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Some generalizations of Knopp's identity*

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Abstract. For integers a, b and $n > 0$, define

$$A_\Gamma(a,b,n) = \sideset{}{'}\sum_{\substack{r=1 \\ n\nmid b}}^{n} \left(\left(\frac{ar}{n}\right)\right) \ln \Gamma\left(\left\{\frac{b\bar{r}}{n}\right\}\right)$$

and

$$B_\Gamma(a,b,n) = \sideset{}{'}\sum_{\substack{r=1 \\ n\nmid b}}^{n} \left(\left(\frac{ar}{n}\right)\right) \frac{\Gamma'\left(\left\{\frac{b\bar{r}}{n}\right\}\right)}{\Gamma\left(\left\{\frac{b\bar{r}}{n}\right\}\right)},$$

where $\sideset{}{'}\sum_{r}$ denotes the summation over all r such that $(r,n)=1$, and $\bar{r}$ is defined by the equation $r\bar{r} \equiv 1 \bmod n$. The two sums are analogous to the homogeneous Dedekind sum $S(a,b,n)$. The functional equations for A_Γ and B_Γ are established. Furthermore, Knopp's identity on Dedekind sum is extended.

Keywords: Dedekind sum, Knopp's identity, homogeneous.

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1 Introduction

For integers a and $n > 0$, the classical Dedekind sum is defined by

$$S(a,n) = \sum_{j=1}^{n} \left(\left(\frac{j}{n}\right)\right)\left(\left(\frac{aj}{n}\right)\right),$$

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where

$$((x)) = \begin{cases} x - [x] - \dfrac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

The sum $S(a, n)$ plays an important role in the transformation theory of the Dedekind η function (see [6] and Chapter 3 of [1] for details).

Knopp [4] established the following arithmetical identity

$$\sum_{cd=m} \sum_{r=1}^{d} S(ac + rn, dn) = \sigma(m) S(a, n),$$

where $\sigma(m) = \sum_{d|m} d$. Many authors gave elementary proofs for Knopp's identity, for example, Goldberg [3], Parson [5] and Zheng [7].

For integers a, b and $n > 0$, the sum

$$S(a, b, n) = \sum_{r=0}^{n-1} \left(\left(\frac{ar}{n}\right)\right) \left(\left(\frac{br}{n}\right)\right)$$

is called a homogeneous Dedekind sum. Zheng [8] obtained the following extension of Knopp's identity

$$\sum_{cd=m} \sum_{r_1, r_2 \in d_*} S(ac + r_1 n, bc + r_2 n, dn) = m\sigma(m) S(a, b, n),$$

where $d_* = \{r \in \mathcal{Z} : 0 \le r < d\}$.

According to [2], for a function F of two complex variables into the complex field C, if for any ordered pair $< x, y >$ in the domain $\mathrm{Dom}(F)$ of F we have

$$\left\{ \left\langle \frac{x + r}{n}, ny \right\rangle : r \in n_* \right\} \subseteq \mathrm{Dom}(F) \quad \text{and} \quad \sum_{r=0}^{n-1} F\left(\frac{x + r}{n}, ny\right) = F(x, y)$$

for every $n = 1, 2, 3, \cdots$, then we call F a uniform function into C.

A uniform function F is said to be periodic if

$$\langle x, y \rangle \in \mathrm{Dom}(F) \Longrightarrow \langle x \pm 1, y \rangle \in \mathrm{Dom}(F) \quad \& \quad F(x \pm 1, y) = F(x, y).$$

We use PUF to denote the class of all periodic uniform functions.

Set

$$\begin{bmatrix} F; x, y \\ G; u, v \end{bmatrix} (a, b, n) = \sum_{r=0}^{n-1} F\left(\frac{x + ar}{n}, ny\right) G\left(\frac{u + br}{n}, nv\right).$$

Chen and Sun [2] gave the following extension of Knopp's identity.

