical points: pseudo-umbilics and semiumbilics. There is an essential difference between the two types: whereas the pseudo-umbilics present generically the Darbouxian configurations D_1, D_2 and D_3 $\left(\text{with indices } \pm\frac{1}{2}\right)$, the semiumbilics appear generically as D^1_{23} points (see [15]). We finally obtain some global results as a consequence of the Poincaré-Hopf index formula for foliations on closed orientable surfaces.

2 Second fundamental form and curvature ellipses

Let S be a surface immersed in $\mathbb{R}^n$, $n \geq 3$, that we can locally consider as the image of an embedding $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$, $\phi(\mathbb{R}^2) = S$. At each point $p \in S$ consider the decomposition $T_p\mathbb{R}^n = T_pS \oplus N_pS$, where N_pS denotes the orthogonal complement of the tangent plane T_pS in $\mathbb{R}^n$, that is the normal subspace of S at p. Let $\bar{\nabla}$ denote the Riemannian connection of $\mathbb{R}^n$. Given two vector fields X and Y, locally defined along S, we can choose local extensions $\bar{X}, \bar{Y}$ over $\mathbb{R}^n$, and define the Riemannian connection on S as $\nabla_X Y = \left(\bar{\nabla}_{\bar{X}}\bar{Y}\right)^\top$, where $\top$ denotes the tangent component of the normal connection $\bar{\nabla}$.

If we denote by $\mathfrak{X}(S)$ and $\mathcal{N}(S)$ respectively the spaces of tangent and normal fields on S, the second fundamental form on S is defined as follows:

$$\begin{array}{rccl} \alpha & : & \mathfrak{X}(S) \times \mathfrak{X}(S) & \longrightarrow & \mathcal{N}(S) \\ & & \left(X, Y\right) & \longmapsto & \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y, \end{array}$$

This is a well defined bilinear symmetric map.

Now, given $p \in S$ any $\nu \in N_pS$, $\nu \neq 0$, induces a bilinear form on the tangent space T_pS given by

$$\begin{array}{rccl} H_\nu & : & T_pS \times T_pS & \longrightarrow & \mathbb{R} \\ & & (v, w) & \longmapsto & \alpha(v, w) \cdot \nu, \end{array}$$

and a quadratic form

$$\begin{array}{rccl} \alpha_\nu & : & T_pS & \longrightarrow & \mathbb{R} \\ & & v & \longmapsto & H_\nu(v, v) = \alpha(v, v) \cdot \nu. \end{array}$$

If we take local coordinates (x, y) and an orthonormal frame $\{w_3, \cdots, w_n\}$ of the normal bundle NS in a neighborhood of $p = \phi(0,0) \in S$, the matrix of the second fundamental form in the frame $\{\phi_x, \phi_y, w_3, \ldots, w_n\}$ is given by

$$\alpha_\phi(p) = \begin{bmatrix} a_1 & b_1 & c_1 \\ & \vdots & \\ a_{n-2} & b_{n-2} & c_{n-2} \end{bmatrix},$$

where

$$a_i = \frac{\partial^2\phi}{\partial x^2}(0,0) \cdot w_{i+2}, \quad b_i = \frac{\partial^2\phi}{\partial x \partial y}(0,0) \cdot w_{i+2}, \quad c_i = \frac{\partial^2\phi}{\partial y^2}(0,0) \cdot w_{i+2},$$

for $i = 1, \cdots, n-2$.

We can complete the orthonormal frame $\{w_3, \cdots, w_n\}$ by means of

$$w_1 = \frac{\phi_u}{\sqrt{E}}, \quad w_2 = \frac{E\phi_v - F\phi_u}{\sqrt{E(EG - F^2)}},$$

where E, F and G are the coefficients of the first fundamental form on S. If $w \in \chi(S)$, we can write $w = \lambda_1 w_1 + \lambda_2 w_2$, for some functions λ_i, $i = 1, 2$ and then we have

$$\alpha(w, w) = \lambda_1^2 \alpha(w_1, w_1) + 2\lambda_1\lambda_2\alpha(w_1, w_2) + \lambda_2^2\alpha(w_2, w_2).$$

Given the tangent unit field ν, the functions $e_\nu = \alpha(w_1, w_1) \cdot \nu$, $f_\nu = \alpha(w_1, w_2) \cdot \nu$ and $g_\nu = \alpha(w_2, w_2) \cdot \nu$ are the *coefficients of the second fundamental form in the direction* ν on the frame (w_1, w_2).

Given $p \in S$, consider the unit circle in T_pS parameterized by the angle $\theta \in [0, 2\pi)$ with respect to w_1. Denote by γ_θ the curve obtained by intersecting S with the hyperplane at p composed by the direct sum of the normal subspace N_pS with the line in T_pS defined by the direction θ. Such curve is called the normal section of $\phi(S)$ in the direction θ. The curvature vector $\eta(\theta)$ of γ_θ at p lies in N_pS. Varying θ from 0 to 2π, this vector describes an ellipse in N_pS, called the *curvature ellipse* of S at p. In fact, the curvature ellipse is the image of the map

$$\eta : S^1 \subset T_pS \longrightarrow N_pS$$

given by

$$\begin{aligned}\eta(w(\theta)) \;=\; & \frac{1}{2}\left(\alpha(w_1, w_1) + \alpha(w_2, w_2)\right) + \frac{1}{2}\left(\alpha(w_1, w_1) - \alpha(w_2, w_2)\right)\cos 2\theta \\ & + \alpha(w_1, w_2)\sin 2\theta,\end{aligned}$$

where $w(\theta) = w_1 \cos(\theta) + w_2 \sin(\theta)$ is a unit vector in T_pS.

