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The Crossed Product by a Partial Endomorphism

Ruy Exel* and Danilo Royer**

Abstract. Given a closed ideal I in a C^*-algebra A, an ideal J (not necessarily closed) in I, a *-homomorphism $\alpha\colon A \to M(I)$ and a map $L\colon J \to A$ with some properties, based on earlier works of Pimsner and Katsura, we define a C^*-algebra $\mathcal{O}(A, \alpha, L)$ which we call the *Crossed Product by a Partial Endomorphism.* We introduce the Crossed Product by a Partial Endomorphism $\mathcal{O}(X, \alpha, L)$ induced by a local homeomorphism $\sigma : U \to X$ where X is a compact Hausdorff space and U is an open subset of X. A bijection between the gauge invariant ideals of $\mathcal{O}(X, \alpha, L)$ and the σ, σ^{-1}-invariant open subsets of X is showed. If (X, σ) has the property that $(X', \sigma_{|_{X'}})$ is topologically free for each closed σ, σ^{-1}-invariant subset X' of X then we obtain a bijection between the ideals of $\mathcal{O}(X, \alpha, L)$ and the open σ, σ^{-1}-invariant subsets of X.

Keywords: partial endomorphism, crossed product.

Mathematical subject classification: 47L65, 37A99.

Introduction

Since the pioneering work of Cuntz [1], many authors, notably Paschke [11], Stacey [15], and Murphy [10], have proposed constructions of crossed products of C^*-algebras by endomorphisms. Those constructions depends essentially on an endomorphism α on a C^*-algebra A. In [3] it was introduced by the first named author the concept of Crossed Product by an Endomorphism, based not only on an endomorphism α but on a C*-dynamical system (A, α, L). Here A is a C*-algebra, α is an endomorphism and L, following [3], is a transfer operator, that is, $L\colon A \to A$ is a continuous linear map such that L is positive and $L(\alpha(a)b) = aL(b)$ for all $a, b \in A$. The Crossed Product by an Endomorphism is a quotient of the universal C*-algebra generated by a copy of A and an element S subject to the relations $Sa = \alpha(a)S$ and $S^*aS = L(a)$ for all $a \in A$.

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See [3] for more details. In this article it was shown that the Cuntz-Krieger algebra is an example of Crossed Product by an Endomorphism. The C^*-dynamical system associated to this example is induced by the Markov subshift (Ω_A, σ), that is, the endomorphism $\alpha\colon C(\Omega_A) \to C(\Omega_A)$ is given by $\alpha(f) = f \circ \sigma$ and $L : C(\Omega_A) \to C(\Omega_A)$ is defined by

$$L(f)(x) = \frac{1}{\#\sigma^{-1}(x)} \sum_{y \in \sigma^{-1}(x)} f(y)$$

for each $x \in X$ and for each $f \in C(\Omega_A)$.

It was defined in [4] by the first named author and M. Laca the Cuntz-Krieger algebra for infinite matrices. This algebra has a topological compact Hausdorff space $\widetilde{\Omega_A}$ associated to it, which can be seen in [4, 4-7]. The difference between this case and the previous one is that the shift σ can not be defined in the whole space $\widetilde{\Omega_A}$, but only in an open subset U of $\widetilde{\Omega_A}$. Then the local homeomorphism $\sigma : U \to \widetilde{\Omega_A}$ induces the *-homomorphism $\alpha\colon C(\widetilde{\Omega_A}) \to C^b(U)$ given by $\alpha(f) = f \circ \sigma$, where $C^b(U)$ is the set of all continuous and bounded functions in U. Moreover, since $\#\sigma^{-1}(x)$ may be infinite for some $x \in \widetilde{\Omega_A}$, the convergence of the sum $\sum_{y\in\sigma^{-1}(x)} f(y)$ is not guaranteed and so $L(f)$ can not be defined by $L(f)(x) = \sum_{y\in\sigma^{-1}(x)} f(y)$ for every $f \in C(\widetilde{\Omega_A})$. However, we will show that for each $f \in C_c(U)$, that is, for each function with compact support in U, $L(f)$ defined by $L(f)(x) = \sum_{y\in\sigma^{-1}(x)} f(y)$ for each $x \in \widetilde{\Omega_A}$ is an element of $C(\widetilde{\Omega_A})$. In this way we obtain a map $L : C_c(U) \to C(\widetilde{\Omega_A})$. Because α is not an endomorphism in $C(\widetilde{\Omega_A})$ and the domain of L is not the whole algebra $C(\widetilde{\Omega_A})$, the triple (A, α, L) (which we also call by C^*-dynamical system) is not a C^*-dynamical system as in [3] and therefore the construction of Crossed Product by an Endomorphism defined in [3] cannot be applied.

In this work we define, making use of the constructions of T. Katsura ([7]) and M. Pimsner ([13]), the *Crossed Product by a Partial Endomorphism*. We show that our construction may be applied to the situation described in the previous paragraph. We study specially the case where the Crossed Product by a Partial Endomorphism, wich we denote by $\mathcal{O}(X, \alpha, L)$, is induced by a local homeomorphism $\sigma : U \to X$, where U is an open subset of a topological compact Hausdorff space X. More specifically, we show a bijection between the gauge invariant ideals of $\mathcal{O}(X, \alpha, L)$ and the σ, σ^{-1}-invariant open subsets of X. Moreover, if (X, σ) has the property that $(X', \sigma_{|_{X'}})$ is topologically free for every closed σ, σ^{-1}-invariant subset X' of X then there exists a bijection between the ideals of $\mathcal{O}(X, \alpha, L)$ and the open σ, σ^{-1}-invariant subsets of X. Finally we present a simplicity criteria for the Cuntz-Krieger algebras for infinite matrices. The choice of the name Crossed Product by a Partial Endomorphism for the algebra $\mathcal{O}(A, \alpha, L)$ defined in this work was motived by the local homeomorphism $\sigma : U \to X$ where U is an open subset of X.

There is a strong relationship between the Crossed Product of a Partial Endomorphism associated to a commutative C^*-dynamical system, and the algebra studied by J. Renault in [14]. However, our approach is completely different from the one used by Renault. Moreover, the construction of the Crossed Product by a Partial Endomorphism introduced in our paper applies also to non commutative C^*-dynamical systems.

In [8], B.K. Kwasniewski defined an algebra which he called *Covariance algebra of a partial dynamical system* based on a partial dynamical system (X, α), that is, a continuous map $\alpha : \Delta \to X$ where X is a compact Hausdorff space and Δ is a clopen subset of X and $\alpha(\Delta)$ is open. In our construction Δ need not be clopen, only open, but we require that α is a local homeomorphism. The possible relationship between these two constructions will be studied in a future paper.

1 The crossed product by a partial endomorphism

In this section we define the crossed product by a partial endomorphism and show some results about its structure. We study the gauge action and gauge-invariant ideals of this algebra.

1.1 Definitions and basic results

Let A be a C^*-algebra and I a closed two-sided ideal in A.

Definition 1.1. *A partial endomorphism is a *-homomorphism $\alpha : A \to M(I)$ where $M(I)$ is the multiplier algebra of I.*

Let J be a two-sided self adjoint idempotent (not necessarily closed) ideal in I and let $\alpha : A \to M(I)$ and $L : J \to A$ be functions. We denote a such situation by (A, α, L).

Definition 1.2. *(A, α, L) is a C^*-dynamical system if (A, α, L) has the following properties:*

- *α is a partial endomorphism,*
- *L is linear, positive and preserves $*$,*
- *$L(\alpha(a)x) = aL(x)$ for all a in A and x in J.*

The function L is positive in the sense that $L(x^*x)$ is a positive element of A for all x in J. Moreover, denoting $\alpha(a)$ by (L^a, R^a), $\alpha(a)x$ is a notation for the element $L^a(x)$. Note that if $x, y \in J$ and $a \in A$ then $L^a(x) \in I$ and so $L^a(xy) = L^a(x)y \in J$. Since J is idempotent we have in general that $\alpha(a)x \in J$ for all $a \in A$ and $x \in J$. Therefore $\alpha(a)x$ lies in fact in the domain of L. Defining $x\alpha(a) = R^a(x)$ for all $x \in J$ and $a \in A$ we have that $(\alpha(a)x)^* = x^*\alpha(a^*)$ for every $x \in J$ and $a \in A$. In fact,

$$(\alpha(a)x)^* = (L^a(x))^* = (R^a)^*(x^*) = R^{a^*}(x^*) = x^*\alpha(a^*)\,.$$

In the same way $(x\alpha(a))^* = \alpha(a^*)x^*$.

If (A, α, L) is a C^*-dynamical system then $L(x\alpha(a)) = L(x)a$ for all $a \in A$ and $x \in J$. In fact, given $a \in A$ e $x \in J$, since $a^* \in A$ and $x^* \in J$ we have that $L(\alpha(a^*)x^*) = a^*L(x^*)$. Therefore $L(x\alpha(a)) = L((x\alpha(a))^*)^* = L(\alpha(a^*)x^*)^* = (a^*L(x^*))^* = L(x)a$.

The next goal is to define a left A-module which is also a right Hilbert A-module. Define the operation

$$\begin{array}{rrcl} .: & J \times A & \to & J \\ & (x, a) & \mapsto & x\alpha(a) \end{array}.$$

It is easy to verify that this operation is bilinear and associative. Thus J is a right A-module. It is also easy to see that the function

$$\begin{array}{rrcl} \langle\, , \rangle : & J \times J & \to & A \\ & (x, y) & \mapsto & L(x^*y) \end{array}$$

is a semi-inner product. Considering the quotient of J by $N_0 = \{x \in J : \langle x, x\rangle = 0\}$ and denoting the elements x of J by $\widetilde{x}$ in J/N_0 (or by $(x)\widetilde{\ }$) we obtain an inner product of J/N_0 in A defined by $\langle \widetilde{x}, \widetilde{y}\rangle = \langle x, y\rangle$. So the function

$$\begin{array}{rrcl} \|\, , \| : & J/N_0 & \to & \mathbb{R}^+ \\ & \widetilde{x} & \mapsto & \sqrt{\|\langle \widetilde{x}, \widetilde{x}\rangle\|} \end{array}$$

defines a norm in J/N_0. Denote by M the right Hilbert A-module $\overline{(J/N_0)}^{\|\,\|}$.

