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Robustly transitive actions of $\mathbb{R}^2$ on compact three manifolds

Ali Tahzibi and Carlos Maquera

Abstract. In this paper, we define robust transitivity for actions of $\mathbb{R}^2$ on closed connected orientable manifolds. We prove that if the ambient manifold is three dimensional and the dense orbit of a robustly transitive action is not planar, then the action is defined by an Anosov flow, i.e. its orbits coincide with the orbits of an Anosov flow.

Keywords: singular action, compact orbit, closing lemma, robust transitivity, Anosov flow.

Mathematical subject classification: Primary: 37C85.

1 Introduction

In some recent works in the theory of dynamical systems robust transitivity of diffeomorphisms and flows has been investigated. It has been shown that, weak forms of hyperbolicity are necessary conditions for robust transitivity of flows and diffeomorphisms of compact manifolds. Bonatti-Díaz-Pujals [1] proved that C^1-robustly transitive diffeomorphisms admit dominated splittings. Previous to their work, Díaz-Pujals-Ures [3] had proved that robustly transitive diffeomorphism on three dimensional manifolds are partially hyperbolic. For C^1-flows, there are parallel results on robust transitivity. See for example a result of Vivier [15] about robustly transitive flows on any dimension and a result

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of Doering [4] in three dimensional case. For more recent results about robustly transitive subsets of flows, see [6] or [8].

Using Kupka-Smale theorem in one dimensional case, one deduces that there does not exist any robustly transitive diffeomorphism on one dimensional manifolds. Also, by a result of Peixoto [9], we know that the Morse-Smale flows form a dense subset of the set of C^1-flows on any surface. As Morse-Smale flows can not be transitive, we conclude that robustly transitive flows may exist only on manifolds with dimension higher than two.

If we consider the diffeomorphisms or flows defined on a manifold as the action of $\mathbb{Z}$, $\mathbb{R}$ on it, a natural question arises: "what about robustly transitive actions of higher dimensional groups?"

In this paper, we begin with the study of robustly transitive actions of $\mathbb{R}^2$ by giving some examples of these actions and proving that in three dimensional manifolds the only robustly transitive actions of $\mathbb{R}^2$ (We do not consider the case when all orbits are planar) are defined by robustly transitive flows (see Theorem 1.3). By a result of Doering, we know that robustly transitive flows on three dimensional manifolds are in fact Anosov.

Let N denote a closed connected orientable three manifold and $\varphi : \mathbb{R}^2 \times N \to N$ be a C^r-action. By definition

$$\varphi(u, \varphi(v, x)) = \varphi(u + v, x), \forall u, v \in \mathbb{R}^2, \forall x \in N.$$

For each $w \in \mathbb{R}^2 \setminus \{0\}, \varphi$ induces a C^r-flow $(\varphi^t_w)_{t\in\mathbb{R}}$ given by $\varphi^t_w(p) = \varphi(tw, p)$ and its corresponding C^{r-1}-vector field X_w is defined by $X_w(p) = D_1\varphi(0, p)\cdot w$. If $\{w_1, w_2\}$ is a base of $\mathbb{R}^2$, the associated vector fields X_{w_1}, X_{w_2} satisfy the commutativity condition $[X_{w_1}, X_{w_2}] = 0$ and determine completely the action φ. They are called *infinitesimal generators* of φ. This condition of commutativity between two vector fields is a necessary and sufficient condition for them to be generators of an action. $X_{(1,0)}$ and $X_{(0,1)}$ are called the *canonical infinitesimal generators*.

