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# Hyperbolicity, heterodimensional cycles and Lyapunov exponents for partially hyperbolic dynamics

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**Abstract.** We prove a dichotomy of $C^2$ partially hyperbolic sets with one-dimensional center direction admitting no zero Lyapunov exponents, either hyperbolicity over the supports of ergodic measures or approximation by a heterodimensional cycle. This provides a partial result to the $C^1$ Palis Conjecture that claims a dichotomy, hyperbolicity or homoclinic bifurcations in a dense subset of the space of $C^1$ diffeomorphisms. Moreover, a theorem of Mañé applied in the proof is modified to have an additional property concerning the Hausdorff distance between a periodic orbit and the support of a hyperbolic ergodic measure.

**Keywords:** partial hyperbolicity, heterodimensional cycles, Lyapunov exponents, hyperbolic measures, Pesin set.

**Mathematical subject classification:** 37C29, 37D30.

## 1 Introduction

Let $M$ be a smooth compact manifold without boundary, and let $\mathrm{Diff}^r(M)$ $(r \geq 1)$ be the space of $C^r$ diffeomorphisms with the $C^r$ topology. In order to understand the dynamics beyond uniform hyperbolicity, Palis has conjectured that every diffeomorphism $f \in \mathrm{Diff}^r(M)$ in the complement of the closure of Axiom A diffeomorphisms (hyperbolicity of the nonwandering set $\Omega(f)$ that is the closure of all periodic points) can be approximated by some $g \in \mathrm{Diff}^r(M)$ exhibiting a homoclinic tangency or a heterodimensional cycle [P]. For the $C^1$ case (that is considered to be the only realistic one in the present situation), Pujals and Sambarino solved it when $\dim M = 2$ [PS]. For higher dimensions, partial results have been obtained by Pujals ([Pu1], [Pu2]) and Wen ([W]). On the other

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hand, the study of partially hyperbolic dynamics is crucial for the understanding of nonhyperbolic dynamics, and has been one of the main subject of dynamical systems (see [BDV]). So, it is reasonable to ask the $C^1$ Palis Conjecture for partially hyperbolic diffeomorphisms. In this paper, we shall give a partial result to this problem.

A *dominated splitting* on a compact invariant set $\Lambda$ of $f \in \mathrm{Diff}^1(M)$ is a continuous, $Df$-invariant splitting

$$TM|\Lambda = E \oplus F$$

such that there exist $m \in \mathbf{Z}^+$ and $0 < \lambda < 1$ satisfying

$$\|(Df^m)|E(x)\| \cdot \|(Df^{-m})|F(f^m(x))\| < \lambda$$

for all $x \in \Lambda$. In particular, if dim $E(x)$ is constant for all $x \in \Lambda$, we call it a *homogeneous* dominated splitting.

We say that $TM|\Lambda = F_1 \oplus F_2 \oplus F_3$ is a *double dominated splitting* if both $F_1 \oplus (F_2 \oplus F_3)$ and $(F_1 \oplus F_2) \oplus F_3$ are dominated splittings. In particular, we say that a subbundle $F_1$ (resp. $F_3$) is *contracting* (resp. *expanding*) if there exist $m \in \mathbb{Z}^+$ and $0 < \lambda < 1$ satisfying

$$\|(Df^m)|F_1(x)\| < \lambda$$

$$(\text{resp. } \|(Df^{-m})|F_3(x)\| < \lambda\ )$$

for all $x \in \Lambda$.

We say that $\Lambda$ is a *partially hyperbolic set with one-dimensional center* of $f \in \mathrm{Diff}^1(M)$ if there exists a continuous $Df$-invariant splitting

$$TM|\Lambda = E^s \oplus E^c \oplus E^u$$

with dim $E^c(x) = 1$ ($x \in \Lambda$), satisfying the following properties:

a) the splitting is double dominated;

b) both subbundles $E^s$ and $E^u$ are not zero;

c) $E^s$ is contracting and $E^u$ is expanding.

Denote by $W^{ss}_{loc}(x)$ (resp. $W^{uu}_{loc}(x)$) the local strong stable (resp. unstable) manifold of $x$ tangent to $E^s(x)$ (resp. $E^u(x)$) at $x$. Note that if $E^c$ is zero then $\Lambda$ is a hyperbolic set. When $\Lambda = M$ is a partially hyperbolic set with one-dimensional center, $f$ is called a *partially hyperbolic diffeomorphism with one-dimensional center*.

We define, for every hyperbolic periodic point $p$, its index $\mathrm{Ind}\,(p)$ by the dimension of the stable subspace $E^s(p)$. A *heterodimensional cycle* is a geometric configuration between two hyperbolic periodic points with different indices such that their stable and unstable manifolds have mutual nonempty intersection; i.e., if $p, q \in \mathrm{Per}(f)$ with $\mathrm{Ind}\,(p) \neq \mathrm{Ind}\,(q)$ satisfy $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$ then we say that $f$ exhibits a heterodimensional cycle. Note that one of the intersections is not transversal. In particular, we say that $f$ exhibits a heterodimensional cycle in $U$ if there are points $x \in W^s(p) \cap W^u(q)$ and $y \in W^u(p) \cap W^s(q)$ such that the closure of the full orbit of $x$ and that of $y$ are both contained in $U$. Since any partially hyperbolic diffeomorphism with one-dimiensional center does not exhibit a homoclinic tangency, the dichotomy in a dense subset of $\mathrm{Diff}^1(M)$, either Axiom A diffeomorphisms or ones with a heterodimensional cycle, is the conjecture in our case.