A shorter expression for the curvature ellipse is given by

$$\eta(w(\theta)) = H + B\cos 2\theta + C \sin 2\theta,$$

where

$$\begin{aligned} H &= \frac{1}{2}\left(\alpha(w_1, w_1) + \alpha(w_2, w_2)\right), \\ B &= \frac{1}{2}\left(\alpha(w_1, w_1) - \alpha(w_2, w_2)\right), \\ C &= \alpha(w_1, w_2). \end{aligned}$$

The vector $H(p)$ is known as the *mean curvature vector* at p. It joins the origin of the normal space N_pS to the center of the ellipse described by the image of the map η. On the other hand, the vectors $B(p)$ and $C(p)$ generate an affine subspace of N_pS, passing by $H(p)$, which is in general an affine plane. Following the nomenclature introduced by Montaldi in [22], if that plane is orthogonal to $H(p)$ we say that p is a *pseudo-umbilic* point. If it degenerates to a line we say that p is *semiumbilic*. If that line passes by the origin, the point is called an *inflection point*. Finally, if it degenerates into a point, p is called *umbilic*. We observe that all the points of surfaces immersed in $\mathbb{R}^3$ are inflection points and that umbilics correspond to the critical points of the principal direction fields. When $H(p) = 0$, we say that p is a *minimal* point.

Remark 2.1. *It can be seen that a point is pseudo-umbilic if and only if it is a umbilical point for the H-principal configuration on S [20].*

3 Generic surfaces in $\mathbb{R}^n$

We analyze in this section the distribution of semiumbilics, inflection, umbilic, pseudo-umbilic and minimal points on generically embedded surfaces in $\mathbb{R}^n$. The main tool used here is the multijet version of Thom's Transversality Theorem ([10]).

Given a point $p \in S$, consider the immersion ϕ in the Monge form in a small enough neighborhood of p,

$$\begin{aligned} \phi : (\mathbb{R}^2, q) &\longrightarrow (\mathbb{R}^n, p = \phi(q)) \\ (x, y) &\longmapsto (x, y, \phi_1(x, y), \ldots, \phi_{n-2}(x, y)), \end{aligned}$$

where we can suppose $q = (0,0)$, $\phi(q) = 0$, $\frac{\partial \phi_i}{\partial x}(q) = \frac{\partial \phi_i}{\partial y}(q) = 0$ for $i = 1, \ldots, n-2$.

The curvature ellipse at $p = \phi(q) = 0$ is given by

$$\eta(w(\theta)) = \sum_{i=1}^{n-2} \left[\frac{1}{2}(a_i + c_i)\, e_{i+2} + \frac{1}{2}(a_i - c_i)\, e_{i+2} \cos 2\theta + b_i e_{i+2} \sin 2\theta \right]_p$$

and we have

$$H = \sum_{i=1}^{n-2} \frac{1}{2}(a_i + c_i)\, e_{i+2}, \quad B = \sum_{i=1}^{n-2} \frac{1}{2}(a_i - c_i)\, e_{i+2} \quad \text{and} \quad C = \sum_{i=1}^{n-2} b_i e_{i+2}$$

where a_i, b_i and c_i are as in the previous section.

Proposition 3.1. *Let S be a surface in $\mathbb{R}^5$. There is a residual subset of immersions $\mathcal{I} \subset \mathrm{Imm}(S, \mathbb{R}^5)$, with Whitney C^∞ topology such that $\forall\, f \in \mathcal{I}$ it verifies*

i) *the semiumbilic points of $f(S)$ are isolated;*

ii) *the pseudo-umbilic points of $f(S)$ are isolated;*

iii) *$f(S)$ has neither inflection points, nor umbilic, nor minimal, nor points that are simultaneously semiumbilic and pseudo-umbilic.*

Proof. With the above notation, we have that the condition that p is semiumbilic is given by $B \wedge C = 0$, i.e.:

$$\begin{cases} (a_1 - c_1)b_2 - b_1(a_2 - c_2) = 0 \\ (a_3 - c_3)b_1 - b_3(a_1 - c_1) = 0 \\ (a_2 - c_2)b_3 - b_2(a_3 - c_3) = 0. \end{cases}$$

These conditions on the second derivatives of the embedding f define a closed algebraic variety V_1 of codimension 2 in the 2-jets space $J^2(\mathbb{R}^2, \mathbb{R}^5)$. Then, as a consequence of Thom's Transversality Theorem, there is a residual subset $\mathcal{I}_1 \subset \mathrm{Imm}(S, \mathbb{R}^5)$ such that $\forall f \in \mathcal{I}_1$, $j^2 f \pitchfork V_1$. But this means that f has only isolated semiumbilic points.

Pseudo-umbilic points are characterized by the conditions $H \cdot B = 0$ and $H \cdot C = 0$, which lead to an algebraic variety V_2 of codimension 2 in $J^2(\mathbb{R}^2, \mathbb{R}^5)$. A further application of Thom's Transversality Theorem implies the existence of a residual subset $\mathcal{I}_2 \subset \mathrm{Imm}(S, \mathbb{R}^5)$, whose maps may only have isolated pseudo-umbilic points.

The condition for an inflection point is $\mathrm{rank}\{H, B, C\} = \mathrm{rank}\{a, b, c\} = 1$. This can be written in the above coordinates as $a_1c_2 - a_2c_1 = a_1b_2 - a_2b_1 = a_1b_3 - a_3b_1 = a_1c_3 - a_3c_1 = 0$ and determines a codimension 4 algebraic variety V_3 in $J^2(\mathbb{R}^2, \mathbb{R}^5)$. In this case, it follows from Thom's Transversality Theorem that there exists a residual subset $\mathcal{I}_3 \subset \mathrm{Imm}(S, \mathbb{R}^5)$ such that $\forall f \in \mathcal{I}_3$, f has no inflection points. Analogously, at an umbilic point $B = C = 0$, or equivalently, $a_1 - c_1 = a_2 - c_2 = a_3 - c_3 = b_1 = b_2 = b_3 = 0$, which determines an algebraic variety V_4 of codimension 6 in $J^2(\mathbb{R}^2, \mathbb{R}^5)$. So we get the existence of a new a residual subset $\mathcal{I}_4 \subset \mathrm{Imm}(S, \mathbb{R}^5)$, whose maps have no umbilics.