Denote by $A^r(\mathbb{R}^2, N)$ $1 \le r \le \omega$ the set of actions of $\mathbb{R}^2$ on N whose infinitesimal generators are of class C^r. Given two actions $\{\varphi; X_{(1,0)}, X_{(0,1)}\}$

and $\{\psi; Y_{(1,0)}, Y_{(0,1)}\}$ define,

$$d_{(1,1)}(\varphi, \psi) = \max\left\{\|X_{(1,0)} - Y_{(1,0)}\|_1, \|X_{(0,1)} - Y_{(0,1)}\|_1\right\}.$$

With this distance $A^r(\mathbb{R}^2, N)$ is a metric space and its corresponding topology is called the $C^{(1,1)}$*-topology.* Note that this topology is finer than the C^2-topology and coarser than the C^1-topology. For any action $\phi \in A^r(\mathbb{R}^2, M)$, $\mathcal{O}_p := \{\phi(\omega, p), \omega \in \mathbb{R}^2\}$ is called the orbit of $p \in M$. The orbit is called *singular* if its dimension is less than two.

Given any C^1-vector field, one can construct an action of $\mathbb{R}^2$ with all orbits singular. Let X be a C^1 vector field on N. Let $X_1 = X$ and $X_2 = fX$ such that $X(f) = 0$. It is clear that X_1 and X_2 commute and consequently define an action on N.

Definition 1.1. *We say that an action $\varphi \in A^1(\mathbb{R}^2, N)$ is* defined by a flow, *if there exists $X \in \mathcal{X}^1(N)$ such that the orbits of φ coincide with the orbits of X.*

Clearly if φ is defined by a flow corresponding to X then any other generator of φ is linearly dependent to X.

Definition 1.2. *An action $\varphi \in A^1(\mathbb{R}^2, N)$ is called* transitive *if it admits a dense orbit. φ is* robustly transitive *if all actions in a $C^{(1,1)}$ neighborhood of it, are transitive.*

Our main result is the following theorem.

Theorem 1.3. *Let N be a closed orientable* 3*-manifold. Assume that $\varphi \in A^1(\mathbb{R}^2, N)$ is robustly transitive with a dense orbit which is not homeomorphic to $\mathbb{R}^2$. Then, φ is defined by an Anosov flow.*

We mention that the hypotheses about the topological type of the dense orbit is important to our result. By this hypotheses, the dense orbit is cylindrical or homeomorphic to $\mathbb{R}$. However, we conjecture that the same result is true without this hypotheses.

We would like to thank C. Bonatti for mentioning us, that Rosenberg had left the stability problem of the action with all leaves planar (which is the case we are avoiding here) as an open problem. More precisely, let ϕ be an action of $\mathbb{R}^2$

on $\mathbb{T}^3$ with all leaves homeomorphic to $\mathbb{R}^2$. It is not known whether in general ϕ is topologically stable or not. It is known that such action is topologically equivalent to a linear action of $\mathbb{R}^2$ (see [2]).

To prove our main result, we study the topological type of the orbits of a robustly transitive action. In general, one can have only three different topological types for non singular (two dimensional) orbits of an action of $\mathbb{R}^2$. The non-singular orbits are homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{S}^1 \times \mathbb{S}^1$. Firstly we show that under the hypotheses of the main theorem, if the dense orbit of the robustly transitive action is homeomorphic to $\mathbb{R}$, then it is defined by an Anosov flow. Finally, by means of a closing lemma for $\mathbb{R}^2$ actions, we prove that a robustly transitive action (whitout dense orbit homeomorphic to $\mathbb{R}^2$) can not have a dense cylinder.

The paper is organized as following: In section 2, we give some examples of robustly transitive actions and prove some topological properties of the orbits of $\mathbb{R}^2$ actions. The lemmas proved in this section are used in section 3. In section 3, firstly, we recall a closing lemma for actions without planar leaves and then prove the main theorem.

2 Examples and basic results

Let us give some examples of robustly transitive actions of $\mathbb{R}^2$.

Example 2.1. Firstly we construct a singular (defined by flow) example of a robustly transitive action. Consider a robustly transitive expansive (we demand robustness of both transitivity and expansiveness) flow defined by vector field X on a manifold N. Any robustly transitive Anosov flow is an example of such flow. Let $\phi \in A^1\big(\mathbb{R}^2, N\big)$ be the action defined by $X_1 := X$ and $X_2 := cX\,(c \in \mathbb{R})$.