Proposition 1.1. *Let $a, b \in \mathcal{Z}$, $m, n \in \mathcal{Z}^+$, $F, G \in PUF$, $\langle x, y\rangle \in \mathrm{Dom}(F)$ and $\langle u, v\rangle \in \mathrm{Dom}(G)$. Then we have the identity*

$$\sum_{cd=m} \sum_{r_1, r_2 \in d_*} \begin{bmatrix} F; x, y \\ G; u, v \end{bmatrix} (ac + r_1 n, bc + r_2 n, dn) = m \sum_{d|m} d \begin{bmatrix} F; x/d, dy \\ G; u/d, dv \end{bmatrix} (a, b, n).$$

For $a, b \in \mathcal{Z}$ and $n \in \mathcal{Z}^+$, define

$$S_\Gamma\,(a, b, n) = \sum_{\substack{r=0 \\ n \nmid br}}^{n-1} \left(\left(\frac{ar}{n}\right)\right) \ln \Gamma \left(\left\{\frac{br}{n}\right\}\right)$$

and

$$T_\Gamma\,(a, b, n) = \sum_{\substack{r=0 \\ n \nmid br}}^{n-1} \left(\left(\frac{ar}{n}\right)\right) \frac{\Gamma'\left(\left\{\frac{br}{n}\right\}\right)}{\Gamma\left(\left\{\frac{br}{n}\right\}\right)},$$

where $\Gamma(x)$ is the well-known gamma function. Applying Proposition 1.1 they obtained the following results.

Proposition 1.2. *Let $a, b \in \mathcal{Z}$ and $m, n \in \mathcal{Z}^+$.*

(i) *For the function T_Γ we have the functional function:*

$$\sum_{cd=m} \frac{1}{d} \sum_{r_1, r_2 \in d_*} T_\Gamma\,(ac + r_1 n, bc + r_2 n, dn) = m d(m) T_\Gamma\,(a, b, n)\,,$$

where $d(m)$ is the divisor function.

(ii) *For S_Γ we have*

$$\sum_{cd=m} \sum_{r_1, r_2 \in d_*} S_\Gamma\,(ac + r_1 n, bc + r_2 n, dn) - m\sigma(m) S_\Gamma\,(a, b, n) = m \sum_{d|m} \Lambda(d) \sigma\left(\frac{m}{d}\right) \left(\frac{S(a, b, n/(d, n))}{d/(d, n)} - S(a, b, n)\right),$$

where (d, n) is the greatest common divisor of d and n, and $\Lambda(n)$ is the Mangoldt function.

In this paper, we shall give some further generalizations of Knopp's identity. Let

$$\left\{ \begin{array}{l} F; x, y \\ G; u, v \end{array} \right\} (a, b, n) = \sum_{r=1}^{n}{}' F\left(\frac{x+ar}{n}, ny\right) G\left(\frac{u+b\bar{r}}{n}, nv\right),$$

where $\sum_{r}{}'$ denotes the summation over all r such that $(r, n) = 1$, and $\bar{r}$ is defined by the equation $r\bar{r} \equiv 1 \bmod n$. In Section 2 we will prove the following:

Theorem 1.1. *Let $a, b \in \mathcal{Z}$, $m, n \in \mathcal{Z}^+$ with $(m, n) = 1$, $F, G \in PUF$, $\langle x, y\rangle \in \mathrm{Dom}(F)$ and $\langle u, v\rangle \in \mathrm{Dom}(G)$. Then we have the identity*

$$\sum_{d|m} \sum_{r_1, r_2 \in d_*} \left\{ \begin{array}{l} F; x, y \\ G; u, v \end{array} \right\} (a + r_1 n, b + r_2 n, dn) = m \left\{ \begin{array}{l} F; x, y \\ G; u, v \end{array} \right\} (a, b, n).$$

Theorem 1.2. *Let $a, b \in \mathcal{Z}$, $m, n \in \mathcal{Z}^+$ with $(m, n) = 1$, $F, G \in PUF$, $\langle x, y\rangle \in \mathrm{Dom}(F)$ and $\langle u, v\rangle \in \mathrm{Dom}(G)$. Then we have*

$$\sum_{cd=m} \sum_{r_1, r_2 \in d_*} \left\{ \begin{array}{l} F; x, y \\ G; u, v \end{array} \right\} (ac + r_1 n, b\bar{c} + r_2 n, dn) = m \left\{ \begin{array}{l} F; x, y \\ G; u, v \end{array} \right\} (a, b, n),$$

where $c\bar{c} \equiv 1 \bmod n$.