Let us now define a left A-module structure for M. Given $a \in A$ and $x \in J$ we have that $x^*a^*ax, \|a\|^2x^*x \in J$. Since $x^*(\|a\|^2 - a^*a)x$ may be written in the form $(bx)^*(bx)$ with $bx \in J$ we have that $L(x^*\|a\|^2x - x^*a^*ax) \geq 0$ and so $L(x^*a^*ax) \leq \|a\|^2L(x^*x)$ from where $\|L(x^*a^*ax)\| \leq \|a\|^2\|L(x^*x)\|$. Therefore

$$\begin{aligned} \|\widetilde{ax}\|^2 &= \|\langle \widetilde{ax}, \widetilde{ax}\rangle\| = \|L(x^*a^*ax)\| \leq \|a\|^2\|L(x^*x)\| \\ &= \|a\|^2\|\langle \widetilde{x}, \widetilde{x}\rangle\| = \|a\|^2\|\widetilde{x}\|^2, \end{aligned}$$

and so, $\|\widetilde{ax}\| \leq \|a\|\|\widetilde{x}\|$. This allows us define the operation

$$\begin{array}{rcl} .:\ A \times M & \to & M \\ (a, m) & \mapsto & am \end{array},$$

where $a\widetilde{x} = \widetilde{ax}$, which is bilinear and associative, and so M is a left A-module. This operation gives rise to a *-homomorphism from A in $L(M)$. In fact, defining $\varphi\colon A \to L(M)$ by $\varphi(a)m = am$ we have:

Proposition 1.3. *φ is a *-homomorphism.*

Proof. For all $a \in A$, $\varphi(a)\colon M \to M$ defined by $\varphi(a)(m) = am$ for all $m \in M$ is a linear function. Moreover, for $x, y \in J$,

$$\langle \varphi(a)\widetilde{x}, \widetilde{y} \rangle = \langle \widetilde{ax}, \widetilde{y} \rangle = L((ax)^* y) = L(x^* a^* y) = \langle \widetilde{x}, \widetilde{a^* y} \rangle = \langle \widetilde{x}, \varphi(a^*)\widetilde{y} \rangle,$$

and since J/N_0 is dense in M it follows that $\langle \varphi(a)m, n\rangle = \langle m, \varphi(a^*)n\rangle$ for all $m, n \in M$. This shows that $\varphi(a)$ is adjointable and $\varphi(a)^* = \varphi(a^*)$. Obviously φ is linear and multiplicative. □

Definition 1.4. *The Toeplitz algebra $\mathcal{T}(A, \alpha, L)$ associated to the C^*-dynamical system (A, α, L) is the universal C^*-algebra generated by $A \cup M$ with the relations of A, of M, the A-bi-module products and $m^* n = \langle m, n\rangle$ for all $m, n \in M$.*

Note that the universal algebra in fact exists, since the relations are admissible. We will denote by $\widehat{K_1}$ the closed sub-algebra of $\mathcal{T}(A, \alpha, L)$ generated by the elements of the form mn^*, for $m, n \in M$.

Definition 1.5. *A redundancy in $\mathcal{T}(A, \alpha, L)$ is a pair (a, k) where $a \in A$, $k \in \widehat{K_1}$ and $am = km$ for all $m \in M$.*

Let $I_0 = \ker(\varphi)^{\perp} \cap \varphi^{-1}(K(M))$ where $\varphi\colon A \to L(M)$ is the *-homomorphism given by the left multiplication.

Definition 1.6. *The Crossed Product by a partial Endomorphism associated to the C^*-dynamical system (A, α, L) is the quotient of $\mathcal{T}(A, \alpha, L)$ by the ideal generated by the elements $a - k$ for all redundancies (a, k) such that $a \in I_0$, and will be denoted by $\mathcal{O}(A, \alpha, L)$.*

It follows from [7] that $A \ni a \to a \in \mathcal{O}(A, \alpha, L)$ is injective. In the following proposition will be showed some consequences of this fact. Let us temporarily denote by $\widehat{a}$ and $\widehat{m}$ the elements of A and M in $\mathcal{T}(A, \alpha, L)$. Define

$$\widehat{K_n} = \overline{\text{span}}\{\widehat{m_1} \cdots \widehat{m_n}\widehat{l_1}^* \cdots \widehat{l_n}^* :\ m_i, l_i \in M\}$$

and denote by q the quotient map from $\mathcal{T}(A,\alpha,L)$ to $\mathcal{O}(A,\alpha,L)$.

Proposition 1.7.

a) $A \ni a \mapsto q(\widehat{a}) \in \mathcal{O}(A,\alpha,L)$ *is an injective *-homomorphism.*

b) $A \ni a \mapsto \widehat{a} \in \mathcal{T}(A,\alpha,L)$ *and* $q_{|_{\widehat{A}}}$ *are injective *-homomorphisms.*

c) $M \ni m \mapsto \widehat{m} \in \mathcal{T}(A,\alpha,L)$ *is an isometry.*

d) $q_{|_{\widehat{M}}}$ *is an isometry.*

e) $M \ni m \mapsto q(\widehat{m}) \in \mathcal{O}(A,\alpha,L)$ *is an isometry.*

f) $q_{|_{\widehat{K_n}}}$ *is an injective *-homomorphism.*

Proof.

a) Is a consequence of [7].

b) Follows from a).

c) Given $m \in M$, $\|\widehat{m}\|^2 = \|\widehat{m}^*\widehat{m}\| = \|\widehat{\langle m, m\rangle}\|$. Since $\langle m, m\rangle \in A$, it follows from b) that $\|\widehat{\langle m, m\rangle}\| = \|\langle m, m\rangle\|$. Moreover $\|m\|^2 = \|\langle m, m\rangle\|$. Then $\|\widehat{m}\|^2 = \|\langle m, m\rangle\| = \|m\|^2$.

d) For all $\widehat{m} \in \widehat{M}$ we have $\widehat{m}^*\widehat{m} \in \widehat{A}$. By a), $q_{|_{\widehat{A}}}$ is injective and therefore an isometry. Then $\|q(\widehat{m})\|^2 = \|q(\widehat{m}^*\widehat{m})\| = \|\widehat{m}^*\widehat{m}\| = \|\widehat{m}\|^2$.

e) Follows from c) and d).

f) Let $k \in \widehat{K_n}$ and suppose $q(k) = 0$. Then $q((\widehat{M}^*)^n k \widehat{M}^n) = 0$. Since $(\widehat{M}^*)^n k \widehat{M}^n \subseteq \widehat{A}$ it follows from b) that $(\widehat{M}^*)^n k \widehat{M}^n = 0$. Then $\widehat{K_n} k \widehat{K_n} = 0$ and so $k = 0$. □

From now on we will identify the elements $\widehat{a} \in \mathcal{T}(A,\alpha,L)$ and $q(\widehat{a}) \in \mathcal{O}(A,\alpha,L)$ with the element a of A. This notation will not cause confusion, by a) and b) of the previous proposition. In the same way, justified by c) and e) we will identify the elements $\widehat{m} \in \mathcal{T}(A,\alpha,L)$ and $q(\widehat{m}) \in \mathcal{O}(A,\alpha,L)$ with the element $m \in M$. With these identifications,

$$\widehat{K_n} = \overline{\text{span}}\left\{m_1 \cdots m_n l_1^* \cdots l_n^* : \ m_i, l_i \in M\right\} \subseteq \mathcal{T}(A,\alpha,L).$$

Define

$$K_n = \overline{\text{span}}\left\{m_1 \cdots m_n l_1^* \cdots l_n^* : \ m_i, l_i \in M\right\} \subseteq \mathcal{O}(A, \alpha, L)$$

and note that $q(\widehat{K_n}) = K_n$. If $(a, k) \in A \times \widehat{K_1}$ is a redundancy and $a \in I_0$ then $q(a) = q(k)$. Since $a = q(a)$ in $\mathcal{O}(A, \alpha, L)$ it follows that $a = q(k)$ in $\mathcal{O}(A, \alpha, L)$.

The spaces K_n e $\widehat{K_n}$ are clearly closed under the sum and are self-adjoint. Moreover, the following proposition shows that they are closed under multiplication, and so are C^*-algebras.

Proposition 1.8.

a) $\widehat{K_n}\widehat{K_m} \subseteq \widehat{K_{\max\{n,m\}}}$ *and also* $K_n K_m \subseteq K_{\max\{n,m\}}$.

b) $A\widehat{K_n} \subseteq \widehat{K_n}$, $\widehat{K_n}A \subseteq \widehat{K_n}$ *and also* $AK_n \subseteq K_n$ *and* $K_n A \subseteq K_n$.

Proof. Since $K_n = q(\widehat{K_n})$ it suffices to show the result for the algebra $\mathcal{T}(A, \alpha, L)$.

a) Taking adjoins we may suppose $n \leq m$. Given $l_1 \ldots l_n t_1^* \ldots t_n^* \in \widehat{K_n}$ and $p_1 \ldots p_m q_1^* \ldots q_m^* \in \widehat{K_m}$, how $a = t_1^* \ldots t_n^* p_1 \ldots p_n \in A$ it follows that $l_n a \in M$. Therefore

$$l_1 \ldots l_n t_1^* \ldots t_n^* p_1 \ldots p_m q_1^* \ldots q_m^* = l_1 \ldots l_n a p_{n+1} \ldots p_m q_1^* \ldots q_m^* \in \widehat{K_m}.$$

This is enough since $\widehat{K_n}$ are generated by elements of this form.

b) Follows by the fact that $am \in M$ for all $a \in A$ and $m \in M$. □

We will denote by $m \otimes n$ the element of $K(M)$ given by $m \otimes n(\xi) = m\langle n, \xi\rangle$, for all $\xi \in M$.

Proposition 1.9. *There exists a *-isomorphism* $S\colon \widehat{K_1} \to K(M)$ *such that* $S(mn^*) = m \otimes n$.

Proof. Given $k \in \widehat{K_1}$ and $m \in M$ then $km \in M$ because M is closed in $\mathcal{T}(A, \alpha, L)$ by the proposition 1.7 c). In $\mathcal{T}(A, \alpha, L)$, $\langle km, n\rangle = (km)^* n = m^* k^* n = \langle m, k^* n\rangle$, and how $\langle m, k^* n\rangle, \langle km, n\rangle \in A$, by 1.7 b) $\langle m, k^* n\rangle = \langle km, n\rangle$ in A. So, defining $S(k)\colon M \to M$ by $S(k)(m) = km$ it follows that $\langle S(k)m, n\rangle = \langle km, n\rangle = \langle m, k^* n\rangle = \langle m, S(k^*)n\rangle$ for all $m, n \in M$. This shows that $S(k)$ is adjointable and $S(k)^* = S(k^*)$. Since $S(k) \in L(M)$ we may define $S\colon \widehat{K_1} \to L(M)$ which is clearly linear and multiplicative, and so S is a *-homomorphism. Obviously $S(mn^*) = m \otimes n$, and therefore $S(k) \in K(M)$ for all $k \in \widehat{K_1}$. Moreover $S(\widehat{K_1})$ is a dense set in $K(M)$ and so $S(\widehat{K_1}) = K(M)$. In order to see that S is injective suppose $S(k) = 0$, that is, $kM = 0$. Then $k\widehat{K_1} = 0$ and since $k \in \widehat{K_1}$ it follows that $k = 0$. $\square$

If (a, k) is a redundancy then $am = km$ for all $m \in M$, from where $\varphi(a)(m) = S(k)(m)$ for each $m \in M$. Since $S(k) \in K(M)$ it follows that $\varphi^{-1}(a) \in K(M)$. So the algebra $\mathcal{O}(A, \alpha, L)$ coincides with the quotient of $\mathcal{T}(A, \alpha, L)$ by the ideal generated by the elements of the form $(a - k)$ for all redundancy (a, k) such that $a \in \ker(\varphi)^{\perp}$.

Given a C^*-dynamical system (A, α, L) and a closed ideal N in A such that $J \subseteq N \subseteq I$, we may consider an other C^*-dynamical system (A, β, L) where the partial endomorphism $\beta\colon A \to M(N)$ is given by $\beta(a) = (L^a_{|_N}, R^a_{|_N})$, considering that $\alpha(a) = (L^a, R^a)$. Since $x\beta(a) = x\alpha(a)$ for all $x \in J$ and $a \in A$ it follows that $\mathcal{O}(A, \alpha, L) = \mathcal{O}(A, \beta, L)$. By this reason we may suppose that J is a dense ideal in I. This situation will occur in the second section.