It is obvious that $\big[X_1, X_2\big] = 0$ and so they define a transitive action of $\mathbb{R}^2$ in N. Clearly, all orbits of this action are singular. We claim that ϕ is a robustly transitive action. Indeed, suppose $\psi \in A^1\big(\mathbb{R}^2, N\big)$ any $C^{(1,1)}$ perturbation of ϕ. By the definition of $C^{(1,1)}$-topology in $A^1\big(\mathbb{R}^2, N\big)$ we conclude that ψ is defined by $\tilde{X}_1, \tilde{X}_2$ such that $\big[\tilde{X}_1, \tilde{X}_2\big] = 0$ and $\tilde{X}_i$ is C^1- close to X_i, $i = 1, 2$. So, $\tilde{X}_1$ is also an expansive transitive vector field. By a result of Masatoshi [7] the centralizer of an expansive flow is trivial. This means that $\tilde{X}_2 = f\tilde{X}_1$ (f is a first integral) and consequently ψ is also defined by a transitive flow. Observe that, in this way, for any $k \geq 2$ we can give example of $\mathbb{R}^k$ robustly transitive actions on $l-$dimensional manifolds, $l \geq k+1$.

Let denote by X^t the flow of a vector field X.

Example 2.2. Let N be a three dimensional manifold supporting a robustly transitive Anosov flow. We construct a robustly transitive action in $M^4 = N \times \mathbb{S}^1$ which is not defined by a flow. Consider the coordinate system (x, θ) in M^4, $x \in N, \theta \in \mathbb{S}^1$. In what follows, for a real function $a(x, \theta)$, by $a(x, \theta)\frac{\partial}{\partial x}$ we mean $a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3}$ where x_1, x_2, x_3 are coordinates in N.

Let $\phi \in A^1\big(\mathbb{R}^2, M^4\big)$ be defined by X_1 and X_2 such that $X_1 = a(x)\frac{\partial}{\partial x}$ is a robustly transitive Anosov flow in N and $X_2 := \frac{\partial}{\partial \theta}$. We claim that ϕ is robustly transitive.

Consider a $C^{(1,1)}$-perturbation ψ of the initial action ϕ. It is generated by two vector fields Y_1 and Y_2 which are respectively C^1 close to X_1 and X_2.

Let $N_0 := \big\{(x, 0) : x \in N\big\}$. By transversality of X_2 to N_0 and closeness of X_2 and Y_2 we conclude that Y_2 is also transverse to N_0.

In our coordinate systems

$$Y_1 = \tilde{a}(x, \theta)\frac{\partial}{\partial x} + b(x, \theta)\frac{\partial}{\partial \theta}.$$

$$Y_2 = c(x, \theta)\frac{\partial}{\partial x} + d(x, \theta)\frac{\partial}{\partial \theta}.$$

where b and c are close to zero in C^1-topology, a and $\tilde{a}$ are close in each coordinates and d is close to constant 1. We define

$$\Pi(Y_1) = dY_1 - bY_2 = (\tilde{a}d - bc)\frac{\partial}{\partial x}.$$

Observe that in N_0, $\Pi(Y_1)$ is a C^1-vector field close to X_1 and consequently it is transitive. The intersection of the orbits of ψ with N_0 coincide with the orbits of $\Pi(Y_1)|_{N_0}$. Let $x_0 \in N_0$ with a $\Pi(Y_1)$ dense orbit. We claim that the orbit of ψ passing through x_0 is dense in M^4. To see this, just observe that N_0 is a global transverse manifold for Y_2. Let $U \in M^4$ an open set and $V = \bigcup_{t \in \mathbb{R}} Y_2^t(U) \cap N_0$.

Then, V is an open subset of N_0 and by density of the orbit of x_0 by $\Pi(Y_1)$ there exists $t \in \mathbb{R}$, $Y_1^t(x_0) \in V$ and consequently for some $s \in \mathbb{R}$, $Y_2^s(Y_1^t(x_0)) \in U$ and this means that the $\psi-$orbit of x_0 is dense in M and ψ is transitive.