Moreover, define

$$A_\Gamma(a, b, n) = \sum_{\substack{r=1 \\ n \nmid b}}^{n}{}' \left(\left(\frac{ar}{n}\right)\right) \ln \Gamma\left(\left\{\frac{b\bar{r}}{n}\right\}\right)$$

and

$$B_\Gamma(a, b, n) = \sum_{\substack{r=1 \\ n \nmid b}}^{n}{}' \left(\left(\frac{ar}{n}\right)\right) \frac{\Gamma'\left(\left\{\frac{b\bar{r}}{n}\right\}\right)}{\Gamma\left(\left\{\frac{b\bar{r}}{n}\right\}\right)}.$$

In Section 3 we shall prove the following:

Theorem 1.3. *Let $a, b \in \mathcal{Z}$ and $m, n \in \mathcal{Z}^+$ with $(m, n) = 1$.*

(i) *For the function B_Γ we have the functional function:*

$$\sum_{d|m} \frac{1}{d} \sum_{r_1, r_2 \in d_*} B_\Gamma\left(a + r_1 n, b + r_2 n, dn\right) = m B_\Gamma\left(a, b, n\right).$$

(ii) *For A_Γ we have*

$$\sum_{d|m} \sum_{r_1, r_2 \in d_*} A_\Gamma\left(a + r_1 n, b + r_2 n, dn\right) - m A_\Gamma\left(a, b, n\right) = \left(m \ln n - \sum_{d|m} \ln(dn)\right) C(a, b, n),$$

where

$$C(a, b, n) = \sum_{r=1}^{n}{}' \left(\left(\frac{ar}{n}\right)\right)\left(\left(\frac{b\bar{r}}{n}\right)\right).$$

Theorem 1.4. *Let $a, b \in \mathcal{Z}$ and $m, n \in \mathcal{Z}^+$ with $(m, n) = 1$.*

(i) *For the function B_Γ we have the functional function:*

$$\sum_{cd=m} \frac{1}{d} \sum_{r_1, r_2 \in d_*} B_\Gamma\left(ac + r_1 n, b\bar{c} + r_2 n, dn\right) = m B_\Gamma\left(a, b, n\right),$$

where $c\bar{c} \equiv 1 \bmod n$.

(ii) *For A_Γ we have*

$$\sum_{cd=m} \sum_{r_1, r_2 \in d_*} A_\Gamma\left(ac + r_1 n, b\bar{c} + r_2 n, dn\right) - m A_\Gamma\left(a, b, n\right) = \left(m \ln n - \sum_{d|m} \ln(dn)\right) C(a, b, n).$$

2 Proofs of Theorem 1.1 and Theorem 1.2

First we need the following lemma.

Lemma 2.1. *Let $a \in \mathcal{Z}$, $n \in \mathcal{Z}^+$, $F \in PUF$ and $\langle x, y\rangle \in \mathrm{Dom}(F)$. Then*

$$\sum_{r=0}^{n-1} F\left(\frac{x+ar}{n}, ny\right) = (a,n)F\left(\frac{x}{(a,n)}, (a,n)y\right).$$

Proof. This is Lemma 2.1 of [2].