It may be showed without much difficulty that the crossed product by endomorphism introduced in [3] in some situations may be seen as crossed products by a partial endomorphism. More specifically, this holds if $\langle\alpha(A)\rangle = A$ and L is faithfull or if $\alpha\colon A \to A$ is injective, $\alpha(A) = \alpha(1)A\alpha(1)$, and $L\colon A \to A$ is given by $L(a) = \alpha^{-1}(\alpha(1)a\alpha(1))$. The first situation occurs in Cuntz-Krieger algebras (see [3, 6]) end the last situation occurs in Pashke's crossed product and in the crossed product proposed by Cuntz (see [3]).

1.2 The gauge action

The next goal is to show that every gauge-invariant ideal of $\mathcal{O}(A, \alpha, L)$ has non-trivial intersection with the fixed point algebra of the gauge action in $\mathcal{O}(A, \alpha, L)$.

By the universal property of $\mathcal{T}(A, \alpha, L)$ it follows that for each $\lambda \in S^1$ there exists a *-homomorphism $\theta_\lambda\colon \mathcal{T}(A, \alpha, L) \to \mathcal{T}(A, \alpha, L)$ which satisfies $\theta_\lambda(a) = a$ for all a in A and $\theta_\lambda(m) = \lambda m$ for all $m \in M$. If (a, k) is a redundancy, because $\theta_\lambda(a) = a$ and $\theta_\lambda(k) = k$ it follows that $(\theta_\lambda(a), \theta_\lambda(k))$ is also a redundancy, and so we may consider $\theta_\lambda\colon \mathcal{O}(A, \alpha, L) \to \mathcal{O}(A, \alpha, L)$. Note that $\theta_{\lambda_1}\theta_{\lambda_2} = \theta_{\lambda_1\lambda_2}$ from where θ_λ is a *-automorphism, with inverse $\theta_{\overline{\lambda}}$. Moreover, given $r \in \mathcal{O}(A, \alpha, L)$, the function $S^1 \ni \lambda \mapsto \theta_\lambda(r) \in \mathcal{O}(A, \alpha, L)$ is continuous. Then we may consider

$$\begin{array}{rcl} E : \mathcal{O}(A, \alpha, L) & \to & \mathcal{O}(A, \alpha, L) \\ r & \mapsto & \displaystyle\int_{S^1} \theta_\lambda(r) d\lambda \end{array} .$$

Proposition 1.10. *The fixed point algebra of θ is $K = \overline{\text{span}} \left\{A, K_n; n \in \mathbb{N}\right\}$ and E is a faithful conditional expectation onto K.*

Proof. It is not difficult to show that E is a faithful conditional expectation onto the fixed point algebra. So it suffices to show that $\text{Im}(E) = K$. The equality holds because

$$E(am_1 \cdots m_k n_1^* \cdots n_l^* b) = \begin{cases} am_1 \cdots m_k n_1^* \cdots n_l^* b & \text{se } k = l \\ 0 & \text{se } k \neq l \end{cases}$$

and the space generated by elements of the form $am_1 \cdots m_i n_1^* \cdots m_j^* b$ is dense in $\mathcal{O}(A, \alpha, L)$. □

Definition 1.11. *A ideal I in $\mathcal{O}(A, \alpha, L)$ is gauge-invariant if $\theta_\lambda(I) \subseteq I$ for each $\lambda \in S_1$.*

If I is gauge-invariant, the gauge action in $\mathcal{O}(A, \alpha, L)/I$ is given by

$$\begin{array}{rcl} \beta_\lambda : \mathcal{O}(A, \alpha, L)/I & \to & \mathcal{O}(A, \alpha, L)/I \\ \pi(r) & \mapsto & \pi(\theta_\lambda(r)) \end{array} ,$$

where π is the quotient map. In this case π is covariant by the gauge actions θ and β, in the sense that $\pi(\theta_\lambda(r)) = \beta_\lambda(\pi(r))$ for all $r \in \mathcal{O}(A, \alpha, L)$ and for each $\lambda \in S^1$. Moreover, the fixed point algebra for β is $\pi(K)$ because the conditional expectation F induced by β is such that $F(\pi(r)) = \pi(E(r))$ for each $r \in \mathcal{O}(A, \alpha, L)$.

Proposition 1.12. *If $0 \neq I \trianglelefteq \mathcal{O}(A, \alpha, L)$ is gauge-invariant then $I \cap K \neq 0$.*

Proof. Since $\theta_\lambda(I) \subseteq I$ for all $\lambda \in S^1$ then $E(r) \in I$ for all $r \in I$. By the fact that E is faithful it follows that, given $0 \neq r \in I$ then $E(r^*r) \neq 0$. Since $E(r^*r) \in K \cap I$, the result is proved. □

Defining

$$L_0 = A \text{ and } L_n = A + K_1 + \cdots + K_n \text{ for every } n \geq 1$$

we have that

$$L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \quad \text{and} \quad K = \overline{\bigcup_{n \in \mathbb{N}} L_n}\,.$$

This form to see the algebra K will be useful in some situations which will appear latter. In some of this situations we will use the fact, given by the following proposition, that the algebras L_n (by the proposition 1.8 L_n are algebras) are closed, for all $n \in \mathbb{N}$.

Proposition 1.13. *For each $n \in \mathbb{N}$ the algebras L_n are closed.*

Proof. The case L_0 follows by 1.7 a). By induction suppose L_n closed. Note that $K_{n+1} \trianglelefteq \overline{L_{n+1}}$ and that L_n is a closed sub-algebra of $\overline{L_{n+1}}$. By [12, 1.5.8], $L_n + K_{n+1}$ is a closed sub-algebra of $\overline{L_{n+1}}$. Therefore

$$L_{n+1} = L_n + K_{n+1} = \overline{L_n + K_{n+1}} = \overline{L_{n+1}}\,. \qquad \square$$

2 The Crossed Product by a Partial Endomorphism induced by a local homeomorphism

Given a topological compact Hausdorff space X and a local homeomorphism $\sigma\colon X \to X$, defining $\alpha\colon C(X) \to C(X)$ by $\alpha(f) = f \circ \sigma$ and $L\colon C(X) \to C(X)$ by $L(f)(x) = \sum_{y \in \sigma^{-1}(x)} f(y)$ for all $x \in X$, we obtain a C^*-dynamical system. This situation occurs in the Cuntz-Krieger algebra in [3]. A more general situation consists in considering an open set $U \subseteq X$ and a local homeomorphism $\sigma\colon U \to X$. In this case, defining α as above, for all $f \in C(X)$ $\alpha(f)$ is an element of $C^b(U)$, where $C^b(U)$ is the set of all continuous and bounded functions in U. Moreover, $\#\sigma^{-1}(x)$ may be infinite for some $x \in X$, and therefore L can not be defined as above.

Although, if $f \in C_c(U)$, that is, $f \in C(X)$ such that

$$\operatorname{supp}(f) = \overline{\{x \in X\colon f(x) \neq 0\}} \subseteq U\,,$$

we will show that $\sum_{y \in \sigma^{-1}(x)} f(y)$ involves finitely many summands for every $x \in X$. We will also show that, for each $f \in C_c(U)$, $L(f)$ defined by $L(f)(x) = \sum_{y \in \sigma^{-1}(x)} f(y)$ is an element in $C(X)$, and so we may define $L\colon C_c(U) \to C(X)$. Moreover, since $C^b(U)$ and $M(C_0(U))$ are *-isomorphic we obtain a partial endomorphism $\widetilde{\alpha}\colon C(X) \to M(C_0(U))$.

We begin this section by showing that $(C(X), \widetilde{\alpha}, L)$ is a C^*-dynamical system which will give us the crossed product by a partial endomorphism $\mathcal{O}(X, \alpha, L)$.

The second part is dedicated to presenting some basic results about the structure of $\mathcal{O}(X, \alpha, L)$, and the most important result of this part is that every ideal of $\mathcal{O}(X, \alpha, L)$ which has nonzero intersection with K (the fixed point algebra of the gauge action) has nonzero intersection with $C(X)$.

In the last part we show that the Cuntz-Krieger algebra for infinite matrices (see [4]) is a crossed product by a partial endomorphism. This is the example which motivated this work.

The choice of the name *Crossed Product by a Partial Endomorphism* for the algebra $\mathcal{O}(A, \alpha, L)$ was motivaded by the local homeomorphism σ.

2.1 The algebra $\mathcal{O}(X, \alpha, L)$

Let X be a topological compact Hausdorff space, $U \subseteq X$ an open subset and $\sigma : U \to X$ a local homeomorphism. Define

$$\begin{aligned} \alpha : C(X) &\to C^b(U) \\ f &\mapsto f \circ \sigma \end{aligned}$$

which is a *-homomorphism. For each $f \in C_c(U)$ define for all $x \in X$,

$$L(f)(x) = \begin{cases} \sum\limits_{\substack{y \in U \\ \sigma(y)=x}} f(y) & \text{if } \sigma^{-1}(x) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

If $K \subseteq U$ is a compact subset, taking an open cover $U_1, \cdots, U_n$ of K in U such that $\sigma_{|U_i}$ is homeomorphism, for every $x \in X$ there exists no more than one element x_i in each $\sigma^{-1}(x) \cap U_i$. Therefore there exists at most n elements in $\sigma^{-1}(x) \cap K$. It follows that the sum which defines $L(f)(x)$ involves finitely many summands for each $x \in X$, and so $L(f)(x)$ in fact may be defined as above.

Lemma 2.1. *For each $f \in C_c(U)$, $L(f)$ is an element of $C(X)$.*

Proof. Let $f \in C_c(U)$ and $K = \operatorname{supp}(f)$. We will show that $L(f)$ is continuous on each point of X. Given $x \in X \setminus \sigma(K)$, since $X \setminus \sigma(K)$ is open and $L(f)y = 0$ for all $y \in X \setminus \sigma(K)$, it follows that $L(f)$ is continuous in x. Let $x \in \sigma(K)$, $\{x_1, \ldots, x_k\} = \sigma^{-1}(x) \cap K$, and U_j open disjoint neighbourhoods of x_j such that $\sigma_{|U_j}$ is a homeomorphism. The U_j may be taken such that $\sigma(U_j)$ are open, because σ is a local homeomorphism.