Let $\mathcal{M}(M)$ denote the set of probabilities on the Borel $\sigma$-algebra $\mathcal{B}$ of $M$ endowed with its usual topology; i.e., the unique metrizable topology such that $\mu_k \to \mu$ if and only if $\int \varphi d\mu_k \to \int \varphi d\mu$ for every continuous function $\varphi : M \to \mathbf{R}$. Denote by $\mathcal{M}_f(M)$ the set of $f$-invariant elements of $\mathcal{M}(M)$ and by $\mathcal{M}_e(f)$ the set of ergodic elements of $\mathcal{M}_f(M)$. If $f \in \mathrm{Diff}^1(M)$, denote by $\Lambda(f)$ the set of regular points; i.e., the set of points $x \in M$ satisfying the following properties: there exists a splitting $T_xM = \bigoplus_{i=1}^s E_i(x)$ (the *Lyapunov splitting* at $x$) and numbers $\lambda_1(x) > \cdots > \lambda_s(x)$ (the *Lyapunov exponents* at $x$) such that $\lim_{n\to\pm\infty} \frac{1}{n} \log \|(D_x f^n)v\| = \lambda_i(x)$ for every $1 \le i \le s$ and $0 \neq v \in E_i(x)$. By Oseledets' theorem, $\Lambda(f)$ has total measure; that is, $\mu(\Lambda(f)) = 1$ for every $\mu \in \mathcal{M}_f(M)$. (See [BP], [M1] or [Po].) Define

$$E^-(x) = \bigoplus_{\lambda_i(x)<0} E_i(x), \quad E^+(x) = \bigoplus_{\lambda_i(x)>0} E_i(x),$$

and

$$E^0(x) = \bigoplus_{\lambda_i(x)=0} E_i(x)$$

at every $x \in \Lambda(f)$. We say that $\mu \in \mathcal{M}_e(f)$ is *hyperbolic* if $E^0(x) = \{0\}$ at $\mu$-a.e. $x$. Similarly to the index of a hyperbolic periodic point, we denote the index of hyperbolic ergodic measure $\mu$ by $\mathrm{Ind}\,(\mu) = \dim E^-(x)$ for $\mu$-a.e. $x \in \Lambda(f)$. Define

$$S(f) = \overline{\big\{x \in \mathrm{supp}\,(\mu) : \mu \in \mathcal{M}_e(f) \ \text{is hyperbolic}\big\}}.$$

Denote by $\mathrm{Per}(f)$ the set of periodic points of $f$ and $\mathrm{Per}_h(f)$ that of hyperbolic ones in $\mathrm{Per}(f)$. Note that $\overline{\mathrm{Per}_h(f)} \subset S(f)$.

The following theorem provides a partial result to the conjecture above.

**Theorem A.** *Let $f \in \mathrm{Diff}^1(M)$ be a $C^2$ diffeomorphism admitting no zero Lyapunov exponents (any Lyapunov exponent of any ergodic measure of $f$ is non-zero) and $\Lambda$ be a partially hyperbolic set with one-dimensional center of $f$. Then, one of the following properties holds:*

a) *$S(f) \cap \Lambda$ is a hyperbolic set;*

b) *given a $C^1$ neighborhood $\mathcal{U}$ of $f$ and a neighborhood $U$ of $S(f) \cap \Lambda$, there exists $g \in \mathcal{U}$ exhibiting a heterodimensional cycle in $U$.*

A theorem of Mañé [M2, Theorem II.1] will be applied in the proof of Theorem A to our partially hyperbolic setting. The following theorem is its modified version, giving us an additional property (which is not necessary to prove Theorem A) concerning the Hausdorff distance between the periodic orbit given in the conclusion and the support of a hyperbolic ergodic measure. The hypothesis is stronger than the original one, but includes the case where $\Lambda$ is the closure of hyperbolic periodic points with the same index to which [M2, Theorem II.1] actually applied in the proof of the $C^1$ Stability Conjecture [M2]. Denote by $\mathcal{O}_f^+(x)$ (resp. $\mathcal{O}_f^-(x)$) the forward (resp. backward) $f$-orbit of $x$, and let $\mathcal{O}_f(x) = \mathcal{O}_f^+(x) \cup \mathcal{O}_f^-(x)$. A finite part of orbit $\{x, f(x), \dots, y\}$ with $y = f^n(x)$ in $\mathcal{O}_f^+(x)$ is called a *string* and written as: $(x, y;\ f)$ or just $(x,\ f^n(x))$ when it is not necessary to specify $f$.

**Theorem B.** *Let $\Lambda$ be a compact invariant set of $g \in \mathrm{Diff}^1(M)$ written as:*

$$\Lambda = \overline{\left\{x \in \mathrm{supp}\,(\mu) : \mu \in \mathcal{M}\right\}}$$

*for some $\mathcal{M} \subset \mathcal{M}_e(g)$ consisting of hyperbolic measures with the same index, and let $TM|\Lambda = E \oplus F$ be a homogeneous dominated splitting with $\dim E(x) = \mathrm{Ind}\,(\mu)$ ($x \in \Lambda$, $\mu \in \mathcal{M}$) such that $E$ is contracting. Suppose that there exists $c > 0$ such that*

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|(Dg^{-1})|F(g^j(x))\| \leq -c \tag{1}$$

*at $\mu$-a.e. $x$ for all $\mu \in \mathcal{M}$. Then either $F$ is expanding or for every sufficiently small neighborhood $V$ of $\Lambda$, every $0 < \gamma < 1$ and $\delta > 0$, there exists a periodic* *point* $p \in M(g, V) \cap \mathrm{Per}(g)$ *with arbitrarily large period* $\ell$ *such that* $\mathcal{O}_g(p)$ *contains a substring* $\delta$*-close to* $\mathrm{supp}\,(\mu)$ *for some* $\mu \in \mathcal{M}$ *with respect to the Hausdorff distance and satisfying*

$$\gamma^{\ell} < \prod_{j=1}^{\ell} \|(Dg^{-1})|\widehat{F}(g^j(p))\| < 1, \tag{2}$$

*where* $\widehat{F}$ *is given by the unique homogeneous dominated splitting* $TM|M(g,V) = \widehat{E} \oplus \widehat{F}$ *that extends* $TM|\Lambda = E \oplus F$*, and* $M(g,V)$ *is the maximal* $g$*-invariant set in* $V$.