Now we prove Theorem 1.1. By Lemma 2.1 we have

$$\begin{aligned}
&\sum_{d|m}\sum_{r_1,r_2\in d_*}\left\{\begin{matrix} F;x,y \\ G;u,v\end{matrix}\right\}(a+r_1n, b+r_2n, dn)\\
&=\sum_{d|m}\sum_{r_1,r_2\in d_*}\sum_{r=1}^{dn}{}' F\left(\frac{x+(a+r_1n)r}{dn}, dny\right) G\left(\frac{u+(b+r_2n)\overline{r}}{dn}, dnv\right)\\
&=\sum_{d|m}\sum_{r=1}^{dn}{}'\sum_{r_1\in d_*} F\left(\frac{x+ar}{dn}+\frac{r_1r}{d}, dny\right)\sum_{r_2\in d_*} G\left(\frac{u+b\overline{r}}{dn}+\frac{r_2\overline{r}}{d}, dnv\right)\\
&=\sum_{d|m}\sum_{r=1}^{dn}{}' F\left(\frac{x+ar}{n}, ny\right) G\left(\frac{u+b\overline{r}}{n}, nv\right).
\end{aligned} \tag{2.1}$$

Since $(m,n)=1$, we get $(d,n)=1$. Let $r=r_1d+r_2n$, then $\overline{r}=\overline{r}_1\overline{d}^2d+\overline{r}_2\overline{n}^2n$, where

$$d\overline{d}\equiv 1 \bmod n,\quad n\overline{n}\equiv 1 \bmod d,\quad r_1\overline{r}_1\equiv 1 \bmod n,\quad r_2\overline{r}_2\equiv 1 \bmod d.$$

Therefore

$$\begin{aligned}
&\sum_{d|m}\sum_{r=1}^{dn}{}' F\left(\frac{x+ar}{n}, ny\right) G\left(\frac{u+b\overline{r}}{n}, nv\right)\\
&=\sum_{d|m}\sum_{r_1=1}^{n}{}'\sum_{r_2=1}^{d}{}' F\left(\frac{x+a(r_1d+r_2n)}{n}, ny\right) G\left(\frac{u+b(\overline{r}_1\overline{d}^2d+\overline{r}_2\overline{n}^2n)}{n}, nv\right)\\
&=\sum_{d|m}\sum_{r_1=1}^{n}{}'\sum_{r_2=1}^{d}{}' F\left(\frac{x+ar_1d}{n}, ny\right) G\left(\frac{u+b\overline{r}_1\overline{d}}{n}, nv\right)\\
&=\sum_{d|m}\phi(d)\sum_{r_1=1}^{n}{}' F\left(\frac{x+ar_1d}{n}, ny\right) G\left(\frac{u+b\overline{r}_1\overline{d}}{n}, nv\right).
\end{aligned} \tag{2.2}$$

When r_1 runs over a reduced residue system modulo n, $r_1 d$ also runs over a reduced residue system modulo n. So we have

$$\begin{aligned}
&\sum_{d|m}\phi(d)\sum_{r_1=1}^{n}{}' F\left(\frac{x+ar_1d}{n}, ny\right) G\left(\frac{u+b\overline{r}_1\overline{d}}{n}, nv\right) \\
&= \sum_{d|m}\phi(d)\sum_{r_1=1}^{n}{}' F\left(\frac{x+ar_1}{n}, ny\right) G\left(\frac{u+b\overline{r}_1}{n}, nv\right) \\
&= m\left\{\begin{array}{l} F; x, y \\ G; u, v \end{array}\right\}(a, b, n).
\end{aligned} \tag{2.3}$$

Theorem 1.1 immediately follows from (2.1), (2.2) and (2.3). Similarly we can get Theorem 1.2.