Now, we outline some basic results about actions of $\mathbb{R}^2$ which will be used in the proof of the main theorem. Recall that, for any action $\phi \in A^1\big(\mathbb{R}^2, N\big)$, $\mathcal{O}_p := \big\{\phi(\omega, p), \omega \in \mathbb{R}^2\big\}$ is the orbit of $p \in N$ and $G_p := \big\{\omega \in \mathbb{R}^2 \colon \phi(\omega, p) = p\big\}$ is called the isotropy group of p. Observe that groups isomorphic to $\mathbb{R}\times\mathbb{R}$, $\mathbb{R}\times\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}\times\mathbb{Z}$ and $\{0\}$, are respectively isotropy groups of orbits homeomorphic to single point, circle, line, cylinder, torus and plane.

Lemma 2.3. *Suppose that $\mathcal{O}_q$ is accumulated by $\mathcal{O}_p$ then $G_p \subseteq G_q$. Consequently, any two dense orbits are homeomorphic.*

Proof. To prove, just observe that for $\omega \in G_p$, by definition of action and isotropy group we have $\phi(\omega, \phi(\eta, p)) = \phi(\eta, p)$, for any $\eta \in \mathbb{R}^2$. So, by continuity of ϕ we conclude that, if z is an accumulation point of $\mathcal{O}_p$ then $\phi(\omega, z) = z$ and consequently we have $\omega \in G_q$.

Finally, observe that if both $\mathcal{O}_p$ and $\mathcal{O}_q$ are dense, we conclude that $G_p = G_q$ and this implies that the two orbits are homeomorphic. □

Using the above lemma we can show that all the dense orbits of φ have the same topological type. In the setting of Theorem 1.3 all the dense orbits are either line or cylinder. In fact, it is easy to see that the existence of a dense line prohibits the existence of any (not necessarily dense) cylinder.

Lemma 2.4. *If $\varphi \in A^1(\mathbb{R}^2, N)$ has a dense orbit homeomorphic to $\mathbb{R}$, then φ is given by a transitive flow.*

Proof. Let X_1, X_2 be two infinitesimal generators for φ. By existence of a dense orbit homeomorphic to $\mathbb{R}$ and continuity of X_1 and X_2, we conclude that for any $x \in N$, $X_1(x)$, $X_2(x)$ are linearly dependent. So, by definition φ is given by a transitive flow. □

Lemma 2.5. *If $\phi \in A^1\big(\mathbb{R}^2, N\big)$ has a dense cylinder orbit, then any two dimensional orbit is either homeomorphic to torus or to cylinder.*

Proof. Suppose that $\mathcal{O}_p$ is a dense cylinder orbit. The isotropy group of p is $\mathbb{Z}u$ for some $0 \neq u \in \mathbb{R}^2$. Let Y be the vector field such that $Y^t = \phi(tu, \cdot)$. It is clear that $Y^1(p) = p$. Let X be the vector field whose flow corresponding is $X^t = \phi(tv, \cdot)$, where v is any linearly independent to u. As X and Y commute every point on $\mathcal{O}_p$ is periodic with period one for Y. Indeed, any $z \in \mathcal{O}_p$ can be written as $z = Y^t(X^s(x))$. So,

$$Y^1(z) = Y^1(Y^t(X^s(x))) = Y^t(X^s(Y^1(x))) = Y^t(X^s(x)) = z.$$

Now, using denseness of $\mathcal{O}_p$ and continuity of Y^1 we conclude that any point of the manifold is a periodic point for Y which finishes the proof of the lemma. $\square$

3 Closing lemma and proof of the main result

First of all let us recall the closing lemma of Pugh ([10, Theorem 6.1]) for the flows in a two dimensional manifold.