In Section I, we consider a partially hyperbolic dynamics and create a heterodimensional cycle from the lack of hyperbolicity of $S(f) \cap \Lambda$ to prove Theorem A. For the creation, we first find a transversal intersection of two hyperbolic periodic points with different indices under the circumstance of Pesin Theory. Then, we apply extended versions of the $C^1$ Connecting Lemma to have also a nontransversal one. In Section II, we prove Theorem B based on the proof of [M2, Theorem II.1].

## I. Proof of Theorem A

In this section, we shall prove Theorem A using Theorem B and extended Connecting Lemmas.

First, we give definitions and notations. By the Ergodic Decomposition Theorem, a Borel set $\Gamma(f)$ defined as the set of $x \in M$ for which we have $\mu_x \in \mathcal{M}_e(f)$ and $x \in \mathrm{supp}\,(\mu_x)$ has total measure, where $\mu_x$ is the unique probability measure on the Borel $\sigma$-algebra of $M$ such that, for every continuous $\varphi : M \to \mathbf{R}$,

$$\int_M \varphi d\mu_x = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

holds, which comes from the Riesz Representation Theorem. (See [M1, Chapter II.6].) Define

$$\Gamma^i(f) = \left\{x \in \Lambda(f) \cap \Gamma(f) : \mathrm{Ind}\,(\mu_x) = i\right\}.$$

It is easy to see that $\Gamma^i(f)$ is a Borel set (see [Po] for instance), and $\Gamma^i(f) \cap \Gamma^j(f) = \emptyset$ if $i \neq j$. For $i \geq 1$, let

$$S^i(f) = \overline{\left\{x \in \mathrm{supp}\,(\mu) : \mu \in \mathcal{M}_e(f) \ \text{ is hyperbolic, } \ \mathrm{Ind}(\mu) = i\right\}}.$$

The following lemma is $C^1$ perturbation results proved in [H2], which are diffeomorphisms versions extended from the Connecting Lemma introduced in [H1]. So, we give the definitions for diffeomorphisms similar to those for flows given in [H2].

For $p$, $q$ and $r$ in $M$, we say that $p$ is *forwardly related to* $q$ if $q \notin \mathcal{O}^+_f(p)$ and there exists a sequence of strings $\{(x_n, y_n;\ f_n) : n \geq 1\}$ with $\lim_{n\to+\infty} f_n = f$, $\lim_{n\to+\infty} x_n = p$ and $\lim_{n\to+\infty} y_n = q$. Moreover, we say that $p$ is *forwardly related to* $q$, or $q$ is *backwardly related to* $p$, *with one jump at* $r$ if $p$ is forwardly related to $r$ and $r$ is forwardly related to $q$.

**Lemma I.1 (Extended Connecting Lemmas [H2]).**

I) *Given a neighborhood* $\mathcal{U}$ *of* $f \in \mathrm{Diff}^1(M)$ *and* $p, q \in M \setminus \mathrm{Per}(f)$ *such that* $p$ *is forwardly related to* $q$ *by* $f_n \to f$, *then there exist* $\eta > 0$ *and* $g \in \mathcal{U}$ *coinciding with* $f$ *outside an arbitrarily small neighborhood of* $(p,\ f^{J^+}(p);\ f) \cup (f^{J^-}(q), q;\ f)$ *for some* $J^+(\mathcal{U}, p,\ f) > 0$ *and* $J^-(\mathcal{U}, q,\ f) < 0$ *and such that there are* $p'$ *and* $q'$, *respectively arbitrarily close to* $p$ *and* $q$ *independent of* $\eta$, *satisfying the following properties:*

   a) $\mathcal{O}^+_{f_n}(p') = q'$ *for arbitrarily large* $n$;

   b) $g^N(p') = q'$ *for some* $N > 0$;

   c) $(B_\eta(p) \cup B_\eta(q)) \cap (p', g^N(p'); g) = \{p', q'\}$.

II) *Let* $p, q \in M \setminus \mathrm{Per}(f)$ *be such that* $p$ *is forwardly (resp. backwardly) related to* $q$ *with one jump at some* $r \in M \setminus \mathrm{Per}(f)$, *then* $p$ *is forwardly (resp. backwardly) related to* $q$ *by some* $f_n \to f$ *coinciding with* $f$ *outside an arbitrarily small neighborhood of* $\mathcal{O}^+_f(r)$.

Properties I) and II) correspond to [H2, Theorem A and Theorem B], respectively.

Let us see how this lemma will be used to create a heterodimensional cycle. First, we will see that if $S(f) \cap \Lambda$ is not hyperbolic, then there appears a sequence of strings with arbitrarily large length and arbitrarily bad hyperbolicity. Since there is no zero Lyapunov exponents by our hypothesis, the bad hyperbolicity comes from the mixing of positive and negative Lyapunov exponents over the one-dimensional center direction. By the partial hyperbolicity and Katok Closing Lemma, the existence of such orbit causes the transversal intersection between two hyperbolic periodic points with different indices. Then, applying Lemma I.1, we create a nontransversal intersection of the stable and unstable manifolds as the counterpart by an arbitrarily small $C^1$ perturbation. Since the previous intersection is robust by the transversality, a heterodimensional cycle is created by the perturbation.