3 Proofs of Theorem 1.3 and Theorem 1.4

According to [2], define $D(x, y) = ((x))$ and $\Gamma^* : \left(\mathcal{R}^+ \bigcup \{0\}\right) \times \mathcal{R}^+ \longrightarrow \mathcal{R}^+$ as follows:

$$\Gamma^*(x, y) = \begin{cases} \Gamma(x)y^x/\sqrt{2\pi y}, & \text{if } x > 0; \\ \sqrt{2\pi y}, & \text{if } x = 0. \end{cases}$$

We know that $D(x, y)$ and $\Gamma_*(x, y) = \ln \Gamma^*(\{x\}, y)$ belong to PUF. Let

$$\Psi(x, y) = \begin{cases} \left(\Gamma'(x)/\Gamma(x) + \ln y\right)/y, & \text{if } x > 0; \\ (-\gamma + \ln y)/y, & \text{if } x = 0. \end{cases}$$

The function $\psi(x, y) = \widetilde{\Psi}(x, y)$ also lies in PUF.

Now we establish the following:

Lemma 3.1. *Let $a, b \in \mathcal{Z}$ and $m, n \in \mathcal{Z}^+$. Then we have*

$$\left\{\begin{array}{l} D; 0, m \\ \Gamma_*; 0, m \end{array}\right\}(a, b, n) = A_\Gamma(a, b, n) + \ln(mn)C(a, b, n)$$

and

$$\left\{\begin{array}{l} D; 0, m \\ \psi; 0, m \end{array}\right\}(a, b, n) = \frac{B_\Gamma(a, b, n)}{mn}.$$

Proof. From the definition of Γ_* we have

$$\left\{ \begin{array}{l} D; 0, m \\ \Gamma_*; 0, m \end{array} \right\} (a, b, n) = \sum_{r=1}^{n}{}' \left(\left(\frac{ar}{n}\right)\right) \Gamma_* \left(\frac{b\bar{r}}{n}, mn\right)$$

$$= \sum_{\substack{r=1 \\ n\nmid b}}^{n}{}' \left(\left(\frac{ar}{n}\right)\right) \left[\ln\Gamma\left(\left\{\frac{b\bar{r}}{n}\right\}\right) + \left(\left\{\frac{b\bar{r}}{n}\right\} - \frac{1}{2}\right)\ln(mn) - \frac{\ln(2\pi)}{2}\right]$$

$$+ \sum_{\substack{r=1 \\ n\mid b}}^{n}{}' \left(\left(\frac{ar}{n}\right)\right) \frac{\ln(2\pi mn)}{2}.$$

By Lemma 2.1 we get

$$\sum_{r=1}^{n}{}' \left(\left(\frac{ar}{n}\right)\right) = \sum_{r=1}^{n} \left(\left(\frac{ar}{n}\right)\right) \sum_{\substack{d\mid n \\ d\mid r}} \mu(d) = \sum_{d\mid n} \mu(d) \sum_{r=1}^{n/d} \left(\left(\frac{ar}{n/d}\right)\right) = 0.$$

Therefore

$$\begin{aligned} \left\{ \begin{array}{l} D; 0, m \\ \Gamma_*; 0, m \end{array} \right\} (a, b, n) &= \sum_{\substack{r=1 \\ n\nmid b}}^{n}{}' \left(\left(\frac{ar}{n}\right)\right) \ln\Gamma\left(\left\{\frac{b\bar{r}}{n}\right\}\right) \\ &\quad + \ln(mn) \sum_{r=1}^{n}{}' \left(\left(\frac{ar}{n}\right)\right)\left(\left(\frac{b\bar{r}}{n}\right)\right) \\ &= A_\Gamma(a, b, n) + \ln(mn) C(a, b, n). \end{aligned}$$

Similarly from Lemma 2.1 we have

$$\left\{ \begin{array}{l} D; 0, m \\ \psi; 0, m \end{array} \right\} (a, b, n) = \sum_{r=1}^{n}{}' \left(\left(\frac{ar}{n}\right)\right) \psi\left(\frac{b\bar{r}}{n}, mn\right)$$

$$= \sum_{\substack{r=1 \\ n\nmid b}}^{n}{}' \left(\left(\frac{ar}{n}\right)\right) \frac{1}{mn} \left(\frac{\Gamma'\left(\left\{\frac{b\bar{r}}{n}\right\}\right)}{\Gamma\left(\left\{\frac{b\bar{r}}{n}\right\}\right)} + \ln(mn)\right)$$