Finally, a point $p \in S$ is minimal if and only if $H(p) = 0$. That is $a_1 + c_1 = a_2 + c_2 = a_3 + c_3 = 0$. The same procedure gives rise to a residual subset $\mathcal{I}_5 \subset \mathrm{Imm}(S, \mathbb{R}^5)$, whose maps have no minimal points.

Finally, it is obvious that a point is both semiumbilic and pseudo-umbilic iff it belongs to the intersection of two independent algebraic varieties of codimension 2. Thus, there is a residual subset $\mathcal{I}_6 \subset \mathrm{Imm}(S, \mathbb{R}^5)$, whose elements have no such points.

The proof is now concluded by taking $\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3 \cap \mathcal{I}_4 \cap \mathcal{I}_5 \cap \mathcal{I}_6$. $\square$

Proposition 3.2. *Let S be a surface in $\mathbb{R}^n$, $n > 5$. There is an open and dense subset of immersions $\mathcal{I} \subset \mathrm{Imm}(S, \mathbb{R}^n)$, with the Whitney C^∞ topology, such that $\forall f \in \mathcal{I}$ the following conditions hold:*

i) *$f(S)$ has neither semiumbilic, nor inflexion, nor umbilic, nor minimal points;*

ii) *the pseudo-umbilic points of $f(S)$ are isolated.*

Proof. When $n \geq 6$, the condition of linear dependence of the vectors B and C at a semiumbilic point gives rise to $n-3$ independent equations which define an algebraic variety V_1 of codimension $n - 3 > 2$ in $J^2(\mathbb{R}^2, \mathbb{R}^5)$. The transversality of $j^2 f$ to V_1 implies that $f(S)$ has no semiumbilics. This leads, as a consequence of Thom's Transversality Theorem, to a residual subset $\mathcal{I}_1$ of $\mathrm{Imm}(S, \mathbb{R}^n)$. An analogous argument implies that there are no inflection, nor umbilic points on the immersed surfaces corresponding to conveniently defined residual subsets $\mathcal{I}_2$ and $\mathcal{I}_3$ of $\mathrm{Imm}(S, \mathbb{R}^n)$. The minimal points are characterized in this case by the $n - 2 \geq 4$ equations $a_1 + c_1 = \cdots = a_{n-2} + c_{n-2} = 0$. And thus, we get that there exists a residual subset $\mathcal{I}_4$ whose immersions have no minimal points.

Finally, the conditions $H \cdot B = 0$ and $H \cdot C = 0$ on pseudo-umbilics lead to an algebraic variety of codimension 2 in $J^2(\mathbb{R}^2, \mathbb{R}^n)$ also in this case. And thus we obtain a residual subset $\mathcal{I}_5 \subset \mathrm{Imm}(S, \mathbb{R}^n)$, whose maps may only have isolated pseudo-umbilic points.

Again, we take $\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3 \cap \mathcal{I}_4 \cap \mathcal{I}_5$. □

In what follows, all the considered immersions will belong to the residual subset $\mathcal{I}$.

4 Relative mean curvature lines: differential equations

We define in this section the direction fields for surfaces immersed in $\mathbb{R}^n$, $n \geq 5$, that generalize, as explained in the Introduction, the mean directionally curved lines on surfaces immersed in $\mathbb{R}^4$ studied by Mello [18].

If S is a surface immersed in $\mathbb{R}^4$, the vector line generated by the mean curvature vector H meets in general the ellipse of curvature at points $\eta(w(\theta))$ that satisfy

$$\eta(w(\theta)) \wedge H = 0. \tag{4.1}$$

These points induce two orthogonal directions on T_pS and, hence, two direction fields on S, called H-direction fields ([18]). The singularities of these fields are either minimal points or inflection points.

By substituting the expression for $\eta(w(\theta))$ obtained in section 2 in the equation 4.1, we get the following expression

$$0 = \eta(w(\theta)) \cdot JH = (hc\,bb - hb\,bc)\cos 2\theta + (hc\,bc - hb\,cc)\sin 2\theta,$$

where $hb = H \cdot B$, etc., and J denotes the rotation of angle $\frac{\pi}{2}$ in the plane T_pS. There are two such rotations, but both give rise to the same equation.

Suppose now that S is a surface immersed into $\mathbb{R}^n$, $n \geq 5$, and denote by $\mathcal{R}$ the open subset of points $p \in S$ for which $B(p)$ and $C(p)$ are linearly independent, and $H(p)$ is not orthogonal to both $B(p)$ and $C(p)$.

Proposition 4.1. *$p \notin \mathcal{R}$ iff p is either a semiumbilic or a pseudo-umbilic point.*

Proof. Let $p \notin \mathcal{R}$. Suppose that $\{B(p), C(p)\}$ is linearly independent; then $H(p) \cdot B(p) = H(p) \cdot C(p) = 0$. Thus, p is pseudo-umbilic. On the other hand, if $\{B(p), C(p)\}$ are linearly dependent, then p is a semiumbilic point. □

If we take $p \in \mathcal{R}$, then there is a unique hyperplane ξ of N_pS containing $H(p)$ and orthogonal to the plane generated by $B(p)$ and $C(p)$. In fact, it is the hyperplane whose normal vectors are those linear combinations of $B(p)$ and $C(p)$ that are orthogonal to $H(p)$.

Definition 4.2. *Let $p \in \mathcal{R}$ and $\theta \in S^1(T_pS)$. We say that θ is a* relative mean curvature direction *if $\eta(w(\theta)) \in \xi$.*

Proposition 4.3. *The tangent direction $\theta \in S^1(T_pS)$ is a relative mean curvature direction if and only if*

$$(bb\,hc - bc\,hb)\cos 2\theta + (bc\,hc - cc\,hb)\sin 2\theta = 0. \tag{4.2}$$

Proof. By the hypotheses, the vector $n = hc\,B - hb\,C$ is orthogonal to H and does not vanish. Thus n is a normal of the hyperplane ξ. It follows that θ is a relative mean curvature direction if and only if $\eta(w(\theta)) \cdot n = 0$, that is iff

$$\begin{aligned} &(H + B\cos 2\theta + C\sin 2\theta) \cdot (hc\,B - hb\,C) \\ &= (bb\,hc - bc\,hb)\cos 2\theta + (bc\,hc - cc\,hb)\sin 2\theta = 0. \end{aligned} \qquad \square$$

Since this equation, for $n = 4$, is the same as the equation for the mean curvature directions studied by Mello, we may regard relative mean curvature directions as a generalization of Mello's ones.