Claim. *There exists an open set $V \ni x$ such that*

$$\sigma^{-1}(V) \cap \left(K \setminus \left(\bigcup_{j=1}^{k} U_j \right) \right) = \emptyset.$$

Suppose $\sigma^{-1}(V) \cap \big(K \setminus \big(\bigcup_{j=1}^{k} U_j\big)\big) \neq \emptyset$ for each open set V which contains x. For every open subset $W \ni x$ define

$$F_W = \sigma^{-1}(\overline{W}) \cap \left(K \setminus \left(\bigcup_{j=1}^{k} U_j \right) \right).$$

Since $\sigma^{-1}(\overline{W})$ is closed in U and $K \setminus \big(\bigcup_{j=1}^{k} U_j\big) \subseteq U$ is compact, it follows that F_W is compact, and therefore closed in X. Moreover F_W is nonempty because

$$\emptyset \neq \sigma^{-1}(W) \cap \left(K \setminus \left(\bigcup_{j=1}^{k} U_j \right) \right) \subseteq F_W.$$

Given $W_1, \ldots, W_m$ open neighbourhoods of x, we have that $F_{\bigcap_{j=1}^{m} W_j} \subseteq F_{W_j}$ for each j from where

$$F_{\bigcap_{j=1}^{m} W_j} \subseteq \bigcap_{j=1}^{m} F_{W_j}, \quad \text{and so} \quad \bigcap_{j=1}^{m} F_{W_j} \neq \emptyset$$

for each finite collection of open neighbourhoods $W_1, \ldots, W_m$ of x. By the fact that X is compact it follows that there exists $y \in \bigcap_{\substack{W \ni x; \\ W \text{ open}}} F_W$. Since

$$\bigcap_{\substack{W \ni x; \\ W \text{ open}}} F_W \subseteq K \setminus \left(\bigcup_{j=1}^{k} U_j \right)$$

it follows that $\sigma(y) \neq x$. Choose an open set $W_x \ni x$ such that $\sigma(y) \notin \overline{W_x}$. Then $y \notin F_{W_x}$, which is an absurd. This proves the claim.

Let $V_0 \ni x$ be an open subset according to the claim and define

$$V = V_0 \bigcap \left(\bigcap_{j=1}^{k} \sigma(U_j) \right).$$

Let $(y_i)_i$ an net such that $y_i \to x$. We may suppose that $(y_i)_i \subseteq V$, and so $\sigma^{-1}(y_i) = \{y_{1,i}, \ldots, y_{k,i}\}$ where $y_{j,i} \in U_j$. How $\sigma_{|U_j}$ is a homeomorphism we have that $y_{j,i} \overset{i\to\infty}{\longrightarrow} x_j$ for each j, and so

$$L(f)(y_i) = \sum_{\substack{z\in U\\ \sigma(z)=y_i}} f(z) = \sum_{j=1}^{k} f(y_{j,i}) \overset{i\to\infty}{\longrightarrow} \sum_{j=1}^{k} f(x_j) = \sum_{\substack{y\in U\\ \sigma(y)=x}} f(y) = L(f)(x).$$

This shows that $L(f)$ is continuous on the points of $\sigma(K)$, and the lemma is proved. □

Now we are in the situation where $C_c(U)$ is an idempotent self-adjoint ideal of $C_0(U)$, which is an ideal of $C(X)$, and by the previous lemma, $L: C_c(U) \to C(X)$ is a function. Moreover, composing α with the *-isomorphism $C^b(U) \ni g \mapsto (L_g, R_g) \in M(C_0(U))$ we obtain the partial endomorphism $\widetilde{\alpha}: C(X) \to M(C_0(U))$. It is easy to verify that $(C(X), \widetilde{\alpha}, L)$ is a C^*-dynamical system.

Since $\widetilde{\alpha}$ is essentialy given by α we will use the notation $(C(X), \alpha, L)$ to us refer to the C^*-dynamical system $(C(X), \widetilde{\alpha}, L)$. Moreover, since $g\widetilde{\alpha}(f) = g\alpha(f)$ for each $g \in C_c(U)$ and $f \in C(X)$, no more references will be made to $\widetilde{\alpha}$. So we have the Toeplitz algebra $\mathcal{T}(C(X), \alpha, L)$ and the crossed product by a partial endomorphism $\mathcal{O}(C(X), \alpha, L)$. From now on we will denote $\mathcal{T}(C(X), \alpha, L)$ by $\mathcal{T}(X, \alpha, L)$ and $\mathcal{O}(C(X), \alpha, L)$ by $\mathcal{O}(X, \alpha, L)$.

2.2 Basic results

Here we will prove some basic results about the crossed product by a partial endomorphism $\mathcal{O}(X, \alpha, L)$.

Lemma 2.2. *Given $f \in C_c(U)$, we have that:*

a) $\widetilde{f} = 0$ *if and only if* $f = 0$.

b) *if $\sigma_{|\text{supp}(f)}$ is a homeomorphism then* $\|f\|_\infty = \|\widetilde{f}\|$.

Proof.

a) Given $f \in C_c(U)$ and $x \in U$ such that $f(x) \neq 0$ then

$$L(f^* f)(\sigma(x)) = \sum_{\sigma(y)=\sigma(x)} f^*(y) f(y) = \sum_{\substack{y\neq x\\ \sigma(y)=\sigma(x)}} |f(y)|^2 + |f(x)|^2 > 0.$$

This shows that L is faithful, and so $\widetilde{f} = 0$ if and only if $f = 0$.

b) Since $\|\widetilde{f}\|^2 = \|L(f^* f)\|_\infty$ it suffices to show that $\|L(f^* f)\|_\infty = \|f\|_\infty^2$. For this note that

$$L(f^* f)(x) = \begin{cases} |f(\sigma^{-1}(x))|^2 & \text{if } x \in \sigma(\operatorname{supp}(f)) \\ 0 & \text{otherwise} \end{cases} .$$

Then $\|L(f^* f)\|_\infty \leq \|f\|_\infty^2$. On the other hand, choose $x \in U$ such that $|f(x)| = \|f\|_\infty$, and note that $L(f^* f)(\sigma(x)) = (f^* f)(x)$, which means that $\|L(f^* f)\|_\infty \geq \|f\|_\infty^2$. □

Consider the *-homomorphism $\varphi\colon C(X) \to L(M)$ given by the left product of A by M. Note that $f \in \ker(\varphi)$ if and only if $fm = 0$ for each $m \in M$, which occurs if and only if $\widetilde{fg} = f\widetilde{g} = 0$ for each $g \in C_c(U)$. By a) of the previuos lemma $\widetilde{fg} = 0$ if and only if $fg = 0$. Therefore $f \in \ker(\varphi)$ if and only if $fg = 0$ for every $g \in C_c(U)$ and so $fg = 0$ for all $g \in C_0(U)$. So, given $g \in C_0(U)$ it follows that $fg = 0$ for every $f \in \ker(\varphi)$ and so $f \in \ker(\varphi)^\perp$. This means that $C_0(U) \subseteq \ker(\varphi)^\perp$.

Lemma 2.3.

a) *If $f, g \in C_c(U)$ and $\sigma_{|\operatorname{supp}(f)\cup\operatorname{supp}(g)}$ is a homeomorphism then $(fg^*, \widetilde{f}\widetilde{g}^*)$ is a redundancy of $\mathcal{T}(X, \alpha, L)$ and $fg^* = \widetilde{f}\widetilde{g}^*$ in $\mathcal{O}(X, \alpha, L)$.*

b) $C_0(U) \subseteq \varphi^{-1}(K(M))$.

c) $C_0(U) \subseteq I_0 \ (= \varphi^{-1}(K(M)) \cap ker(\varphi)^\perp)$.

d) $C_0(U) \subseteq K_1$.

Proof.

a) Let $f, g \in C_c(U)$ such that $\sigma_{|\operatorname{supp}(f)\cup\operatorname{supp}(g)}$ is a homeomorphism and $h \in C_c(U)$. Notice that $\widetilde{f}\widetilde{g}^*\widetilde{h} = (f\alpha(L(g^*h)))^{\sim}$. Since $\sigma_{|\operatorname{supp}(f)\cup\operatorname{supp}(g)}$ is a homeomorphism, for each element $x \in \operatorname{supp}(f)$ we have that

$$f(x) \sum_{\substack{y\in U \\ \sigma(y)=\sigma(x)}} (g^*h)(y) = f(x)g(x)^*h(x) .$$

Therefore for these x,

$$f\alpha(L(g^*h))(x) = f(x)L(g^*h)(\sigma(x)) = f(x) \sum_{\substack{y\in U \\ \sigma(y)=\sigma(x)}} (g^*h)(y)$$
$$= f(x)g^*(x)h(x) = (fg^*h)(x).$$

If $x \notin \text{supp}(f)$ then $(f\alpha(L(g^*h)))(x) = 0 = (fg^*h)(x)$. Therefore $f\alpha(L(g^*h)) = fg^*h$. Then $\widetilde{f}\,\widetilde{g}^*\widetilde{h} = (f\alpha(L(g^*h)))^{\sim} = \widetilde{fg^*h} = fg^*\widetilde{h}$ for every $h \in C_c(U)$, from where $\widetilde{f}\,\widetilde{g}^*m = fg^*m$ for all $m \in M$. It follows that $(fg^*, \widetilde{f}\,\widetilde{g}^*)$ is a redundancy. Since $fg^* \in C_0(U) \subseteq \ker(\varphi)^{\perp}$ we have that $fg^* = \widetilde{f}\,\widetilde{g}^*$ in $\mathcal{O}(X, \alpha, L)$.

b) It is enough to show that $C_c(U) \subseteq K(M)$. Let $f \in C_c(U)$, choose a cover $V_1, \cdots, V_n$ of $\text{supp}(f)$ such that $\sigma_{|V_i}$ is a homeomorphism. Let ξ_i'' be a partition of unity relative to this cover. Define $\xi_i = f\sqrt{\xi_i''}$ and $\xi_i' = \sqrt{\xi_i''}$. Then $f = \sum_{i=1}^n \xi_i \xi_i'^*$. By a), $(\xi_i\xi_i'^*, \widetilde{\xi_i}\,\widetilde{\xi_i'}^*)$ is a redundancy from where (f, k) is a redundancy where $k = \sum_{i=1}^n \widetilde{\xi_i}\,\widetilde{\xi_i'}^* \in \widehat{K_1}$. In this way $fm = km$ for all $m \in M$ and so $\varphi(f)(m) = fm = km = S(k)(m)$ for every $m \in M$, where S is the *-isomorphism of 1.9. It follows that $\varphi(f) = S(k)$ and so $f \in \varphi^{-1}(K(M))$. Therefore $C_c(U) \subseteq \varphi^{-1}(K(M))$.

c) Follows by b) and by the fact that $C_0(U) \subseteq \ker(\varphi)^{\perp}$.

d) Given $f \in C_c(U)$, by b) it follows that (f, k) is a redundancy for some $k \in \widehat{K_1}$. Since $f \in C_0(U) \subseteq I_0$ it follows that $f = q(k) \in K_1$. So $C_c(U) \subseteq K_1$ from where $C_0(U) \subseteq K_1$. □

The following lemma will be used several times in this work.

Lemma 2.4. *If* $(k_0, k_1, \ldots, k_n) \in C(X) \times K_1 \times \cdots \times K_n$ *such that*

$$g \sum_{i=0}^n k_i = 0$$

for each $g \in C_0(U)$ *then:*

a) $k_{0|_{\partial(U)}} = 0, k_0 = f_1 + f_2$ *where* $f_1 \in C_0(U)$ *and* $f_2 \in C_0(X \setminus \overline{U})$.

b) $\sum_{i=0}^n k_i = f_2$.