$$+ \sum_{\substack{r=1 \\ n\mid b}}^{n}{}' \left(\left(\frac{ar}{n}\right)\right) \frac{1}{mn} \left(-\gamma + \ln(mn)\right)$$

$$= \frac{1}{mn} \sum_{\substack{r=1 \\ n \nmid b}}^{n}{}' \left(\left(\frac{ar}{n} \right) \right) \frac{\Gamma' \left(\left\{ \frac{b\bar{r}}{n} \right\} \right)}{\Gamma \left(\left\{ \frac{b\bar{r}}{n} \right\} \right)} = \frac{B_\Gamma(a, b, n)}{mn}.$$

This proves Lemma 3.1.

Now we prove Theorem 1.3 and Theorem 1.4. By Theorem 1.1,

$$\sum_{d|m} \sum_{r_1, r_2 \in d_*} \left\{ \begin{array}{l} D; 0, 1 \\ \psi; 0, 1 \end{array} \right\} (a + r_1 n, b + r_2 n, dn) = m \left\{ \begin{array}{l} D; 0, 1 \\ \psi; 0, 1 \end{array} \right\} (a, b, n),$$

then from Lemma 3.1 we have

$$\sum_{d|m} \frac{1}{d} \sum_{r_1, r_2 \in d_*} B_\Gamma \left(a + r_1 n, b + r_2 n, dn \right) = m B_\Gamma \left(a, b, n \right).$$

Similarly from Theorem 1.2 and Lemma 3.1 we can get

$$\sum_{cd=m} \frac{1}{d} \sum_{r_1, r_2 \in d_*} B_\Gamma \left(ac + r_1 n, b\bar{c} + r_2 n, dn \right) = m B_\Gamma \left(a, b, n \right),$$

where $c\bar{c} \equiv 1 \bmod n$. This proves (i) of Theorem 1.3 and Theorem 1.4.

By Theorem 1.1 we also have

$$\sum_{d|m} \sum_{r_1, r_2 \in d_*} \left\{ \begin{array}{l} D; 0, 1 \\ \Gamma_*; 0, 1 \end{array} \right\} (a + r_1 n, b + r_2 n, dn) = m \left\{ \begin{array}{l} D; 0, 1 \\ \Gamma_*; 0, 1 \end{array} \right\} (a, b, n),$$

then from Lemma 3.1 we get

$$\begin{aligned} &\sum_{d|m} \sum_{r_1, r_2 \in d_*} A_\Gamma \left(a + r_1 n, b + r_2 n, dn \right) \\ &\qquad + \sum_{d|m} \sum_{r_1, r_2 \in d_*} \ln(dn) C \left(a + r_1 n, b + r_2 n, dn \right) \\ &= m A_\Gamma \left(a, b, n \right) + m \ln n C(a, b, n). \end{aligned}$$

From the proof of Theorem 1.1 we easily have

$$\sum_{d|m} \sum_{r_1, r_2 \in d_*} \ln(dn) C \left(a + r_1 n, b + r_2 n, dn \right) = \sum_{d|m} \ln(dn) C(a, b, n).$$

Therefore

$$\sum_{d|m} \sum_{r_1,r_2 \in d_*} A_\Gamma (a + r_1 n, b + r_2 n, dn) - m A_\Gamma (a, b, n)$$
$$= \left(m \ln n - \sum_{d|m} \ln(dn) \right) C(a, b, n).$$

Similarly from Theorem 1.2 and Lemma 3.1 we get

$$\sum_{cd=m} \sum_{r_1,r_2 \in d_*} A_\Gamma (ac + r_1 n, b\bar{c} + r_2 n, dn) - m A_\Gamma (a, b, n)$$
$$= \left(m \ln n - \sum_{d|m} \ln(dn) \right) C(a, b, n).$$

This completes the proofs of (ii) of Theorem 1.3 and Theorem 1.4.

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