We apply Theorem B in the following setting to $g = f^{-1}$. Let

$$TM|\Lambda = E^s \oplus E^c \oplus E^u \tag{1}$$

be the partially hyperbolic splitting with $\dim E^c(x) = 1$ $(x \in \Lambda)$. Taking a subset of $\Lambda$ if necessary, we let the dimension of $E^s(x)$ be constant for all $x \in \Lambda$. Suppose that $f \in \mathrm{Diff}^1(M)$ is a $C^2$ diffeomorphism admitting no zero Lyapunov exponents. Without loss of generality, we may assume that

$$S^{i_0}(f) \cap \Lambda = S(f) \cap \Lambda, \quad i_0 = \dim E^s(x) + 1 \tag{2}$$

for all $x \in S(f) \cap \Lambda$; for otherwise $S^{i_0-1}(f) \cap \tilde{\Lambda} \neq \emptyset$ for some compact invariant subset $\tilde{\Lambda} \subset S(f) \cap \Lambda$ with $\tilde{\Lambda} \cup (S^{i_0}(f) \cap \Lambda) = S(f) \cap \Lambda$ and then it is enough to consider $f|\tilde{\Lambda}$ as well as $f^{-1}|(S^{i_0}(f) \cap \Lambda)$. Let

$$TM|(S^{i_0}(f) \cap \Lambda) = E \oplus F \tag{3}$$

be a homogeneous dominated splitting with $E = (E^s \oplus E^c)|(S^{i_0}(f) \cap \Lambda)$ and $F = E^u|(S^{i_0}(f) \cap \Lambda)$. Then, by (1), $F$ is an expanding subbundle. In order to prove Theorem A, it suffices to show that if $S^{i_0}(f) \cap \Lambda$ is not hyperbolic then we can find $g$ exhibiting a heterodimensional cycle in a given neighborhood $U_0$ of $S^{i_0}(f) \cap \Lambda$ by a $C^1$ small perturbation.

By Theorem B for $g = f^{-1}$, if $S^{i_0}(f) \cap \Lambda$ is not hyperbolic, then either the hypothesis corresponding to (1) of Theorem B does not hold or one of the two options of the conclusion in which $E$ of (3) is not contracting for $f$ holds. First suppose that the hypothesis does not hold. Then, there exist sequences $\mu_n \in \mathcal{M}_e(f)$ of index $i_0$, $\mu_n$-a.e. points $x_n$, $c_n > 0$ with $c_n \to 0$ and $\ell_n \in \mathbf{Z}^+$ with $\lim_{n\to+\infty} \ell_n = +\infty$ that can be arbitrarily large with $x_n$ fixing, satisfying

$$\frac{1}{\ell_n} \sum_{j=0}^{\ell_n-1} \log \|(Df)|E(f^j(x_n))\| > -c_n. \tag{4}$$

By domination property, we have

$$\prod_{j=0}^{\ell_n-1} \|(Df)|E(f^j(x_n))\| = \prod_{j=0}^{\ell_n-1} \|(Df)|E^c(f^j(x_n))\|$$

for sufficiently large $n$. From this, (2) and (4) together with hypothesis that $E^c$ is one-dimensional, we have a choice of $\ell_n$ such that

$$e^{-c_n \ell_n} \leq \prod_{j=0}^{\ell_n - 1} \|(Df)|E(f^j(x_n))\| = \|(Df^{\ell_n})|E^c(x_n)\| < 1 \tag{5}$$

for large $n$. Define a continuous function $\varphi : \Lambda \to \mathbb{R}$ by:

$$\varphi(x) = \log \|(Df)|E^c(x)\|$$

and a sequence of probabilities $\nu_n$, $n \geq 1$, by:

$$\nu_n = \frac{1}{\ell_n} \sum_{j=0}^{\ell_n - 1} \delta_{f^j(x_n)}.$$

Then, from (5), we have

$$\begin{aligned} -c_n \leq \int \varphi d\nu_n &= \frac{1}{\ell_n} \sum_{j=0}^{\ell_n - 1} \log \|(Df)|E^c(f^j(x_n))\| \\ &= \frac{1}{\ell_n} \log \|(Df^{\ell_n})|E^c(x_n)\| < 0. \end{aligned}$$

Taking an $f$-invariant measure $\nu \in \mathcal{M}_f(M)$ as an accumulation point of $\{\nu_n : n \geq 1\}$ in $\mathcal{M}(M)$ and applying Birkhoff's Ergodic Theorem, we get

$$\begin{aligned} 0 = \int \varphi d\nu &= \int \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df)|E^c(f^j(p))\| d\nu(p) \\ &= \sum_{i=1,2} \int_{\Gamma^i} \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df)|E^c(f^j(p))\| d\nu(p), \end{aligned} \tag{6}$$

where $\Gamma^1 = \bigcup_{i=1}^{i_0 - 1} \Gamma^i(f)$ and $\Gamma^2 = \bigcup_{i \geq i_0} \Gamma^i(f)$.

For $j = 1, 2$ and $\kappa \in \mathbf{Z}^+$, let

$$\Gamma^j_\kappa = \Gamma^j \cap \bigcup_{k=1}^{\kappa} \Lambda_k,$$

where $\bigcup_{k=1}^{\infty} \Lambda_k$ is the Pesin set (see [BP] or [Po]). Since $f$ does not admit zero Lyapunov exponents, (6) implies that there is $\kappa \in \mathbf{Z}^+$ such that both $\nu(\Gamma^1_\kappa) > 0$

and $\nu(\Gamma_\kappa^2) > 0$ hold. By the regularity of $\nu$, $\nu(B) = \sup\{\nu(C) : C$ is closed, $C \subset B\}$ for every $B \in \mathcal{B}$. So, we can take compact sets (that may not be invariant):

$$S \subset \operatorname{supp}(\nu) \cap \Gamma_\kappa^1 \subset \Lambda \quad \text{and} \quad T \subset \operatorname{supp}(\nu) \cap \Gamma_\kappa^2 \subset \Lambda$$