Definition 4.4. *A curve $\gamma : (-\varepsilon, \varepsilon) \to S$ will be said to be a* relative mean curvature line *provided its tangent $\gamma'(t)$ is parallel to a relative mean curvature direction of S at the point $\gamma(t)$, $\forall t \in (-\varepsilon, \varepsilon)$.*

Theorem 4.5. *Let the surface S immersed in $\mathbb{R}^n$, $n \geq 5$, be parameterized by the isothermal coordinates $\phi : (u, v) \in U \mapsto \phi(u, v) \in S$ with first fundamental form $E(du^2 + dv^2)$ and let $\gamma(t) = \phi(u(t), v(t))$ be a smooth curve in S. The differential equation that γ must satisfy for being a relative mean curvature line is given by*

$$N(u, v)(u'^2 - v'^2) + 2P(u, v)u'v' = 0, \tag{4.3}$$

where $N \equiv bb\,hc - bc\,hb$ and $P \equiv bc\,hc - cc\,hb$ should be computed by means of $w_1 = \phi_u/\sqrt{E}$, $w_2 = \phi_v/\sqrt{E}$.

Proof. We consider the orthonormal frame $\left(w_1 = \frac{\phi_u}{\sqrt{E}}, w_2 = \frac{\phi_v}{\sqrt{E}}\right)$. Then we have

$$B = \frac{1}{2E}\left(\phi_{uu}^{\perp} - \phi_{vv}^{\perp}\right), \quad C = \frac{1}{E}\phi_{uv}^{\perp} \quad \text{and} \quad H = \frac{1}{2E}\left(\phi_{uu}^{\perp} + \phi_{vv}^{\perp}\right),$$

and the curvature ellipse is given by the equation

$$\eta(w(\theta)) = \frac{1}{2E}\left(\phi_{uu}^{\perp} + \phi_{vv}^{\perp}\right) + \frac{1}{2E}\left(\phi_{uu}^{\perp} - \phi_{vv}^{\perp}\right)\cos 2\theta + \frac{1}{E}\phi_{uv}^{\perp}\sin 2\theta$$

so that

$$H = \frac{1}{2E}\left(\phi_{uu}^{\perp} + \phi_{vv}^{\perp}\right), \quad B = \frac{1}{2E}\left(\phi_{uu}^{\perp} - \phi_{vv}^{\perp}\right), \quad C = \frac{1}{E}(\phi_{uv}^{\perp}).$$

We can write

$$\gamma' = m(w_1\cos\theta + w_2\sin\theta) = u'\phi_u + v'\phi_v.$$

Thus $\cos\theta = \frac{\sqrt{E}}{m}u'$, $\sin\theta = \frac{\sqrt{E}}{m}v'$, and the result follows by substituting these expressions for $\cos\theta$ and $\sin\theta$ in equations 4.2. □

The coefficients that appear in the above differential equations are well defined differentiable functions at any point of U, and vanish simultaneously exactly at the pseudo-umbilic and the semiumbilic points of S. That is, its singularities are exactly the points away from $\mathcal{R}$. As we have seen in the previous section, for a generic immersion of the surface S in $\mathbb{R}^5$, that is for $S \in \mathcal{I}$, the subset $\mathcal{R}$ is open and dense and its complement is made of isolated pseudo-umbilic and isolated semiumbilic points (that are not pseudo-umbilic). When the surface is generically immersed into $\mathbb{R}^n$, $n > 5$, the only critical points are isolated pseudo-umbilics with no other special property (umbilics, etc.).

We observe that in the case $n = 4$, we obtain the equation of the mean curvature lines studied by Mello (see [18]) as a particular case. In this case the coefficients vanish exactly at the inflection points and the minimal points, which occur generically as isolated points on S.

If $p \in \mathcal{R}$, then the discriminant of equation 4.3, $\triangle(p) = (N^2 + P^2)(p)$, is positive. Therefore there exist two orthogonal solutions of the differential equation of the relative mean curvature lines. In a neighborhood of p there exist two families of orthogonal curves. These two families determine two foliations, denoted by L_1 and L_2, on the open subset $\mathcal{R}$. Each isolated singularity defines an isolated singularity of both foliations. Under the orientability hypothesis on the surface it is possible to distinguish the foliation L_1 from L_2 all over $\mathcal{R}$, (see [16]).

5 Generic configurations for the relative mean curvature lines

5.1 Some basic tools

We denote by PS the projective tangent fiber bundle over S, and by $\Pi : PS \to S$ the natural projection. For any isothermal chart (u, v) on an open neighborhood

U of S there are two charts $(u, v; p = \frac{dv}{du})$ and $(u, v; q = \frac{du}{dv})$, which cover $\Pi^{-1}(U)$. The differential equation of relative mean curvature lines 4.3 defines a surface $\mathbb{F}$ over PS. In the chart $(u, v; p = \frac{dv}{du})$ the surface is given by $F^{-1}(0)$, where $F(u, v; p) = N(u, v)(1 - p^2) + 2P(u, v)p$. Suppose that $(0, 0)$ is a critical point of the equation 4.3, that is $N(0, 0) = P(0, 0) = 0$. The projective line $\Pi^{-1}(0, 0)$ is contained in $\mathbb{F}$, because

$$F(0, 0, p) = N(0, 0)(1 - p^2) + 2P(0, 0)p = 0.$$

We have

$$dF = \big(N_u(1 - p^2) + 2P_u p,\ N_v(1 - p^2) + 2P_v p,\ -2Np + 2P\big).$$

The value of dF at $(0, 0, p)$, $dF_{(0,0,p)}$, is equal to $\big(N_u(0, 0)(1 - p^2) + 2P_u(0, 0)p,\ N_v(0, 0)(1 - p^2) + 2P_v(0, 0)p,\ 0\big)$. If $\frac{\partial(N,P)}{\partial(u,v)}(0, 0) \neq 0$ then $dF_{(0,0,p)} \neq 0$ for all p. In this case, there is a neighborhood V of $(0, 0)$, such that the surface $\mathbb{F}$ is regular in $\Pi^{-1}(V)$.