Proof. Let $\varepsilon > 0$ be fixed. For every $i \geq 1$ choose

$$k_i' = \sum_{j=1}^{N_i} m_{j,1}^i \cdots m_{j,i}^i \left(l_{j,1}^i\right)^* \cdots \left(l_{j,i}^i\right)^* \in K_i$$

such that $m_{j,k}^i = \widetilde{f_{j,k}^i}$ with $f_{j,k}^i \in C_c(U)$ and $\|k_i - k_i'\| \leq \frac{\varepsilon}{n}$. Define

$$k_\varepsilon = k_1' + \cdots + k_n' \quad \text{and} \quad K_\varepsilon = \bigcup_{i,j,k} \operatorname{supp}\left(f_{j,k}^i\right) \subseteq U$$

which is compact. Given $x \in U \setminus K_\varepsilon$ take $f \in C_0(U)$ such that $f(x) = 1$, $0 \leq f \leq 1$ and $f_{|K_\varepsilon} = 0$. Then $f k_\varepsilon = 0$ by the choice of f and $f k_0 = -f \sum_{i=1}^n k_i$ by hypothesis. It follows that

$$\|fk_0\| = \left\|-f\sum_{i=1}^n k_i + f_x k_\varepsilon\right\| = \left\|f\left(-\sum_{i=1}^n k_i + k_\varepsilon\right)\right\| = \left\|f\sum_{i=1}^n \left(k_i' - k_i\right)\right\| \leq \varepsilon$$

from where $|k_0(x)| \leq \varepsilon$. In this way we have showed that $|k_0(x)| \leq \varepsilon$ for all $x \in U \setminus K_\varepsilon$. Given $y \in \partial(U)$, take a net $(x_l)_l \subseteq U$ such that $x_l \to y$. Since $y \notin K_\varepsilon$ and $U \setminus K_\varepsilon$ is open we may suppose $(x_l)_l \subseteq U \setminus K_\varepsilon$ from where $|k_0(x_l)| \leq \varepsilon$ for each l. By continuity of k_0, $|k_0(y)| \leq \varepsilon$. This shows (taking ε sufficiently small) that $k_{0|_{\partial(U)}} = 0$. Defining $f_1 = k_0 1_U$ and $f_2 = k_0 1_{U^c}$, we obtain a).

We will show b). For each $\varepsilon > 0$ choose $g_\varepsilon \in C_0(U)$ such that $0 \leq g \leq 1$ and $g_{|K_\varepsilon} = 1$. Define $h_\varepsilon = g_\varepsilon k_0$. So we obtain a set of functions $(h_\varepsilon)_\varepsilon \subseteq C_0(U)$.

Claim. $\lim_{\varepsilon \to 0} h_\varepsilon = f_1$.

For each ε, given $x \in X$,

$$|(h_\varepsilon - f_1)(x)| = |(g_\varepsilon - 1_U)(x) k_0(x)| = \begin{cases} |g_\varepsilon(x) - 1|\,|k_0(x)| & \text{if } x \in U \setminus K_\varepsilon \\ 0 & x \in K_\varepsilon \cup U^c \end{cases}$$

For $x \in U \setminus K_\varepsilon$ it holds that $|k_0(x)| \leq \varepsilon$ and so for such elements

$$|g_\varepsilon(x) - 1|\,|k_0(x)| \leq 2\varepsilon.$$

So $\|h_\varepsilon - f_1\| \leq 2\varepsilon$. This shows the claim.

Notice that $g_\varepsilon k_\varepsilon = k_\varepsilon$ and $h_\varepsilon = g_\varepsilon k_0 = -g_\varepsilon(k_1 + \cdots + k_n)$ because $g_\varepsilon \in C_0(U)$. Then

$$\begin{aligned} h_\varepsilon + (k_1 + \cdots + k_n) &= h_\varepsilon + k_\varepsilon - k_\varepsilon + (k_1 + \cdots + k_n) \\ &= -g_\varepsilon(k_1 + \cdots + k_n) + k_\varepsilon - k_\varepsilon + (k_1 + \cdots + k_n) \\ &= -g_\varepsilon(k_1 + \cdots + k_n - k_\varepsilon) - k_\varepsilon + (k_1 + \cdots + k_n), \end{aligned}$$

and so

$$\begin{aligned}\|h_\varepsilon + (k_1 + \cdots + k_n)\| &= \|g_\varepsilon(-(k_1 + \cdots + k_n) + k_\varepsilon) + ((k_1 + \cdots + k_n) - k_\varepsilon)\| \\ &\leq \|g_\varepsilon(-(k_1 + \cdots + k_n) + k_\varepsilon)\| + \|(k_1 + \cdots + k_n) - k_\varepsilon\| \\ &\leq 2\varepsilon.\end{aligned}$$

This shows that $\lim_{\varepsilon\to 0} h_\varepsilon = -(k_1 + \cdots + k_n)$. By the claim $\lim_{\varepsilon\to 0} h_\varepsilon = f_1$, and so $f_1 = -(k_1 + \cdots + k_n)$. Then

$$\sum_{i=0}^{n} k_i = f_1 + f_2 + k_1 + \cdots + k_n = f_2,$$

proving b). □

Corollary 2.5. $K_1 \cap C(X) = C_0(U)$

Proof. Let $r \in K_1 \cap C(X)$. Then $r = f = k$ where $f \in C(X)$ and $k \in K_1$. Then $f - k = 0$ and so $g(f - k) = 0$ for all $g \in C_0(U)$, and so by 2.4, $f = f_1 + f_2$ with $f_1 \in C_0(U)$, $f_2 \in C_0(X \setminus \overline{U})$ and $f - k = f_2$. How $f - k = 0$ it follows that $f_2 = 0$. Therefore $f = f_1$, which means that $r = f_1 \in C_0(U)$. In this way $K_1 \cap C(X) \subseteq C_0(U)$. The other inclusion is the lema 2.3 d). □

In the construction of $\mathcal{O}(X, \alpha, L)$ we have considered the ideal $I_0 = \varphi^{-1}(K(M)) \cap \ker(\varphi)^\perp$. The previous corollary allows us to identify this ideal.

Corollary 2.6. $I_0 = C_0(U)$

Proof. Given $f \in I_0$ then $\varphi(f) = k \in K(M)$. Choose $k' \in \widehat{K_1}$ such that $S(k') = k$ where S is the *-isomorphism of 1.9. Then $fm = \varphi(f)(m) = k(m) = S(k')(m) = k'm$ for all $m \in M$. Therefore (f, k') is a redundancy. Since $f \in I_0$ it follows that $f = q(k') \in K_1$ in $\mathcal{O}(X, \alpha, L)$. By the previous corollary we have that $f \in C_0(U)$. So $I_0 \subseteq C_0(U)$. The reverse inclusion follows by 2.3 c). □

Recall that K is the fixed point algebra of the gauge action and that

$$K = \overline{\bigcup_{n\in\mathbb{N}} L_n}$$

where $L_n = C(X) + K_1 + \cdots + K_n$ for $n \geq 1$ and $L_0 = C(X)$.

Proposition 2.7. *Every ideal of $\mathcal{O}(X,\alpha,L)$ which has nonzero intersection with K has nonzero intersection with $C(X)$.*

Proof. Let I be an ideal of $\mathcal{O}(X,\alpha,L)$ such that $I\cap K\neq 0$. By [2, III.4.1] there exists $n\in\mathbb{N}$ such that $I\cap L_n\neq 0$. Let $n_0=\min\{n\in\mathbb{N}\colon I\cap L_n\neq 0\}$ and choose $0\neq k\in I\cap L_{n_0}$. Suppose $n_0\neq 0$. Supposing $m^*kk^*l=0$ for all $m,l\in M$ we have that $m^*k=0$ for all $m\in M$. So $K_1k=0$ and by the fact that $C_0(U)\subseteq K_1$ it follows that $fk=0$ for all $f\in C_0(U)$. By 2.4, $k\in C(X)=L_0$, which is a contradiction because we are supposing $n_0\neq 0$. So there exists $m,l\in M$ such that $m^*kk^*l\neq 0$. Notice that $m^*kk^*l\in I\cap L_{n_0-1}$ which again is an absurd because $n_0=\min\{n\in\mathbb{N}\colon I\cap L_n\neq 0\}$. Therefore $n_0=0$, that is, $k\in L_0=C(X)$. □

By this proposition and by 1.12 follows the corollary:

Corollary 2.8. *If $0\neq I$ is a gauge-invariant ideal of $\mathcal{O}(X,\alpha,L)$ then $I\cap C(X)\neq 0$.*

2.3 The Cuntz-Krieger algebra for infinite matrices

We show that that the Cuntz-Krieger algebra for infinite matrices, introduced in [4], is an example of crossed product by partial endomorphism. We begin by presenting a short summary of the construction of this algebra.

Ler G be a set and $A=A(i,j)_{i,j\in G}$ a matrix where each $A(i,j)\in\{0,1\}$. Define the universal C^*-algebra $\widetilde{O_A}$ generated by a set of partial isometries $\{S_x\}_{x\in G}$ with the following relations:

1. $S_i^*S_i$ and $S_j^*S_j$ commute,
2. $S_i^*S_j=0$ for all $i\neq j$,
3. $S_i^*S_iS_j=A(i,j)S_j$,
4. $\prod_{x\in X}S_x^*S_x\prod_{y\in Y}(1-S_y^*S_y)=\sum_{j\in G}A(X,Y,j)S_jS_j^*$, whenever X,Y are finite subsets of G such that
$$A(X,Y,j):=\prod_{x\in X}A(x,j)(1-\prod_{y\in Y}A(y,j))\neq 0$$
only for finitely many $j\in G$.

The Cuntz-Krieger algebra for infinite matrices was defined in [4] as the subalgebra O_A of $\widetilde{O_A}$ generated by the S_x.

Let $\mathbb{F}$ be the free group generated by G and let $\{0,1\}^{\mathbb{F}}$ be the topological space (with the product topology), which can also be seen as the set of the subsets of $\mathbb{F}$. In $\{0,1\}^{\mathbb{F}}$ consider the set $\Omega_e = \{\xi \subseteq \mathbb{F};\, e \in \xi\}$, which is compact. For each $t \in \mathbb{F}$ define $\Delta'_t = \{\xi \in \Omega_e;\, t \in \xi\}$, which is an clopen subset. Denoting by 1_t the characteristic function of Δ'_t consider the set $R_A \subseteq C(\Omega_e)$ formed by the following functions:

1. $1_x 1_y$ for all $x \neq y$, $x, y \in G$,
2. $1_{x^{-1}} 1_y - A(x,y)1_y$ for all $x, y \in G$,
3. $1_{ts}1_t - 1_{ts}$ for $t, s \in \mathbb{F}$ such that $|ts| = |t| + |s|$, (where $|s|$ is the number of generators of the reduced form of s),
4. $\prod_{x\in X} 1_{x^{-1}} \prod_{y\in Y}(1 - 1_{y^{-1}}) - \sum_{j\in G} A(X,Y,j)1_j$ where X, Y are finite subsets of G such that $A(X,Y,j) \neq 0$ only for finitely many $j \in G$.

In Ω_e consider the closed set $\widetilde{\Omega_A} = \{\xi \in \Omega_e;\, f(t^{-1}\xi) = 0 \ \forall\, t \in \xi,\ f \in R_A\}$. In [4, 7.3] it was showed that $\widetilde{\Omega_A}$ is the closure in $\Omega_A^{\mathcal{T}}$ of the set of the elements which have an infinite stem (see [4, 5.5]), where

$$\Omega_A^{\mathcal{T}} = \left\{ \begin{array}{ll} \xi \in \Omega_e : & e \in \xi, \xi \text{ is convex} \\ & \text{if } t \in \xi \text{ there is at most one } x \in G \text{ such that } tx \in \xi \\ & \text{if } t \in \xi,\, y \in G \text{ and } ty \in \xi \text{ then } tx^{-1} \in \xi \Leftrightarrow A(x,y) = 1 \end{array} \right\}$$

The homeomorphisms $h_t\colon \Delta'_{t^{-1}} \to \Delta'_t$ given by $h_t(\xi) = t\xi$ induces a partial action $(\{D_t\}_{t\in\mathbb{F}}, \{\theta_t\})$ (see [5] and [9]) of $\mathbb{F}$ in $C(\widetilde{\Omega_A})$ where $D_t = C(\Delta_t)$, $\Delta_t = \Delta'_t \cap \widetilde{\Omega_A}$ and $\theta_t\colon D_{t^{-1}} \to D_t$ is given by $\theta(f) = f \circ h_{t^{-1}}$ and so we may consider the partial crossed product $C(\widetilde{\Omega_A}) \rtimes_\theta \mathbb{F}$ (see [5] and [9]).