Theorem 3.1. *Let $X \in \mathcal{X}^1(M^2)$ have a nontrivial recurrent trajectory through $p^* \in M$, let U be a neighborhood of p^* and $\epsilon > 0$ be given. Then, there exists $Z \in \mathcal{X}(M)$ such that:*

1. *$X - Z$ vanishes on $M \setminus U$,*
2. *the C^1-size of $X - Z$ is less than ϵ respecting the U-coordinates,*
3. *Z has a closed orbit through p^*.*

In [13], Roussarie and Weil proved a closing lemma for the action of $\mathbb{R}^2$ on three manifolds. More precisely one of their results is the following:

Theorem 3.2. *Let N be an orientable compact closed C^r $(r > 2)$, 3-manifold and φ a locally free C^r-action. If all orbits of φ are not planar, then there is a locally free action $\varphi_1 \in A^r\left(\mathbb{R}^2, N\right)$ with a compact orbit and C^1-close to φ.*

To prove the above theorem, the authors firstly observe that either φ has a compact orbit or all the orbits are dense. In the latter case just take $\varphi = \varphi_1$. In the former case, the denseness of all orbits is a corollary of a result of

Sacksteder [14] about the minimal sets of $\mathbb{R}^{n-1}$ actions on $n-$manifolds. The result of Sacksteder states that there is no exceptional minimal set for locally free actions. Using the denseness of a cylinder, one can show that all other orbits are cylindrical. In this setting (all the orbits are cylindrical) the proof of Pugh closing lemma for flows on surfaces can be carried on to prove that φ can be perturbed to give a compact orbit.

Let us mention that the above theorem is not the main result of Roussarie and Weil´s paper. In fact, their paper is mainly dedicated to the proof of the following theorem [13, Theorem 2 (1)].

Theorem 3.3. *Let ϕ be a C^r-action. For all non-planar and recurrent orbit Λ and for all $\epsilon > 0$ there exists a submanifold diffeomorphic to $\mathbb{T}^2$, ϵ-close to Λ such that the plane field tangent to this submanifold can be extended to a plane field C^1 near to plane field corresponding to ϕ.*

The main issue in this result is to find a nearby torus to the recurrent leaf.

Here we have a general action which can have singularities. However, we suppose that there exists a dense cylinder and claim a closing lemma.

Let $\mathcal{O}_p$ be a cylindrical orbit of $\varphi \in A^1\big(\mathbb{R}^2, N\big)$ and $\big\{w_1, w_2\big\}$ be a base of $\mathbb{R}^2$ such that w_2 is a generator of the group G_p . Write $X = X_{w_1}$, $Y = X_{w_2}$, then Y has periodic orbit through p. We may suppose that it has period one.

Theorem 3.4. *Let N be an orientable compact closed, 3-manifold and $\varphi \in A^1\big(\mathbb{R}^2, N\big)$. If there exists a dense orbit $\mathcal{O}_p$ of φ homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$, then there is an action $\varphi_1 \in A^1\big(\mathbb{R}^2, N\big)$ with a compact orbit and $C^{(1,1)}$-close to φ. Moreover, the perturbation is supported on a neighborhood of the periodic orbit of Y through p.*

Proof. To use the closing lemma of Pugh, we should adapt the ideas for the case of actions. We emphasize that the lack of planar orbits is crucial to obtain such a closing lemma. Whenever, we have a dense cylinder we choose a closed orbit of one of the infinitesimal generating vector fields and take an adequate system of coordinates around this closed orbit. Firstly, we introduce this coordinate system.

3.1 Infinitesimal generators adapted to a $S^1 \times \mathbb{R}$-orbit.

Note that if $q \in \mathcal{O}_p$, then the orbit of Y passing through q is periodic of period one too (see proof of Lemma 2.5). Put a Riemannian metric on N and let ξ be the norm one vector field defined in a neighborhood of the Y_p^I that is orthogonal to the orbits of φ.

Let c be the circle orbit of Y through p. For small $\epsilon > 0$, define the ring $A_\epsilon = \left\{Y^I(\xi^t(c)), |t| \leq \epsilon\right\}$ (see figure 1). As the action is φ is orientable and ϵ is small, A_ϵ is diffeomorphic to $S^1 \times (-\epsilon, \epsilon)$. We parametrize c with $\theta \in [0, 1]$ such that $\frac{\partial}{\partial\theta} = Y|c$. We put a coordinate system (x, θ, z) in a small neighborhood of c diffeomorphic to $S^1 \times (-\epsilon, \epsilon) \times (-1, 1)$ such that:

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial \theta}, \quad \xi = \frac{\partial}{\partial z}.$$

In this new coordinates system the (pieces of) orbits of φ inside such neighborhood are $z =$ constant.