Definition 5.1. *We say that the singularity at* $(0, 0)$ *verifies the transversality condition if* $\frac{\partial(N,P)}{\partial(u,v)}(0, 0) \neq 0$.

The transversality condition is equivalent to the transversality of the curves $N = 0$, $P = 0$ at $(0, 0)$. If that condition does not hold at $(0, 0)$ then there are exactly two critical points of F in $\Pi^{-1}(0, 0)$.

Away from the critical points of 4.3 the surface $\mathbb{F}$ is regular and in fact is a double covering of S.

Let $\zeta : \mathbb{F} \to T\mathbb{F}$ be the Lie-Cartan vector field corresponding to equation 4.3. It is tangent to $\mathbb{F}$ and its components are given by

$$\zeta(u, v; p) = \left(\frac{\partial F}{\partial p}, p\frac{\partial F}{\partial p}, -\left(\frac{\partial F}{\partial u} + p\frac{\partial F}{\partial v}\right)\right)$$

The function F is a first integral of ζ. The projections of the integral curves of ζ by $\Pi(u, v; p) = (u, v)$ are the relative mean curvature lines. Namely, the singularities of $d\Pi(\zeta)$ occur only at the critical points of 4.3, and in addition, if $(u, v; p_j) \in \mathbb{F}$, then $d\Pi(\zeta(u, v; p_j))$ defines a mean relatively curved direction with slope p_j, $j = 1, 2$. The singularities of the field ζ, lying on the projective line $\Pi^{-1}(0, 0)$, are given by the roots of the cubic polynomial

$$\varphi(p) = \frac{\partial F}{\partial u}(0, 0; p) + p\frac{\partial F}{\partial v}(0, 0; p).$$

Definition 5.2. *We say that the singularity at* $(0,0)$ *verifies the hyperbolicity condition if the polynomial* φ *has only simple roots.*

Both conditions, transversality and hyperbolicity, imply that the vector field ζ has only singularities of saddle or node type, that induce in S configurations known as Darbouxian types D_1, D_2 or D_3, according to there is only one root of φ (type D_1), or three roots (saddle-node-saddle, D_2; saddle-saddle-saddle, D_3) (see [13] for a detailed description).

In what follows, S will be a surface immersed in $\mathbb{R}^n$, $n \geq 5$ and $p \in S$. If $n = 4$, it is enough to consider that S is contained on the subspace given by $x_5 = \cdots = x_n = 0$.

Proposition 5.3. *Given any point* $p \in S$*, there is an orthonormal basis of* T_pS *such that* $B(p) \cdot C(p) = 0$, $|B(p)| \geq |C(p)|$.

Proof. Let w_1, w_2 be an orthonormal basis for T_pS. Hence, if α is the second fundamental form of S, we have

$$B = \frac{1}{2}(\alpha(w_1, w_1) - \alpha(w_2, w_2)), \quad C = \alpha(w_1, w_2).$$

Given $\psi \in [0, 2\pi)$, the vectors

$$u_1 = w_1 \cos\psi + w_2 \sin\psi, \quad u_2 = -w_1 \sin\psi + w_2 \cos\psi$$

also form an orthonormal basis of T_pS and we can write

$$\begin{aligned}
\tilde{B} &= \frac{1}{2}\,(\alpha(u_1, u_1) - \alpha(u_2, u_2)) \\
&= \frac{1}{2}\,(\alpha(w_1, w_1)\cos^2\psi + \alpha(w_1, w_2)\sin 2\psi + \alpha(w_2, w_2)\sin^2\psi \\
&\quad - \alpha(w_1, w_1)\sin^2\psi + \alpha(w_1, w_2)\sin 2\psi - \alpha(w_2, w_2)\cos^2\psi) \\
&= \frac{1}{2}\,(\alpha(w_1, w_1) - \alpha(w_2, w_2))\cos 2\psi + \alpha(w_1, w_2)\sin 2\psi \\
&= B\cos 2\psi + C\sin 2\psi.
\end{aligned}$$

Analogously

$$\begin{aligned}
\tilde{C} &= \alpha(u_1, u_2) \\
&= -\frac{1}{2}\alpha(w_1, w_1) - \alpha(w_2, w_2)\sin 2\psi + \alpha(w_1, w_2)\cos 2\psi \\
&= -B\sin 2\psi + C\cos 2\psi.
\end{aligned}$$

From here, we obtain $\tilde{B} \cdot \tilde{C} = \frac{1}{2}(C \cdot C - B \cdot B) \sin 4\psi + B \cdot C \cos 4\psi$. If $\psi = \frac{\pi}{4}$, then one has $\tilde{B} = C, \ \tilde{C} = -B$, and hence we can choose the larger vector among B and C. We call $m = \sqrt{(\frac{1}{2}(C \cdot C - B \cdot B))^2 + (B \cdot C)^2}$. If $m = 0$ Then $B \cdot C = 0$ and we only need to make an interchange as indicated. If $m \neq 0$, it is enough take

$$\sin 4\psi = -\frac{B \cdot C}{m}, \quad \cos 4\psi = \frac{C \cdot C - B \cdot B}{2m}$$

in order to get $\tilde{B} \cdot \tilde{C} = 0$. □

Given a surface $S \subset \mathbb{R}^n$, suppose that $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$ and let $p \in S$. By applying an affine isometry of $\mathbb{R}^n$ if necessary, we can consider without loss of generality that p is the origin of $\mathbb{R}^n$ and that the basis $\{w_1, w_2\}$ of T_pS determined by the above proposition coincides with $\{e_1, e_2\}$. Moreover, since the vectors $B(p)$ and $C(p)$ lie in N_pS, we can also rotate the axes e_3, e_4 so that $B(p) = be_3$ and $C(p) = ce_4, \ b \geq c$, where $b, c \in \mathbb{R}$ are the respective lengths of the vectors $B(p)$ and $C(p)$. As for the mean curvature vector $H(p)$ we can write $H(p) = \sum_{i=3}^5 h_i e_i$. For this, it is enough to choose e_5 so that $H(p)$ is contained in the 3-space spanned by $\{e_3, e_4, e_5\}$, or in other words, the first normal space N_p^1S at p is spanned by the vectors e_3, e_4 and e_5.