such that $\nu(S) > 0$ and $\nu(T) > 0$. Then $S \cap T = \emptyset$. Let $\delta(\kappa) > 0$ be such that local stable and unstable manifolds $W^\sigma_{\delta(\kappa)}(x)$, $x \in \Gamma_\kappa^1 \cup \Gamma_\kappa^2$ ($\sigma = s, u$) are defined (see [BP] or [Po]). By continuity, there exists $0 < \delta < \delta(\kappa)$ such that we have transversal intersections

$$W^{ss}_{loc}(y) \pitchfork W^u_{\delta(\kappa)/2}(S) \neq \emptyset \quad \text{and} \quad W^{uu}_{loc}(z) \pitchfork W^s_{\delta(\kappa)/2}(T) \neq \emptyset$$

for all $y \in U_\delta(S) \cap \Lambda$ and $z \in U_\delta(T) \cap \Lambda$ with

$$\overline{U_\delta(S)} \cap \overline{U_\delta(T)} = \emptyset$$

and

$$\overline{U_\delta(S^{i_0}(f) \cap \Lambda)} \subset U_0, \tag{7}$$

where $U_\rho(G) = \{x \in M : d(x, G) < \rho\}$. By Katok Closing Lemma (see [K] or [Po]), we can find $q \in U_{\delta/2}(S) \cap \mathrm{Per}_h(f)$ and $r \in U_{\delta/2}(T) \cap \mathrm{Per}_h(f)$ approximating some points $\bar{q} \in S$ and $\bar{r} \in T$, respectively, and such that

$$\mathcal{O}_f(q) \cup \mathcal{O}_f(r) \subset U_{\bar{\delta}}(S^{i_0}(f) \cap \Lambda) \subset V_0 \tag{8}$$

for any small $\bar{\delta} > 0$, where $V_0 = U_{\delta/2}(S^{i_0}(f) \cap \Lambda)$. Then,

$$W^{ss}_{loc}(y_n) \pitchfork W^u_{\delta(\kappa)}(q) \neq \emptyset \quad \text{and} \quad W^{uu}_{loc}(z_n) \pitchfork W^s_{\delta(\kappa)}(r) \neq \emptyset \tag{9}$$

for some $y_n, z_n \in (x_n, f^{\ell_n}(x_n))$ with large $n$ because $\{(x_n, f^{\ell_n}(x_n)) : n \geq 1\}$ accumulates on both $S$ and $T$. Hence, by the partial hyperbolicity, we get

$$W^u(q, f) \pitchfork W^s(r, f) \neq \emptyset,$$

which is preserved by a small perturbation. Note that if $S$ or $T$ is a periodic orbit, we don't need Katok Closing Lemma, as $S = \mathcal{O}_f(\bar{q}) = \mathcal{O}_f(q)$ or $T = \mathcal{O}_f(\bar{r}) = \mathcal{O}_f(r)$. In order to prove Theorem A, it is enough to show that $f$ can be perturbed to have $g$ such that

$$W^s(q_g, g) \cap W^u(r_g, g) \neq \emptyset$$

by an arbitrarily small $C^1$ perturbation, where $q_g$ and $r_g$ are the continuations of $q$ and $r$ for $g$. Then, if $g$ is sufficiently close to $f$, $g$ exhibits a heterodimensional cycle associated to $q_g$ and $r_g$.

We may suppose that both $S$ and $T$ are not periodic orbits for otherwise the problem becomes easier and a slight modification of the proof below gives a proof. For an open set $U$ of $\mathcal{O}_f(p)$ with a hyperbolic periodic saddle $p$, denote by $H_f(p, U)$ the closure of transversal homoclinic points whose orbits are contained in $U$ associated to $p$, and let $H_f(p) = H_f(p, M)$. It is easy to see from the $\lambda$-lemma that given points $x, y \in H_f(p, U)$ and $\varepsilon > 0$, there exists a string $(z, f^n(z))$ contained in $U$ such that $d(x, z) < \varepsilon$ and $d(y, f^n(z)) < \varepsilon$.

Let $\{r_i : i \geq 1\}$ and $\{q_i : i \geq 1\}$ be sequences of $r$ and $q$ obtained by Katok Closing Lemma converging to nonperiodic points $\bar{r}$ and $\bar{q}$, respectively. Then, it is easy to see from the proof of Katok Closing Lemma through the Lyapunov neighborhoods that $H_f(r_i) = H_f(r_{i'})$ and $H_f(q_i) = H_f(q_{i'})$ for all $i, i' \geq 1$ sufficiently large. By (8), we can fix some large $i$ such that $\bar{r} \in H_f(r_i, V_0)$ and $\bar{q} \in H_f(q_i, V_0)$. To simplify the notations, set $r = r_i$ and $q = q_i$. Then, as seen above, there exists a string $(w, \bar{w}\,; f) \subset V_0$ such that $w$ and $\bar{w}$ approximate $r$ and $\bar{r}$, respectively. Take a substring $(w^u, \bar{w}; f) \subset (w, \bar{w}; f)$ such that $w^u$ approximate some $p^u \in W^u(r) \setminus \mathcal{O}_f(r)$. Then, given $\varepsilon > 0$, we get a finite part of $\varepsilon$-pseudo-orbit of $f$,

$$(w^u, \bar{w};\ f) \cup (z'_n, y'_n;\ f)$$

for some $y'_n, z'_n \in (x_n,\ f^{\ell_n}(x_n))$ with $y'_n$ and $z'_n$ approximating $\bar{q}$ and $\bar{r}$, respectively. By considering $f^{-1}$ and $S$ instead of $f$ and $T$, we get a similar finite part of $\varepsilon$-pseudo-orbit of $f^{-1}$,

$$(w^s, \tilde{w},\ f^{-1}) \cup (y'_n, z'_n;\ f^{-1})$$

with $(w^s, \tilde{w};\ f^{-1}) \subset V_0$, where $w^s$ and $\tilde{w}$ approximate some $p^s \in W^s(q) \setminus \mathcal{O}_f(q)$ and $\bar{q}$, respectively. Here, we may assume that $y'_n \notin \mathcal{O}^+_f(w^u)$ and $z'_n \notin \mathcal{O}^-_f(w^s)$. Thus, we obtain finite parts of $\varepsilon$-pseudo-orbits,

$$(w^u, \bar{w};\ f) \cup (z'_n, y'_n;\ f)$$

by which $p^u$ is forwardly related to $y'_n$ with one jump at $\bar{r}$, and

$$(w^s, \tilde{w};\ f^{-1}) \cup (y'_n, z'_n;\ f^{-1})$$

by which $p^s$ is backwardly related to $z'_n$ with one jump at $\bar{q}$. Then, applying Lemma I.1, II) twice, we have $p^u$ forwardly related to $p^s$. Moreover, by using