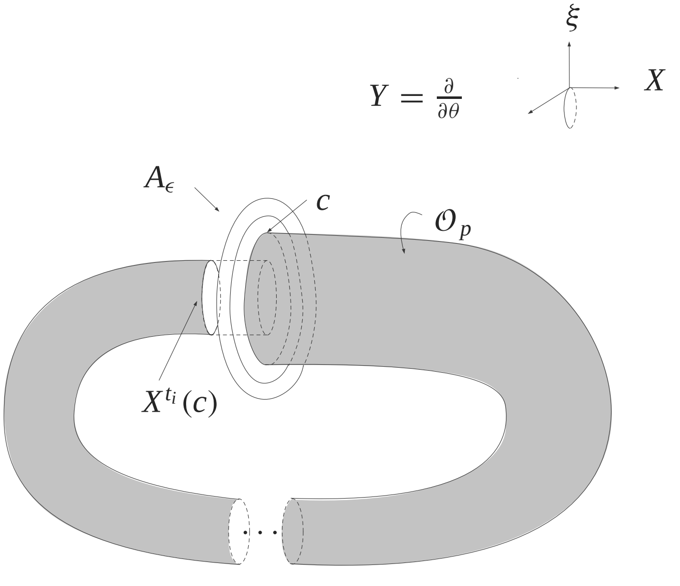

Figure 1: Adapted coordinates near c.

An small box in our new coordinates will serve as a flow box of the closing lemma of Pugh. All the orbits passing through this box are two dimensional.

Like in the closing lemma for flows, we have a transversal section, which is a ring in our case. We construct this ring foliated by closed orbits of Y. Now, we should take care about the returns of the dense orbit in our neighborhood. More precisely, we show that the dense orbit returns and intersects the transversal section in closed orbits.

Lemma 3.5. *Let $\mathcal{O}_p$ be a dense cylindrical orbit of $\phi \in A^1(\mathbb{R}^2, N)$ and c (homeomorphic to S^1) be the periodic orbit of Y passing through p. Then, for any neighborhood $U(c) \subset N$ of c there exists an unbounded sequence $t_i \in \mathbb{R}$ such that $X^{t_i}(c) \subset U(c)$.*

This lemma was announced in [12] in the non-singular actions context. For completeness of the proof we show that singularities do not matter. In what follows X and Y are two generating infinitesimal vector fields for $\phi \in A^1(\mathbb{R}^2, M)$.

Proof. Let $\mathcal{U}_\epsilon$ be an ϵ-neighborhood of c in M such that $Y^t(z) \in U(c)$ for any $z \in \mathcal{U}, t \in [0,1]$. By density of $\mathcal{O}_p$ there exist $z \in c, t \in \mathbb{R}$ such that $X^t(z) \in \mathcal{U}_\epsilon$. It comes out that $Y^I(X^t(z)) \in U(c)$ where $Y^I(\cdot)$ stands for $\{Y^s(\cdot), s \in I = [0,1]\}$. But by commutativity

$$Y^I(X^t(z)) = X^t(Y^I(z)) = X^t(c) \in U(c).$$

As ϵ can be any small number, we conclude that there is a sequence $t_i \to \infty$ such that $X^{t_i}(c) \in U(c)$. $\square$

So, we can carry the proof of the closing lemma for flows to the case of actions of $\mathbb{R}^2$ whose orbits are not planar. $\square$

3.2 Proof of the main theorem

Let $\mathcal{U}$ be a $C^{(1,1)}$ neighborhood of φ such that every action in $\mathcal{U}$ is transitive.