Let $\psi : U \to \mathbb{R}^n$ be an isothermal chart of S such that $\psi(0,0) = p = 0, \ \psi_u(0,0) = e_1, \ \psi_v(0,0) = e_2$. We observe that if $h : \mathbb{C} \to \mathbb{C}$ is a holomorphic function, then it is a conformal function too, and thus the composition $\phi = \psi \circ h$ is also an isothermal chart. We shall take h, in a neighborhood of the origin, as a complex polynomial, namely $h(z) = z + c_2z^2 + c_3z^3 + \ldots$, where $c_2, c_3, \cdots \in \mathbb{C}$. This will allow us to simplify the Taylor series of S on the considered chart by conveniently choosing the complex coefficients $c_2, c_3, \ldots$ so that the composition $\psi \circ h$ satisfy additional conditions at the origin. The choice of the coefficient 1 for the term of degree one guarantees that the property $e_1 = \psi_u(0,0), e_2 = \psi_v(0,0)$ will also hold for the new chart ϕ.

In the remaining part of this section we study the generic configurations near the critical points.

5.2 Generic configurations for the relative mean curvature lines at semiumbilic points

In this subsection, we will consider an immersion S in the subset $\mathcal{I} \subset \mathrm{Imm}(\mathbb{R}^2; \mathbb{R}^5)$ and we shall study the configuration of the lines of relative mean curvature in the proximity of a semiumbilic point. Thus that point will not be umbilic, nor pseudo-umbilic, nor minimal, nor an inflection point. We shall obtain a reduced form for the expansion of the binary differential equation near that critical point.

Proposition 5.4. *Let m be any point of S. Then, we can find an orthonormal affine basis of $\mathbb{R}^5$ and an isothermal chart ϕ of S in a neighborhood of m such that the fourth order expansion of $\phi(u, v)$ in that affine base verifies:*

1. *The second degree terms and the terms inu^3 and in u^4 of the first two components are zero;*
2. *The expansion of the third, fourth and fifth component has neither constant nor linear terms;*
3. *The remaining terms of the first two components are determined by the coefficients of the terms of the third, fourth and fifth components.*

Proof. Most of the following statements of a computing nature have been obtained with the aid of a symbolic computation program (Mathematica®).

Consider an affine basis of $\mathbb{R}^5$ and a chart ψ of S as in the last section and let:

$$\psi(u, v) = (X(u, v), Y(u, v), Z(u, v), W(u, v), T(u, v)).$$

Firstly we consider the change of variable given by:

$$(x, y) = z + c_2 z^2 + c_3 z^3 + c_4 z^4, \quad z = u + iv, \quad c_k = a_k + ib_k \in \mathbb{C}, \quad k = 2, 3, 4$$

We compute the derivatives of the resulting chart ϕ at $(0, 0)$ and observe that we can pick out the coefficients $a_2, b_2, a_3, b_3, a_4, b_4$ in order that:

$$X_{uv} = Y_{uv} = X_{uuu} = Y_{uuu} = X_{uuuu} = Y_{uuuu} = 0.$$

Here, and in the following, symbols as X_{uv} denote the corresponding derivatives at $u = v = 0$. Next, since the chart is isothermal, we must have

$$E - G = \phi_u \cdot \phi_u - \phi_v \cdot \phi_v \equiv 0,$$
$$F = \phi_u \cdot \phi_v \equiv 0.$$

Therefore, the functions $E - G$ and F and its first and second derivatives vanish at the origin. Then we obtain a system of eighteen equations that are linear in eighteen of the coefficients, and may be solved uniquely so that:

- the coefficients $X_{uu}, X_{vv}, Y_{uu}, Y_{vv}$ are zero
- the remaining non-null coefficients of X, Y up to fourth degree can be written as function of the coefficients of Z, W, T. □

Proposition 5.5. *Let m be a semiumbilic point of S. Then, we can find an orthonormal affine basis of $\mathbb{R}^5$ and an isothermal chart ϕ of S in a neighborhood of m such that the function:*

$$\phi(u, v) = (X(u, v), Y(u, v), Z(u, v), W(u, v), T(u, v)).$$

obtained in the preceeding result verifies:

$$\begin{aligned} &T_{uu} = T_{vv} = T_{uv} = 0, \\ &W_{uu} = W_{vv} \neq 0, \quad W_{uv} = 0, \\ &|Z_{uu}| \neq |Z_{vv}|, \quad Z_{uv} = 0. \end{aligned}$$

Proof. We have:

$$\begin{aligned} H(0,0) &= \left(0, 0, \frac{Z_{uu} + Z_{vv}}{2}, \frac{W_{uu} + W_{vv}}{2}, \frac{T_{uu} + T_{vv}}{2}\right) \\ B(0,0) &= \left(0, 0, \frac{Z_{uu} - Z_{vv}}{2}, \frac{W_{uu} - W_{vv}}{2}, \frac{T_{uu} - T_{vv}}{2}\right) \\ C(0,0) &= (0, 0, Z_{uv}, W_{uv}, T_{uv}). \end{aligned}$$

Since the minor semiaxis of the curvature ellipse is zero, $C(0,0) = 0$. Thus we can suppose that the 4th component of $B(0,0)$ and the 5th components of $B(0,0)$ and $H(0,0)$ vanish, and this implies the proposition. As a consequence, we have:

$$\begin{aligned} H(0,0) &= \left(0, 0, \frac{Z_{uu} + Z_{vv}}{2}, W_{uu}, 0\right), \\ B(0,0) &= \left(0, 0, \frac{Z_{uu} - Z_{vv}}{2}, 0, 0\right), \\ C(0,0) &= (0, 0, 0, 0, 0). \end{aligned}$$