It was showed in [4, 7.10] that there exists a *-isomorphism $\Phi\colon \widetilde{O_A} \to C(\widetilde{\Omega_A}) \rtimes_\theta \mathbb{F}$ such that $\Phi(S_x) = 1_x\delta_x$.

Based on these informations we will show that $\widetilde{O_A}$ is an example of crossed product by a partial endomorphism.

Let

$$U \subseteq \widetilde{\Omega_A}\,,\ U = \bigcup_{x\in G} \Delta_x\,.$$

By the fact that each Δ_x is open it follows that U is open. Moreover, U is dense in $\widetilde{\Omega_A}$, because U contains all the elements of $\Omega_A^{\mathcal{T}}$ which have an infinite stem, and these elements form a dense set in $\widetilde{\Omega_A}$. Since each $\xi \in U$ contains a unique $x \in G$, we may define the continuous function $\sigma : U \to \widetilde{\Omega_A}$ given by $\sigma(\xi) = x^{-1}\xi$ where x is the unique element of G which lies in ξ. This function is a local homeomorphism (in fact, $\sigma_{|\Delta_x} : \Delta_x \to \Delta_{x^{-1}}$ is a homeomorphism). Defining

$$\alpha : C(\widetilde{\Omega_A}) \to C^b(U) \quad \text{by} \quad \alpha(f) = f \circ \sigma$$

and

$$L : C_c(U) \to C(\widetilde{\Omega_A}) \quad \text{by} \quad L(f)(\xi) = \sum_{\substack{\eta \in U \\ \sigma(\eta)=\xi}} f(\eta)$$

we have that $(C(\widetilde{\Omega_A}), \alpha, L)$ is a C^*-dynamical system, and so we obtain the algebra $\mathcal{O}(\widetilde{\Omega_A}, \alpha, L)$ (see section 2.1).

The next step is to show that the algebras $\mathcal{O}(\widetilde{\Omega_A}, \alpha, L)$ and $\widetilde{O_A}$ are isomorphic.

Lemma 2.9.

a) $L(1_x) = 1_{x^{-1}}$ *for each* $x \in G$.

b) $f 1_x \alpha L(1_x g) = 1_x fg$ *for each* $x \in G$ *and* $f, g \in C(\widetilde{\Omega}_A)$.

Proof. Both a) and b) follow by direct calculation. To prove the first part notice that $\sigma^{-1}(\xi) = \{x\xi : x^{-1} \in \xi\}$. □

Proposition 2.10. *There exists an unitary *-homomorphism*

$$\psi : \widetilde{O_A} \to \mathcal{O}(\widetilde{\Omega_A}, \alpha, L)$$

such that $\psi(S_x) = \widetilde{1_x}$.

Proof. We will show that ψ preserves the relations 1-4 which defines $\widetilde{O_A}$. The first relation follows by the fact that $\psi(S_x)^*\psi(S_x) = \widetilde{1_x}^*\widetilde{1_x} \in C(\widetilde{\Omega}_A)$. To verify the second relation note that $1_x 1_y = 0$ for $x, y \in G$ and $x \neq y$, from where $\psi(S_x)^*\psi(S_y) = \widetilde{1_x}^*\widetilde{1_y} = L(1_x 1_y) = 0$. The third relation follows by 2.9 a) and by the fact that $1_{x^{-1}} 1_y = A(x, y) 1_y$ in $\widetilde{\Omega_A}$. In fact,

$$\begin{aligned} \psi(S_x)^*\psi(S_x)\psi(S_y) &= \widetilde{1_x}^*\widetilde{1_x}\,\widetilde{1_y} = L(1_x)\widetilde{1_y} = 1_{x^{-1}}\widetilde{1_y} \\ &= \widetilde{1_{x^{-1}} 1_y} = A(x, y)\widetilde{1_y} = A(x, y)\psi(S_y). \end{aligned}$$

Let us verify the fourth relation. By 2.3 a) $1_x = \widetilde{1_x}\,\widetilde{1_x}^*$ in $\mathcal{O}(\widetilde{\Omega_A}, \alpha, L)$. Therefore, also

$$\sum_{i=1}^{n} 1_{x_i} = \sum_{i=1}^{n} \widetilde{1_{x_i}}\,\widetilde{1_{x_i}}^* \quad \text{in} \quad \mathcal{O}(\widetilde{\Omega_A}, \alpha, L).$$

Let $X, Y \subseteq G$ finite such that $A(X, Y, x_i) \neq 0$ only for $i = 1, \cdots, n$. Then

$$\prod_{x\in X} 1_x^{-1} \prod_{y\in Y}(1 - 1_y^{-1}) = \sum_{i=1}^{n} 1_{x_i} \quad \text{in} \quad \widetilde{\Omega}_A$$

and so

$$\prod_{x\in X} \psi(S_x)^*\psi(S_x) \prod_{y\in Y}(1 - \psi(S_y)^*\psi(S_y)) = \prod_{x\in X} 1_x^{-1} \prod_{y\in Y}(1 - 1_y^{-1})$$

$$= \sum_{i=1}^{n} 1_{j_i} = \sum_{i=1}^{n} \widetilde{1_{x_i}}\, \widetilde{1_{x_i}}^{\,*} = \sum_{i=1}^{n} \psi(S_{x_i})\psi(S_{x_i})^* \qquad \square$$

$$= \sum_{x\in G} A(X, Y, x)\psi(S_x)\psi(S_x)^*.$$

We will show that the *-homomorphism defined in this proposition is a *-isomorphism. The following lemma will be useful to show that this *-homomorphism is surjective.

Lemma 2.11. *The C^*-algebra B generated by $\widetilde{1_x} : x \in G$ in $\mathcal{O}(\widetilde{\Omega_A}, \alpha, L)$ contains all the elements of $C(\Omega_e)$ of the form $1_r : e \neq r \in \mathbb{F}$ and moreover B coincides with the C^*-algebra generated by M.*

Proof. By 2.9 a), $\widetilde{1_x}^{\,*}\widetilde{1_x} = 1_{x^{-1}}$. Given $\beta = x_1^{-1} \cdots x_n^{-1} \in \mathbb{F}$ with $x_i \in G$, by induction

$$\widetilde{1_{x_n}}^{\,*} \cdots \widetilde{1_{x_1}}^{\,*}\widetilde{1_{x_1}} \cdots \widetilde{1_{x_n}} = \widetilde{1_{x_n}}^{\,*} 1_{x_{n-1}^{-1}\cdots x_1^{-1}} \widetilde{1_{x_n}} = L(1_{x_n} 1_{x_{n-1}^{-1}\cdots x_1^{-1}}) = 1_{x_n^{-1}\cdots x_1^{-1}}.$$

If $b = yr^{-1}$ with $r = x_1 \cdots x_n$ and $x_i, y \in G$ then

$$\widetilde{1_y}\, \widetilde{1_{x_n}}^{\,*} \cdots \widetilde{1_{x_1}}^{\,*}\widetilde{1_{x_1}} \cdots \widetilde{1_{x_n}}\, \widetilde{1_y}^{\,*} = \widetilde{1_y}\, 1_{r^{-1}} \widetilde{1_y}^{\,*} = (1_y\alpha(1_{r^{-1}}))^{\widetilde{}}\, \widetilde{1_y}^{\,*}.$$

By 2.3 a) $(1_y\alpha(1_{r^{-1}}))^{\widetilde{}}\, \widetilde{1_y}^{\,*} = 1_y\alpha(1_{r^{-1}})$, and by direct calculation $1_y\alpha(1_{r^{-1}}) = 1_{yr^{-1}}$. Therefore $1_{yr^{-1}} \in B$ for all $y \in G$, $r = x_1 \cdots x_n$ with $x_i \in G$. The general case, $\beta = sr^{-1}$, with $s = x_1 \cdots x_n, r = y_1 \cdots y_m$ and $x_i, y_i \in G$ follows by induction. If $t \in \mathbb{F}$ and t is not of the form $\beta = sr^{-1}$ like above, then $1_t = 0$ em $\widetilde{\Omega_A}$ by [4, 5.8]. Therefore $1_t \in B$ for all $e \neq t \in \mathbb{F}$. We will show that B is the algebra generated by M. For each $x \in G$, span$\{1_x \prod_s 1_s\}$ is dense in D_x and by 2.2 b), since $\sigma_{|\Delta_x}$ is a homeomorphism, it follows that span$\{(1_x \prod_s 1_s)^{\widetilde{}}\}$ is dense in $\widetilde{D_x}$. Since $(1_x \prod_s 1_s)^{\widetilde{}} = 1_x \prod_s 1_s \widetilde{1_x} \in B$ we have that $\widetilde{D_x} \subseteq B$, because B is closed. So $\widetilde{C_c(U)} \subseteq B$ and since B is closed it follows that $M \subseteq B$. This shows that B contains the algebra generated by M. On the other hand, since $\widetilde{1}_x \in M$ for each $x \in G$, it is clear that the algebra generated by M contains B, and this concludes the proof. □

Proposition 2.12. *There exists a *-homomorphism*

$$\phi\colon \mathcal{O}(\widetilde{\Omega_A}, \alpha, L) \to C(\widetilde{\Omega}_A) \rtimes_\theta \mathbb{F}$$

such that $\phi(f) = f\delta_e$ *for all* $f \in C(X)$ *and* $\phi(\widetilde{f_x}) = f_x\delta_x$ *for all* $f \in D_x$ *and* $x \in G$.

Proof. Let us define initially a homomorphism from the Toeplitz algebra $\mathcal{T}(\widetilde{\Omega_A}, \alpha, L)$ to $C(\widetilde{\Omega}_A) \rtimes \mathbb{F}$. Define

$$\phi' : C(\widetilde{\Omega_A}) \to C(\widetilde{\Omega_A}) \rtimes_\theta \mathbb{F} \quad \text{by} \quad \phi'(f) = f\delta_e$$

and

$$\phi'' : \widetilde{C_c(U)} \to C(\widetilde{\Omega_A}) \rtimes_\theta \mathbb{F} \quad \text{by} \quad \phi''(\widetilde{f_x}) = f_x\delta_x$$

for $f_x \in D_x$. Clearly ϕ' is a *-homomorphism. By 2.2 a) ϕ'' is well defined. Moreover ϕ'' is linear and given $g = \sum g_x$ and $f = \sum f_x$ in $C_c(U)$, where $f_x, g_x \in D_x$, we have that

$$\begin{aligned}
\phi''(\widetilde{g})^*\phi''(\widetilde{f}) &= \Big(\sum g_x\delta_x\Big)^*\Big(\sum f_y\delta_y\Big) = \Big(\sum \theta_{x^{-1}}(g_x^*)\delta_{x^{-1}}\Big)\Big(\sum f_y\delta_y\Big)\\
&= \sum_{x,y} \theta_{x^{-1}}(g_x^*)\delta_{x^{-1}} f_y\delta_y = \sum_{x,y} \theta_{x^{-1}}(g_x^* f_y)\delta_{x^{-1}y}\\
&= \sum_x \theta_{x^{-1}}(g_x^* f_x)\delta_e.
\end{aligned}$$

Claim. $L(g^* f) = \sum \theta_{x^{-1}}(g_x^* f_x)$.