Lemma I.1, I), we easily get $g$ arbitrarily $C^1$ close to $f$ such that

$$W^s(q_g, g) \cap W^u(r_g, g) \neq \emptyset$$

as required. Here, from (7), (8) and (9), Lemma I.1 can be applied to have $g$ exhibiting a heterodimensional cycle in $U_0$.

Next, let us consider the case where the option of the conclusion in which $E$ of (3) is not contracting for $f$ occurs. Let $V$ be a neighborhood of $S^{i_0}(f) \cap \Lambda$ such that we have $TM|M(f, V) = \widehat{E} \oplus \widehat{F}$ that extends $TM|\Lambda = E \oplus F$. Then, we have sequences of positive numbers $0 < \gamma_n < 1$ with $\lim_{n\to+\infty} \gamma_n = 1$, neighborhoods $V_n \subset V$ of $S^{i_0}(f) \cap \Lambda$ with

$$\bigcap_{n\geq 1} V_n = S^{i_0}(f) \cap \Lambda, \tag{10}$$

and periodic points $p_n \in \mathrm{Per}(f)$ such that $\mathcal{O}_f(p_n) \subset V_n$ and

$$\gamma_n^{\ell_n} < \prod_{j=1}^{\ell_n} \|(Df)|\widehat{E}(f^j(p_n))\| < 1$$

for all $n \geq 1$, where $\ell_n$ is the period of $p_n$ with $\lim_{n\to+\infty} \ell_n = +\infty$. Then, similarly to (5), we have

$$\gamma_n^{\ell_n} < \|(Df^{\ell_n})|\widehat{E}^c(p_n)\| < 1$$

for large $n$, where $\widehat{E}^c(p_n)$ is the eigenspace associated to the eigenvalue of $(Df^{\ell_n})|\widehat{E}(p_n)$ with modulus closest to 1. From this together with (10) it follows that an accumulation point $\tilde{\nu} \in \mathcal{M}_f(M)$ in $\mathcal{M}(M)$ of the sequence of probabilities $\tilde{\nu}_n$, $n \geq 1$, defined by:

$$\tilde{\nu}_n = \frac{1}{\ell_n} \sum_{j=0}^{\ell_n - 1} \delta_{f^j(p_n)},$$

can play the same role as $\nu \in \mathcal{M}_f(M)$ in (6). Hence, by the same argument as in the previous case using $\nu$, we obtain a heterodimensional cycle in $U_0$. This completes the proof of Theorem A.

## II. Proof of Theorem B

We prepare the so-called Pliss Lemma (see [M1, Lemma II.8] for the proof). For a string $(x, g^n(x))$ in a compact invariant set $\Lambda$ admitting a dominated splitting

$TM|\Lambda = E \oplus F$, we say that $(x, g^n(x))$, $n > 0$, is *$\gamma$-string* if

$$\prod_{j=1}^{n} \|(Dg^{-1})|F(g^j(x))\| \leq \gamma^n$$

and we say that it is a *uniform $\gamma$-string* if $(g^k(x), g^n(x))$ is a $\gamma$-string for all $0 \leq k < n$.

**Lemma II.1 (Pliss Lemma [Pl]).** *For all $0 < \gamma < \hat{\gamma} < 1$ there exist $N(\gamma, \hat{\gamma}) > 0$ and $0 < c(\gamma, \hat{\gamma}) < 1$ such that if $(x, g^n(x))$ is a $\gamma$-string and $n \geq N(\gamma, \hat{\gamma})$, then there exist $0 < n_1 < \cdots < n_k \leq n$, $k > 1$, such that $k \geq nc(\gamma, \hat{\gamma})$ and $(x, g^{n_i}(x))$ is a uniform $\hat{\gamma}$-string for all $1 \leq i \leq k$.*

The essential part of Theorem B corresponds to [M2, Lemma II.6]. We modify the proof of [M2, Lemma II.6] to have the additional property concerning the Hausdorff distance. A compact invariant set $\Sigma_0 \subset \Lambda$ is a *$(t, \gamma)$-set* ($t \in \mathbf{Z}^+$, $0 < \gamma < 1$) if for every $x \in \Sigma_0$ there exists $-t < t_0 < t$ such that $(g^{t_0-n}(x), g^{t_0})$ is a $\gamma$-string for all $n > 0$. Note that $(t, \gamma)$-set is a hyperbolic set.