We will prove that φ is defined by an Anosov flow. As previously we mentioned (Lemma 2.4), if the dense orbit of a transitive action is one-dimensional then the action is defined by a flow. In what follows we will show that a robustly transitive action can not have a dense cylinder. So, we conclude that in fact φ is given by a robustly transitive flow and by a result of Doering [4], it comes out that φ is an Anosov flow.

First of all, we state a technical lemma which is standard in algebraic topology.[1]

Lemma 3.6. *Let N be a three dimensional compact orientable manifold. There exists $k \in \mathbb{N}$ such that if $T_1, T_2, \ldots, T_k$ are submanifolds homeomorphic to torus $\mathbb{T}^2$, then they form the boundary of a three dimensional submanifold of N.*

Having in mind the above lemma, we conclude that if A has k compact orbits then there can not exist any dense orbit. Indeed, any dense two dimensional submanifold should intersect one of these k tori.

Suppose that φ has a dense cylinder $\mathcal{O}_p$. Let c be the periodic orbit (homeomorphic to S^1) through p and A_ϵ the ring defined in 3.1. Recall that $\{z = 0\} \cap A_\epsilon = c$ and all $\{z = t\} \cap A_\epsilon$, $|t| \leq \epsilon$ are periodic orbits of the generating vector field Y. By Lemma 2.5 all two dimensional orbits are either cylindrical or homeomorphic to torus.

Take $\epsilon > 0$ such that all the orbits passing through A_ϵ are cylindrical. In fact, if there does not exist such an ϵ we conclude that there are more than k torus and using the above lemma we contradict the denseness of $\mathcal{O}_p$. For $0 \leq i \leq k-1$ Let

$$A_i := \left\{ \frac{\epsilon i}{k} < z < \frac{(i+1)\epsilon}{k} \right\}.$$

By the denseness of $\mathcal{O}_p$ and Lemma 3.5 there exists a return time $\bar{t}$ such that $X^{\bar{t}}(c) \in \left\{ |z| < \frac{\epsilon}{k} \right\}$. As $X^{\bar{t}}(p) \in \left\{ |z| < \frac{\epsilon}{k} \right\}$ we project $X^{\bar{t}}(p)$ along the orbit of X and find out t such that $X^t(p) \in A_0$. By definition, A_ϵ is foliated by the orbits of Y and by the commutativity of X and Y one concludes that $X^t(c) \in A_0$ for some t. Indeed,

$$X^t(Y^s(p)) = Y^s(X^t(p)) \in A_0 \quad \text{for all} \quad s \in [0, 1]$$

which means that $X^t(c) \in A_0$.

[1] The authors would like to thank C. Biasi for usefull comments and a proof on this lemma. Later, we find out that a similar lemma was proved in [11]. So, we omit the similar proof.

Now, we use the closing lemma (Theorem 3.4) and perturb φ inside $\left\{|z| < \frac{\epsilon}{k}\right\}$ and find a new action φ_1 and $C^{(1,1)}$-close to φ with a compact orbit. As φ_1 is also transitive, it has a dense orbit which we claim it is of cylindrical type. To see this remember that our perturbation is supported on $\left\{|z| < \frac{\epsilon}{k}\right\}$ and consequently the orbits passing through A_i, $i > 0$ remains cylindrical. So, the dense orbit of φ_1 which necessarily intersect $\left\{\frac{1}{k} < z < \frac{2}{k}\right\}$ is cylindrical. Perturbing again by the closing lemma (Theorem 3.4) we obtain another invariant torus and by induction we find $\varphi_k \in A^1(\mathbb{R}^2, N)$ with k compact leaves which by Lemma 3.6 form the frontier of a compact three manifold with boundary inside N and consequently no dense orbit can exist which gives a contradiction.

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Ali Tahzibi and **Carlos Maquera**
Universidade de São Paulo – São Carlos
Instituto de Ciências Matemáticas e de Computação
Av. do Trabalhador São-Carlense, 400
13560-970 São Carlos, SP
BRAZIL

E-mails: tahzibi@icmc.usp.br /
cmaquera@icmc.usp.br