It is enough to show that $L(g_x^* f_x) = \theta_x^{-1}(g_x^* f_x)$ because $g_x^* f_y = 0$ for $x \neq y$. For this notice that if $x^{-1} \notin \xi$ then $L(g_x^* f_x)(\xi) = 0 = \theta_x^{-1}(g_x^* f_x)(\xi)$. Moreover, if $x^{-1} \in \xi$ then we have

$$L(g_x^* f_x)(\xi) = (g_x^* f_x)(x\xi) = (g_x^* f_x)(h_x(\xi)) = \theta^{-1}(g_x^* f_x)(\xi).$$

So the claim is proved. □

Then $\sum_x \theta_{x^{-1}}(g_x^* f_x)\delta_e = L(g^* f)\delta_e = \phi'(\langle \widetilde{g}, \widetilde{f} \rangle)$, and so, $\phi''(\widetilde{g})^* \phi''(\widetilde{f}) = \phi'(\langle \widetilde{g}, \widetilde{f} \rangle)$. Therefore

$$\|\phi''(\widetilde{f})\|^2 = \|\phi''(\widetilde{f})^* \phi''(\widetilde{f})\| = \|\phi'(\langle \widetilde{f}, \widetilde{f} \rangle)\| \leq \|\langle \widetilde{f}, \widetilde{f} \rangle\| = \|\widetilde{f}\|_M^2$$

from where we may extend ϕ'' to M. In this way we obtain a function

$$\phi\colon C(\widetilde{\Omega_A}) \cup M \to C(\widetilde{\Omega}_A) \rtimes_\theta \mathbb{F}$$

defined by $\phi(f) = \phi'(f)$ if $f \in C(\widetilde{\Omega_A})$ and $\phi(m) = \phi''(m)$ for $m \in M$.

Claim. *ϕ satisfies the relations which defines $\mathcal{T}(\widetilde{\Omega_A}, \alpha, L)$.*

By density of $\widetilde{C_c(U)}$ in M it suffices to verify if ϕ satisfies the relations for elements of the form $\widetilde{f} = \sum \widetilde{f_x}$, $\widetilde{g} = \sum \widetilde{g_y} \in \widetilde{C_c(U)}$, where $f_x, g_x \in D_x$, and $h \in C(\widetilde{\Omega_A})$. We already know that ϕ preserves the relations of $C(\widetilde{\Omega_A})$, of M and that $\phi(\widetilde{f})^* \phi(\widetilde{g}) = \phi(\langle \widetilde{f}, \widetilde{g} \rangle)$. Moreover,

$$\phi(h)\phi(\widetilde{f}) = h\delta_e \sum f_x \delta_x = \sum h f_x \delta_x = \phi(\widetilde{hf}) = \phi(h\widetilde{f})$$

and

$$\phi(\widetilde{f})\phi(h) = \left(\sum f_x \delta_x\right) h\delta_e = \sum \theta_x(\theta_x^{-1}(f_x)h)\delta_x = \sum f_x \alpha(h)\delta_x$$

$$= \phi(\widetilde{f\alpha(h)}) = \phi(\widetilde{f}h).$$

This proves the claim. □

So we may extend ϕ to $\mathcal{T}(\widetilde{\Omega_A}, \alpha, L)$. We will show that if (a, k) is a redundancy then $\phi(a) = \phi(k)$. For each

$$f_x \in D_x,\ \phi(\widetilde{f_x}\, \widetilde{1_x}^{\,*}) = f_x \delta_x 1_{x^{-1}} \delta_{x^{-1}} = f_x \delta_e = \phi(f_x)$$

and so if $f = \sum_x f_x$ with $f_x \in D_x$ then $\phi(f) = \sum_x \phi(\widetilde{f_x}\, \widetilde{1_x}^{\,*})$. Given a redundancy (f, k) with $f \in I_0$, and so $f \in C_0(U)$ by 2.6, choose $(f_n)_n \subseteq C_c(U)$ such that $f_n \to f$, and $(k_n)_n \subseteq \widehat{K_1}$ such that

$$k_n \to k \quad \text{and} \quad k_n = \sum_{i=1}^{t_n} m_{i,n} r_{i,n}^* \quad \text{with} \quad m_{i,n}, r_{i,n} \in M\,.$$

Since $f_n \in C_c(U)$ for each n, we have that $f_n = \sum_{i=1}^{l_n} f_{x_{i,n}}$ and so $\phi(f_n) = \sum_{i=1}^{l_n} \phi(\widetilde{f_{x_{i,n}}}\ \widetilde{1_{x_{i,n}}}^{\,*})$. Then

$$\begin{aligned}\phi(f-k)\phi(f-k)^* &= \lim_n \phi(f-k)(\phi(f_n)^* - \phi(k_n^*)) \\ &= \lim_n \phi(f-k)\left(\phi\left(\sum \widetilde{1_{x_{i,n}}}\ \widetilde{f_{x_{i,n}}}^{\,*}\right) - \phi\left(\sum_{i=1}^{n_i} r_{i,n} m_{i,n}^*\right)\right) \\ &= \lim_n \phi\left((f-k)\left(\sum \widetilde{1_{x_i}}\ \widetilde{f_{x_i}}^{\,*} - \sum_{i=1}^{n_i} r_{i,n} m_{i,n}^*\right)\right) = 0.\end{aligned}$$

The last equality follows by the fact that $(f-k)m = 0$ for each $m \in M$, because (f, k) is a redundancy. This shows that $\phi(f) = \phi(k)$. □

Proposition 2.13. *The *-homomorphism* $\psi: \widetilde{O_A} \to \mathcal{O}(\widetilde{\Omega_A}, \alpha, L)$ *defined in* 2.10 *is a *-isomorphism.*

Proof. To prove that ψ is surjective it is anough to prove that $M \cup C(\widetilde{\Omega_A}) \subseteq \mathrm{Im}(\psi)$. By the lemma 2.11, $M \subseteq \mathrm{Im}(\psi)$. By the same lemma, the elements of the form $1_r : e \neq r \in \mathbb{F}$ are in the range of ψ and moreover, $\psi(1) = 1 = 1_e$. The algebra generated by the elements $\{1_r : r \in F\}$ is self-adjoint, contains the constant functions and separate points, and so is dense in $C(\widetilde{\Omega_A})$. It follows that $C(\widetilde{\Omega_A}) \subseteq \mathrm{Im}(\psi)$. In order to see that ψ is injective, note that $\Phi^{-1}\phi\psi = Id_{\widetilde{O}_A}$ where ϕ is the *-homomorphism of 2.12 and Φ is the *-isomorphism between $\widetilde{O_A}$ and $C(\widetilde{\Omega_A}) \rtimes_\theta \mathbb{F}$ such that $\Phi(S_x) = 1_x\delta_x$. □

By this proposition and by 2.11 it follows that the Cuntz-Krieger algebra for infinite matrices O_A is isomorphic to the algebra B, generated by M. Note that the algebra generated by M coincides with the ideal $\langle M \rangle$ of $\mathcal{O}(\widetilde{\Omega_A}, \alpha, L)$.

3 Relationship between the gauge-invariant ideals of $\mathcal{O}(X, \alpha, L)$ and open sets of X

We show in this section a bijection between the gauge-invariant ideals of $\mathcal{O}(X, \alpha, L)$ and the σ, σ^{-1}-invariant subsets of X. In particular, we prove that every gauge-invariant ideal of $\mathcal{O}(X, \alpha, L)$ is generated by the set $C_0(V)$ for some $V \subseteq X$ whith is σ, σ^{-1}-invariant.

Definition 3.1.

a) *A set* $V \subseteq X$ *is* σ*-invariant if* $\sigma(V \cap U) \subseteq V$.

b) *A set $V \subseteq X$ is σ^{-1}-invariant if $\sigma^{-1}(V) \subseteq V$.*

c) *A set $V \subseteq X$ is σ, σ^{-1}-invariant if it is σ-invariant and σ^{-1}-invariant.*

Let $V \subseteq X$ be an open set. We say that $C_0(V)$ is L-invariant if $L(C_0(V) \cap C_c(U)) \subseteq C_0(V)$.

Proposition 3.2.

a) An open set $V \subseteq X$ is σ-invariant if and oly if $C_0(V)$ is L-invariant.

b) An open set $V \subseteq X$ is σ^{-1}-invariant if and only if $f\alpha(g) \in C_0(V)$ for all $f \in C_c(U)$ and $g \in C_0(V)$.

Proof.

a) Suppose V σ-invariant. Given $f \in C_0(V) \cap C_c(U)$, choose $x \notin V$. Supposing $y \in \sigma^{-1}(x) \cap V$, we have $x = \sigma(y) \in V$ because V is σ-invariant. So there does not exists a such y, and therefore $L(f)(x) = 0$. This shows that $L(f) \in C_0(V)$. On the other hand, suppose $C_0(V)$ L-invariant. Suppose $x \in U \cap V$ and choose $f_x \in C_c(U) \cap C_0(V)$ such that $f_x(x) \neq 0$. Then $L(f_x^* f_x) \in C_0(V)$ and $L(f_x^* f_x)(\sigma(x)) \neq 0$, which shows that $\sigma(x) \in V$.

b) Suppose V σ^{-1}-invariant. Let $f \in C_c(U)$, $g \in C_0(V)$ and $x \notin V$. If $x \notin U$, then $f(x) = 0$ and so $(f\alpha(g))(x) = 0$. If $x \in U$, since V is σ^{-1}-invariant then $\sigma(x) \notin V$ and therefore $f\alpha(g)(x) = f(x)g(\sigma(x)) = 0$. So $f\alpha(g) \in C_0(V)$. On the other hand, let $x \in \sigma^{-1}(y)$, $y \in V$. Choose $g \in C_0(V)$ such that $g(y) \neq 0$ and $f \in C_c(U)$ such that $f(x) \neq 0$. Then, since $f\alpha(g) \in C_0(V)$ and $(f\alpha(g))(x) = f(x)g(y) \neq 0$ it follows that $x \in V$. So V is σ^{-1}-invariant. □

If $V \subseteq X$ is an open σ, σ^{-1}-invariant set then $X' = X \setminus V$ is a compact σ, σ^{-1}-invariant set. Define $U' = U \cap X'$ $(= U \setminus V)$ and consider $\sigma' := \sigma_{|'_U} : U' \to X'$ which is a local homeomorphism. Consider the C^*-dynamical system (X', α', L') where α' and L' are defined as α and L in the section 2.1. Denote by M' the Hilbert module generated by $C_c(U')$, by $\langle C_0(V) \rangle$ the ideal generated by $C_0(V)$ in $\mathcal{O}(X, \alpha, L)$ and by $\overline{\overline{b}}$ the image of the elements $b \in \mathcal{O}(X, \alpha, L)$ by the quotient map of $\mathcal{O}(X, \alpha, L)$ on $\mathcal{O}(X, \alpha, L)/\langle C_0(V) \rangle$.