Take $\gamma_1, \gamma_2, \bar{\gamma}_2, \gamma_3$ with

$$0 < e^{-c} < \gamma_1 < \gamma_2 < \bar{\gamma}_2 < \gamma_3 < 1 \tag{1}$$

and $N = N(\bar{\gamma}_2, \gamma_3)$, where $c > 0$ is given in the hypothesis of Theorem B and $N(\bar{\gamma}_2, \gamma_3)$ is given by Lemma II.1. We say that $(y, g^n(y))$ is an *$(N, \gamma_2)$-obstruction* if $(y, g^j(y))$ is not a $\gamma_2$-string for all $N \leq j \leq n$. Denote by $\Lambda(N)$ the set of points $y \in \Lambda$ such that $(y, g^n(y))$ is an $(N, \gamma_2)$-obstruction for all $n > N$. Then, observe that given $\varepsilon > 0$ there exists $N(\varepsilon) > N$ such that when $(y, g^n(y))$ is an $(N, \bar{\gamma}_2)$-obstruction and $n > N(\varepsilon)$, then $d(y, \Lambda(N)) < \varepsilon$. Let $\Sigma$ be the set of the union of all the $(N(\varepsilon), \gamma_3)$-sets. Then, its closure $\overline{\Sigma}$ is an $(N(\varepsilon), \gamma_3)$-set. For $n \geq 1$ and $\mu$-a.e. $x \in \Gamma(g)$ for some $\mu \in \mathcal{M}$, denote by $\mathcal{L}(x, n)$ the set of $m \geq n$ such that $(x, g^m(x))$ is a uniform $\gamma_3$-string. Let

$$\mathcal{L}(x, n) = \{m_1 < m_2 < \dots\}.$$

Since $\operatorname{supp}(\mu_x) = \overline{\mathcal{O}_g^+(x)}$, if

$$\sup_{i \geq 1}(m_{i+1} - m_i) \leq N(\varepsilon)$$

then $\operatorname{supp}(\mu_x)(= \operatorname{supp}(\mu))$ is an $(N(\varepsilon), \gamma_3)$-set. Therefore, when $\operatorname{supp}(\mu_x)$ is not an $(N(\varepsilon), \gamma_3)$-set, for arbitrarily large $n$ there exist $m_i, m_{i+1} \in \mathcal{L}(x, n)$ such that $m_{i+1} - m_i > N(\varepsilon)$. Then, by Lemma II.1, $(g^{m_i}(x), g^{m_{i+1}}(x))$ is an $(N, \bar{\gamma}_2)$-obstruction and therefore the above observation implies that $(g^{m_i}(x), \Lambda(N)) < \varepsilon$. Thus, we have proved the following claim:

**Claim.** *For every $\varepsilon > 0$ and $\mu$-a.e. $x \in \Gamma(g) \cap \Lambda$ for some $\mu \in m$, either* $\operatorname{supp}(\mu_x)$ *is an $(N(\varepsilon), \gamma_3)$-set or there exist $y \in \Lambda$ and arbitrarily large $m > 0$ satisfying the following properties:*

a) $(x, g^m(x))$ *is a uniform $\gamma_3$-string;*

b) $d(g^m(x), y) < \varepsilon$*;*

c) $(y, g^n(y))$ *is an $(N, \gamma_2)$-obstruction for all $n > N$.*

Here, given $\delta > 0$, we choose sufficiently large $m > 0$ so that $(x, g^m(x))$ is $\delta/2$-close to $\operatorname{supp}(\mu)$ with respect to the Hausdorff distance. The next step is approximating $y$ by a point $x_2 \in \Gamma(g) \cap \Lambda$ taken from $\nu$-a.e. points for some $\nu \in \mathcal{M}$ so that $(x_2, g^{n_2}(x_2))$ is a uniform $\gamma_3$-string but not a $\gamma_1$-string for arbitrarily large $n_2$. This is possible by Lemma II.1, (1) of Theorem B and the Claim. (See the proof of [M2, Theorem II.1] for the details.) It is important that $n_2$ goes to $+\infty$ as $x_2$ approaches to $y$.

Suppose that we can take $x_2 \notin \overline{\Sigma}$. Then, repeat this choice of two strings to get the other two strings $(x_3, g^{n_3}(x_3))$ and $(x_4, g^{n_4}(x_4))$ with $d(x_3, g^{n_2}(x_2)) < 2\varepsilon$ satisfying the same property as in the previous two strings if we can take $x_4 \notin \overline{\Sigma}$. Inductively, continue this process until we have a $4\varepsilon$-pseudo-periodic orbit written as:

$$\bigcup_{i=j}^{k} (x_{2i-1}, g_{2i-1}(x_{2i-1})) \cup (x_{2i}, g_{2i}(x_{2i}))$$

by setting $x = x_1$, $m = n_1$ and $g_l = g^{n_l}$ $(1 \leq l \leq 2k)$ for some $0 < j < k$. Here, observe that $n_{2i}$ can be chosen arbitrarily larger than $n_{2i-1}$. Given $0 < \gamma < 1$ and $\delta > 0$, take $\gamma_1$ in (1) with $\gamma < \gamma_1$ and $\varepsilon > 0$ sufficiently small depending on these constants. Then, if $n_{2i}$ is much larger than $n_{2i-1}$ for all $j \leq i \leq k$, the $4\varepsilon$-pseudo-periodic orbit gives a $\delta/2$-shadowing periodic orbit $\mathcal{O}_g(p)$ satisfying (2) of Theorem B as in the proof of [M2, Lemma II.6]. By our construction, $\mathcal{O}_g(p)$ contains a substring $\delta$-close to $\operatorname{supp}(\mu_{x_{2i-1}})$ for every $j \leq i \leq k$ with respect to the Hausdorff distance.