Thus, $Z_{uu} \neq Z_{vv}$ because m is not umbilic; $W_{uu} \neq 0$ because m is not a point of inflection; and $Z_{uu} \neq -Z_{vv}$ because m is not a pseudo-umbilic. $\square$

Theorem 5.6. *Let m be a semiumbilic point of S. The Taylor expansion at m of the differential binary equation of the relative mean curvature lines of S is:*

$$(N_{01}u + N_{10}v + \mathcal{O}(2))(du^2 - dv^2) + \mathcal{O}(2)du dv = 0,$$

where

$$\begin{aligned} N_{01} &= W_{uu}W_{uuv}(Z_{uu} - Z_{vv})^2, \\ N_{10} &= W_{uu}W_{uvv}(Z_{uu} - Z_{vv})^2, \\ W_{uu} &\neq 0, \\ |Z_{uu}| &\neq |Z_{vv}| \end{aligned} \tag{5.1}$$

and the coefficients W_{uuv}, W_{uvv} are arbitrary.

Proof. The linear part comes from the expression of ϕ and after some calculus with a symbolic computing program it is obtained that each coefficient of the quadratic term can be controlled by a different coefficient of the expansion of the function ϕ. □

Clearly, the semiumbilic points are not of Darbouxian type for the equation of the mean relative curvature lines, because they do not satisfy the transversality condition. The leaves of the linearized equation consist of an orthogonal net.

We shall see now that, though the transversality conditions fail at a semiumbilic point $p \in S$, it is possible to analyze, following the method developed in [15], the configuration of the relative mean curvature lines around a generic semiumbilic point. By generic we mean here that S must not satisfy some (non-necessary) equality at m.

Definition 5.7 ([15]). *Let m be a singular point of a binary differential equation. It is said to be of type $D^1_{2,3}$ if the following conditions hold:* (1) *The transversality condition* 5.1 *fails at m;* (2) *In the two critical points of the function F on $\Pi^{-1}(m)$ (see subsection* 5.2*), the function F is of Morse type.*

The topological index of a singularity of type $D^1_{2,3}$ is zero and its configuration is described in the figure 1. For details see [15] and [11].

One of the foliations near a $D^1_{2,3}$ point has two semiumbilic separatrices and two hyperbolic sectors. The other has three semiumbilic separatrices, one parabolic and two hyperbolic sectors.

Theorem 5.8. *Let m be a generic semiumbilic point of S. Then, as a singular point of the binary differential equation of the relative mean curvature lines of S, it is of type $D^1_{2,3}$.*

Proof. Consider the preceding chart ϕ of S, around m. The polynomial $\varphi(p)$ whose zeroes give the singularities of the vector field $\mathcal{T}$ is given by

$$\varphi(p) = \frac{1}{4} W_{uu}(Z_{uu} - Z_{vv})^2 (W_{uuv} + p W_{uvv})(p^2 - 1).$$

It is of third degree if, as we assume by genericity, $W_{uvv} \neq 0$. Then, its roots are

$$p_0 = -\frac{Wuuv}{Wuvv}, \quad p_1 = -1, \quad p_2 = 1.$$

If by genericity we assume that

$$\left| -\frac{W_{uuv}}{W_{uvv}} \right| \neq 1,$$

we see that they are simple and the hiperbolicity condition holds.

The critical points of F in the fibre over m are given by the equation

$$\left(\frac{1}{4}(1 - p^2) W_{uu} W_{uuv} (Z_{uu} - Z_{vv})^2,\ \frac{1}{4}(1 - p^2) W_{uu} W_{uvv} (Z_{uu} - Z_{vv})^2,\ 0 \right) = 0.$$

Since $W_{uu} \neq 0$, $Z_{uu} - Z_{vv} \neq 0$ and we have assumed that $W_{uuv} \neq W_{uvv}$, we see that the critical points are $(0, 0, \pm 1)$. The corresponding values of the Hessian of F, computed with Mathematica® are

$$\pm \frac{1}{4} W_{uu}^2 (T_{uvv} W_{uuv} - T_{uuv} W_{uvv})^2 (Z_{uu} - Z_{vv})^5 (Z_{uu} + Z_{vv})$$

and they are non-zero if, as we assume by genericity of m, that $T_{uvv} W_{uuv} - T_{uuv} W_{uvv} \neq 0$. □

The figure below shows an example illustrating the generic configuration of the relative mean curvature lines around a semiumbilic point of a surface in $\mathbb{R}^5$. The drawing has been produced with the aid of the program "ParametricasR5" due to the third author, which is available on request.

Example 5.9. *In this figure the map $\phi : \mathbb{R}^2 \to \mathbb{R}^5$ is given by*

$$\begin{aligned} \phi(u, v) \ = \Big(& u - u^3 v - \frac{5uv^2}{2} + 14u^2 v^2 - \frac{uv^3}{3} - \frac{3v^4}{2}, v + u^2 v \\ & - 4u^3 v - u^2 v^2 - \frac{5v^3}{3} + 12uv^3 - \frac{v^4}{3}, \frac{u^2}{2} - \frac{4v^3}{3} \\ & + \frac{u^3 v}{3} + v^2 - 2uv^2 + \frac{uv^3}{2}, \frac{u^2}{2} + u^2 v + \frac{v^2}{2} - 2uv^2 \\ & + \frac{v^3}{3} + \frac{uv^3}{2}, 2u^2 v + \frac{u^3 v}{3} - uv^2 - \frac{2v^3}{3} \Big). \end{aligned}$$

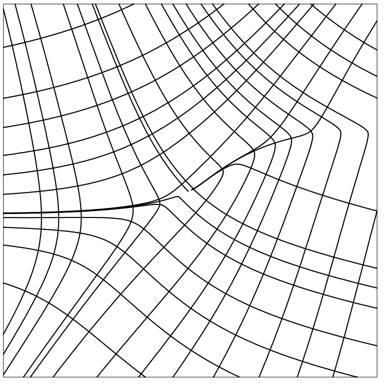

Figure 1: Relative mean curvature lines configuration around semiumbilic point, for a surface on $\mathbb{R}^5$.