Now let us consider the case where we cannot take $x_2 \notin \overline{\Sigma}$. (Other cases for $x_{2i}$ can be treated similarly, so it is enough to consider only this case.) Let $\{x_2(n) : n \geq 1\}$ be a sequence of the choices of $x_2$ approximating $y \in \Lambda(N)$ such that $\lim_{n\to+\infty} x_2(n) = y$ and $x_2(n) \in \overline{\Sigma}$. Then, $y \in \overline{\Sigma} \cap \Lambda(N)$ because $\overline{\Sigma}$ is compact, satisfying

$$\prod_{j=1}^{n} \|(Dg^{-1})|F(g^j(y))\| > \gamma_2^n \tag{2}$$

for all $n > N$. Define $\nu_n \in \mathcal{M}(M)$ by:

$$\nu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{g^j(y)},$$

and let $\bar{\nu} = \lim_{i\to+\infty} \nu_{n_i}$, an accumulation point of $\{\nu_n : n \geq 1\}$ in $\mathcal{M}(M)$. Then, using (2) and taking a subsequence of $i = 1, 2, \ldots$ if necessary, we have

$$\lim_{i\to+\infty} \int \psi \, d\nu_{n_i} = \int \psi \, d\bar{\nu} \geq \log \gamma_2, \tag{3}$$

where $\psi : \Lambda \to \mathbf{R}$ is a continuous function defined by:

$$\psi(x) = \log \|(Dg^{-1})|F(x)\|.$$

By (3) and Birkhoff's Ergodic Theorem, we get

$$\log \gamma_2 \leq \int \lim_{n\to+\infty} \frac{1}{n} \log \sum_{j=0}^{n-1} \|(Dg^{-1})|F(g^j(x))\| d\bar{\nu}(x).$$

Note that $\operatorname{supp}(\bar{\nu}) \subset \overline{\Sigma}$ because $y$ is in a compact invariant set $\overline{\Sigma}$. Hence, there exists $\bar{y} \in \Gamma(g) \cap \overline{\Sigma}$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \log \sum_{j=0}^{n-1} \|(Dg^{-1})|F(g^j(\bar{y}))\| \geq \log \gamma_2. \tag{4}$$

This implies that there exists $N_1 > 0$ such that $(\bar{y}, g^n(\bar{y}))$ is an $(N_1, \gamma_1)$-obstruction for all $n > N_1$, and $\mu_{\bar{y}} \in \mathcal{M}_e(g)$. Then, $\operatorname{supp}(\mu_{\bar{y}}) \subset \overline{\Sigma}$. Let us suppose that there is $\nu \in \mathcal{M}$ such that $\nu = \mu_{\bar{y}}$, and proceed assuming that $\bar{y} \notin \operatorname{Per}(g)$. Since $\operatorname{supp}(\mu_{\bar{y}}) = \overline{\mathcal{O}_g^+(\bar{y})}$, given $0 < \varepsilon < \delta$, there is a string $(\bar{y}, g^{\ell_1}(\bar{y})) \subset \operatorname{supp}(\mu_{\bar{y}})$, $\ell_1 > N_1$, $\delta/2$-close to $\operatorname{supp}(\mu_{\bar{y}})$ with respect to the Hausdorff distance. Moreover, we can find $\ell_2 > \ell_1$ such that $(\bar{y}, g^{\ell_2}(\bar{y}))$ is an $\varepsilon$-pseudo-periodic orbit, which is not a $\gamma_1$-string by the choice of $N_1$ coming from (4). Since $\overline{\Sigma}$ is a hyperbolic set, if $\varepsilon > 0$ has been chosen small enough, Anosov Closing Lemma ([S]) gives us the required periodic orbit $\mathcal{O}_g(p)$, satisfying (2) of Theorem B and $\delta$-close to $\operatorname{supp}(\nu) = \operatorname{supp}(\mu_{\bar{y}})$ in the Hausdorff distance. When $\bar{y} \in \operatorname{Per}(g)$, this periodic orbit $\mathcal{O}_g(\bar{y})$ itself (in the hyperbolic set $\overline{\Sigma}$) is the required one making $0 < \gamma < 1$ larger if necessary to have a large period. Indeed, property (2) of Theorem B is trivially holds and if the periods were uniformly bounded when $\gamma \to 1$, there would exist a nonhyperbolic periodic orbit in $\overline{\Sigma}$, contradicting the hyperbolicity of $\overline{\Sigma}$.

On the other hand, if there is no $\nu \in \mathcal{M}$ such that $\nu = \mu_{\bar{y}}$, recall that $y$ can be approximated by some $\mu_n$-a.e. point $x_2(n)$ for some $\mu_n \in \mathcal{M}$ with supp $(\mu_n) \subset \overline{\Sigma}$. Then, we can find $\bar{y}$ on which some $\mu_n$-a.e. point $y_n \in$ supp $(\mu_n)$, $n \geq 1$, accumulate. As before, take an $\varepsilon$-pseudo-periodic orbit $O_g(\mu_n) \subset$ supp $(\mu_n)$ as a string from $y_n$, $\delta/2$-close to supp $(\mu_n)$ with respect to the Hausdorff distance. Fix $n$ so large that $d(y_n, \bar{y}) < \varepsilon$. Then, for any $k \geq 1$,

$$O_g(\mu_n) \cup O_g(\mu_{\bar{y}}) \underbrace{\cup \cdots \cup}_{k \text{ times}} O_g(\mu_{\bar{y}})$$

forms a $4\varepsilon$-pseudo-periodic orbit in $\overline{\Sigma}$, where $O_g(\mu_{\bar{y}}) = \mathcal{O}_g(\bar{y})$ when $\bar{y} \in \text{Per}(g)$ and $O_g(\mu_{\bar{y}}) = (\bar{y}, g^{\ell_2}(\bar{y}))$ otherwise. Therefore we get a periodic orbit $\mathcal{O}_g(p^k)$ containing a substring $\delta$-close to supp $(\mu_n)$ by Anosov Closing Lemma if $\varepsilon > 0$ is small enough. Observe that the average contraction rate of $Dg^{-1}$ over $F$ along

$$O_g(\mu_n) \cup O_g(\mu_{\bar{y}}) \cup \overset{k}{\cdots} \cup O_g(\mu_{\bar{y}})$$

can be arbitrarily close to that of $O_g(\mu_{\bar{y}})$ as $k \to +\infty$. Hence, for sufficiently large $k$, the periodic orbit $\mathcal{O}_g(p)$ with $p = p^k$ satisfies also property (2) of Theorem B as required. This concludes the proof of Theorem B.

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