Its coefficients have been obtained by choosing more or less at random the coefficients in the expressions of X, Y, Z, W, T that do not depend on other coefficients.

5.3 Generic configurations at pseudo-umbilic points

In this section we will see that a generic pseudo-umbilic point m of a surface $S \in \mathcal{I} \subset \mathrm{Imm}(\mathbb{R}^2, \mathbb{R}^5)$ is of Darbouxian type. We recall that at these points the mean curvature vector is perpendicular to the plane of the curvature ellipse, which is not degenerate. Thus, in this case the conditions $H \neq 0$, $B \wedge C \neq 0$, $H \cdot B = H \cdot C = 0$ hold at $(0, 0)$.

Proposition 5.10. *Let m be a pseudo-umbilic point of S. Then, we can find an orthonormal affine base of $\mathbb{R}^5$ and an isothermal chart ϕ of S in a neighborhood of m such that the function:*

$$\phi(u, v) = (X(u, v), Y(u, v), Z(u, v), W(u, v), T(u, v)),$$

obtained in the preceding section verifies:

$$W_{uu} = W_{vv} = Z_{uv} = T_{uv} = 0, \quad T_{vv} = T_{uu}, \quad Z_{vv} = -Z_{uu}.$$

Proof. The proof proceeds as in 5.2, taking the affine reference so that $B(0, 0) = be_3$, $C(0, 0) = ce_4$ and $H(0, 0) = he_5$. $\square$

Theorem 5.11. *Let m be a generic pseudo-umbilic point of S. Then, as a singular point of the binary differential equation of the relative mean curvature lines of S, it is of Darbouxian type.*

Proof. The differential equation of the relative mean curvature lines up to degree one is given by

$$(Ju + Lv + \mathcal{O}(2))(du^2 - dv^2) + 2(Pu + Qv + \mathcal{O}(2))dudv = 0,$$

where

$$\begin{aligned} J &= Z_{uu}^2(2T_{uu}T_{uuv} + W_{uv}(W_{uuu} + W_{uvv})); \\ L &= Z_{uu}^2(2T_{uu}T_{uvv} + W_{uv}(W_{uuv} + W_{vvv})); \\ P &= W_{uv}^2(T_{uu}(T_{uvv} - T_{uuu}) - Z_{uu}(Z_{uuu} + Z_{uvv})); \\ Q &= W_{uv}^2(T_{uu}(T_{vvv} - T_{uuv}) - Z_{uu}(Z_{uuv} + Z_{vvv})). \end{aligned}$$

The coefficient W_{vvv} appears linearly in the product $P(0,0)L(0,0)$, whereas it does not appear in the product $J(0,0)Q(0,0)$. Conversely, the coefficient W_{uuu} appears linearly in $J(0,0)Q(0,0)$ and not at all in $P(0,0)L(0,0)$. Hence, if m is generic, $P(0,0)L(0,0) - J(0,0)Q(0,0)$ does not vanish, and the transversality condition is verified.

We check now the hyperbolicity condition.

In the chart $\left(u, v, p = \frac{dv}{du}\right)$ on PS around $\Pi^{-1}(0,0)$, the singularities of the Lie-Cartan vector field are determined by the roots of the cubic polynomial

$$\varphi(p) = Lp^3 + (J - 2Q)p^2 - (2P + L)p - J.$$

This polynomial has only simple roots provided its discriminant does not vanish, which is a generic condition. □

5.4 Some global consequences

Application of the Poincaré-Hopf index formula for foliations on closed oriented surfaces leads to the following:

Corollary 5.12. *The number $\mathcal{N}_{ps}$ of pseudo-umbilic points of a closed oriented surface S generically immersed into $\mathbb{R}^n$, $n \geq 5$, satisfies the relation*

$$\mathcal{N}_{ps} \geq 2|\chi(S)|,$$

where $\chi(S)$ denotes de Euler number of S.

Proof. Just observe that the index of the relative mean curvature foliations is zero at generic semiumbilics and $\pm\frac{1}{2}$ at generic pseudo-umbilics. $\square$

Then from the definition of pseudo-umbilic point it follows:

Corollary 5.13. *Any generic immersion of a* 2*-sphere into* $\mathbb{R}^n$ *has at least* 4 *points at which the mean curvature vector* H *is orthogonal to the normal subspace determined by the curvature ellipse.*

In the general case of non necessarily generic immersions we can assert:

Corollary 5.14. *Closed oriented surfaces with non vanishing Euler number immersed into* $\mathbb{R}^n$, $n \geq 5$, *always have either some semiumbilic, pseudo-umbilic, inflection, or minimal point.*

On the other hand, we can consider the special subset of 2-regular immersions of surfaces in $\mathbb{R}^n$, $n \geq 5$. These were introduced independently by E.A. Feldman [5] and W. Pohl [23]. They are characterized by the fact that the normal subspace spanned by the second fundamental form has maximal dimension at every point (or in other words, dim $N_p^1 S = 3$, $\forall p \in S$). This means in our context that the vectors H, B and C are linearly independent at every point. It was shown by Feldman [5] that the subset of 2-regular immersions of any closed surface in $\mathbb{R}^n$ is open and dense (in the Whitney C^∞-topology over the set of immersions) provided $n \geq 7$. A 2-regular immersion of the 2-sphere into $\mathbb{R}^5$ was described in [3] and the existence of a wider class of such immersions is discussed in [25]. Nevertheless the existence of 2-regular immersions of surfaces with non zero genus into $\mathbb{R}^5$ still remains as a conjecture.

The above considerations imply the following.

Corollary 5.15. *Closed oriented* 2*-regular surfaces with non vanishing Euler number in* $\mathbb{R}^n$, $n \geq 5$, *always have pseudo-umbilic points (minimal points considered as a particular case).*

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