Theorem 3.3. *There exists a* *-*isomorphism* $\Psi: \mathcal{O}(X, \alpha, L)/\langle C_0(V)\rangle \to \mathcal{O}(X', \alpha', L')$ *such that* $\Psi(\overline{\overline{f}}) = f_{|_{X'}}$ *for each* $f \in C(X)$.

Proof. Define $\Psi_1: C(X) \to C(X')$ by $\Psi_1(f) = f_{|_{X'}}$ which is a *-homomorphism and is surjective, by Tietze's theorem. Moreover, for every $\widetilde{f} \in \widetilde{C_c(U)} \subseteq M$ define $\Psi_2(\widetilde{f}) = \widetilde{f_{|_{X'}}}$, which is a linear and contractive map of $\widetilde{C_c(U)} \subseteq M$ to M' and so we may extend it to M. So we may define in an obvious manner $\Psi_3: C(X) \cup M \to \mathcal{T}(X', \alpha', L')$. It is easy to verify that Ψ_3 satisfies the relations that defines $\mathcal{T}(X, \alpha, L)$ and so Ψ_3 has an extension to $\mathcal{T}(X, \alpha, L)$, which will be denoted by Ψ_3. We will show that Ψ_3 is surjective. Given $h \in C_c(U')$, choose $g \in C_c(U)$ such that $g_{|_{\mathrm{supp}(h)}} = 1$ and $f \in C(X)$ such that $\Psi_3(f) = h$. Then $fg \in C_c(U)$ and $\Psi_3(f)\Psi_3(\widetilde{g}) = h\widetilde{g_{|_{X'}}} = \widetilde{hg_{|_{X'}}} = \widetilde{h}$. This shows that $\Psi_3(M)$ is dense in M', and with the fact that $C(X') \subseteq \mathrm{Im}(\Psi_3)$, it follows that Ψ_3 is surjective.

Claim. *If* (f, k) *is a redundancy of* $\mathcal{T}(X, \alpha, L)$ *and* $f \in I_0$ *then* $(\Psi_3(f), \Psi_3(k))$ *is a redundancy of* $\mathcal{T}(X', \alpha', L')$ *and* $\Psi_3(f) \in I_0'$.

Let (f, k) be a redundancy of $\mathcal{T}(X, \alpha, L)$ and $f \in I_0$. Then $fm = km$, from where $\Psi_3(f)\Psi_3(m) = \Psi_3(k)\Psi_3(m)$. Since $\Psi_3(f) \in C(X')$ and $\Psi_3(k) \in \widehat{K_1'}$ and moreover $\Psi_3(M)$ is dense in M' it follows that $(\Psi_3(f), \Psi_3(k))$ is a redundancy. Since $f \in I_0$, and $I_0 = C_0(U)$ by 2.6, it follows that $f \in C_0(U)$ and therefore $\Psi_3(f) = f_{|_{X'}} \in C_0(U') = I_0'$.

If q is the quotient map of $\mathcal{T}(X', \alpha', L')$ on $\mathcal{O}(X', \alpha', L')$ then the composition $q \circ \Psi_3$ is a *-homomorphism of $\mathcal{T}(X, \alpha, L)$ on $\mathcal{O}(X', \alpha', L')$ which by the claim above vanishes on the elements $(a - k)$ for all redundancies (a, k) such that $a \in I_0$. By passage to the quotient we obtain a *-homomorphism of $\mathcal{O}(X, \alpha, L)$ to $\mathcal{O}(X', \alpha', L')$ which will be denoted by Ψ_0. Moreover, given $f \in C_0(V)$ note that $\Psi_0(f) = f_{|_{X'}} = 0$, and again passing to the quotient we obtain an other *-homomorphism of $\mathcal{O}(X, \alpha, L)/\langle C_0(V)\rangle$ to $\mathcal{O}(X', \alpha', L')$, which will be called Ψ. It remains to show that Ψ is injective. Note that $\langle C_0(V)\rangle$ is gauge-invariant. Consider the gauge action on $\mathcal{O}(X, \alpha, L)/\langle C_0(V)\rangle$ whose fixed point algebra is

$$\overline{\overline{K}} = \overline{\bigcup_{n\in\mathbb{N}} \overline{\overline{L_n}}}$$

(see paragraph following 1.11) and the gauge action on $\mathcal{O}(X', \alpha', L')$. Since Ψ is covariant by these actions, by [5, 2.9] it is enough to show that Ψ restricted to

$\overline{\overline{K}}$ is injective. For this we will show that Ψ restricted to $\overline{\overline{L_n}}$ is injective for all $n \in \mathbb{N}$.

Claim 1. *Let $\overline{\overline{k_0}} + \overline{\overline{k_1}} + \cdots + \overline{\overline{k_n}} \in \overline{\overline{L_n}}$. If $\phi(\overline{\overline{k_0}} + \overline{\overline{k_1}} + \cdots + \overline{\overline{k_n}}) = 0$ then $\overline{\overline{k_0}} \in \overline{\overline{K_1}}$.*

Let $k_i' = \Psi(\overline{\overline{k_i}})$ and notice that $k_0' \in C(X')$ and $k_i' \in K_i'$ for $i \geq 1$. Then $k_0' + k_1' + \cdots + k_n' = 0$ and so $g(k_0' + k_1' \cdots + k_n') = 0$ for all $g \in C_0(U')$. By 2.4 it follows that $k_0' = f_1 + f_2$ where $f_1 \in C_0(U')$ and $k_0' + k_1' \cdots + k_n' = f_2$ from where $f_2 = 0$. Then $k_0' \in C_0(U')$ and so $k_0 \in C_0(U \cup V)$ from where

$$\overline{\overline{k_0}} \in \overline{\overline{C_0(U \cup V)}} = \overline{\overline{C_0(U)}} + \overline{\overline{C_0(V)}} \subseteq \overline{\overline{K_1}}\,.$$

Claim 2. Ψ *restricted to $\overline{\overline{C(X)}}$ is faithful, and also Ψ restricted to $\overline{\overline{K_n}}$ is faithful.*

If $f \in C(X)$ and $\Psi(\overline{\overline{f}}) = 0$ then $f \in C_0(V)$ and so $\overline{\overline{f}} = 0$. This shows the first part. To prove the second assertion let $\overline{\overline{k_n}} \in \overline{\overline{K_n}}$ and suppose that $\Psi(\overline{\overline{k_n}}) = 0$. Then $\Psi(\overline{\overline{M}}^{*n}\,\overline{\overline{k_n}}\,\overline{\overline{M}}^{n}) = 0$ and how $\overline{\overline{M}}^{*n}\,\overline{\overline{k_n}}\,\overline{\overline{M}}^{n} \subseteq \overline{\overline{C(X)}}$ and Ψ restricted to $\overline{\overline{C(X)}}$ is faithful it follows that $\overline{\overline{M}}^{*n}\,\overline{\overline{k_n}}\,\overline{\overline{M}}^{n} = 0$ from where $\overline{\overline{K_n}}\,\overline{\overline{k_n}}\,\overline{\overline{K_n}} = 0$ and so $\overline{\overline{k_n}} = 0$. □

We will prove now the following claim which will conclude the proof of the theorem.

Claim 3. *For all $n \in \mathbb{N}$, Ψ restricted to $\overline{\overline{L_n}}$ is faithful*

By claim 2 Ψ restricted to $\overline{\overline{L_0}}$ is faithful. By induction, suppose that Ψ restricted to $\overline{\overline{L_n}}$ is faithful, take $\overline{\overline{k_0}} + \overline{\overline{k_1}} + \cdots + \overline{\overline{k_{n+1}}} \in \overline{\overline{L_{n+1}}}$ and suppose that $\Psi(\overline{\overline{k_0}} + \overline{\overline{k_1}} + \cdots + \overline{\overline{k_{n+1}}}) = 0$. Then

$$\Psi(\overline{\overline{M}}^{*}(\overline{\overline{k_0}} + \overline{\overline{k_1}} + \cdots + \overline{\overline{k_{n+1}}})^{*}(\overline{\overline{k_0}} + \overline{\overline{k_1}} + \cdots + \overline{\overline{k_{n+1}}})\overline{\overline{M}}) = 0$$

and by the induction hypothesis,

$$\overline{\overline{M}}^{*}(\overline{\overline{k_0}} + \overline{\overline{k_1}} + \cdots + \overline{\overline{k_{n+1}}})^{*}(\overline{\overline{k_0}} + \overline{\overline{k_1}} + \cdots + \overline{\overline{k_{n+1}}})\overline{\overline{M}} = 0,$$

from where $(\overline{\overline{k_0}} + \overline{\overline{k_1}} + \cdots + \overline{\overline{k_{n+1}}})\,\overline{\overline{M}} = 0$ and so

$$(\overline{\overline{k_0}} + \overline{\overline{k_1}} + \cdots + \overline{\overline{k_{n+1}}})(\overline{\overline{K_1}} + \cdots + \overline{\overline{K_{n+1}}}) = 0\,.$$

By claim 1, $\overline{\overline{k_0}} \in \overline{\overline{K_1}}$, from where $\overline{\overline{k_0}} + \overline{\overline{k_1}} + \cdots + \overline{\overline{k_{n+1}}} \in (\overline{\overline{K_1}} + \cdots + \overline{\overline{K_{n+1}}})$ and therefore $\overline{\overline{k_0}} + \overline{\overline{k_1}} + \cdots + \overline{\overline{k_{n+1}}} = 0$. □

Given an ideal I in $\mathcal{O}(X,\alpha,L)$, the set $I\cap C(X)$ is an ideal of $C(X)$ and so it is of the form $C_0(V)$ for some open set $V\subseteq X$. The following proposition shows a feature of these open sets.

Proposition 3.4. *Let $I\trianglelefteq\mathcal{O}(X,\alpha,L)$ and $V\subseteq X$ the open set such that $I\cap C(X)=C_0(V)$. Then V is a σ,σ^{-1}-invariant set.*

Proof. Given $f\in C_c(U)\cap C_0(V)$, take $g\in C_c(U)$ such that $g_{|\mathrm{supp}(f)}=1$. Then $f\widetilde{g}=\widetilde{f}\in I$ and so $L(f)=\widetilde{g}^*\widetilde{f}\in I\cap C(X)=C_0(V)$. By 3.1 a) it follows that V is σ-invariant. We will show that V is a σ^{-1}-invariant set. Let x be an element of V and $y\in\sigma^{-1}(x)$. Choose $f_x\in C_0(V)$ such that $f_x(x)=1$ and $f_y\in C_c(U)$ such that $f_y(y)=1$ and $\sigma_{|\mathrm{supp}(f_y)}$ is a homeomorphism. Then $(f_y\alpha(f_x))\widetilde{}=\widetilde{f_y}\,f_x\in I\cap M$ and therefore $(f_y\alpha(f))\widetilde{}\;\widetilde{f_y}^{\,*}\in I$. By 2.3 a), $f_y\alpha(f_x)f_y^*=(f_y\alpha(f_x))\widetilde{}\;\widetilde{f_y}^{\,*}$ and so $f_y\alpha(f_x)f_y^*\in I\cap C(X)=C_0(V)$. Note that

$$(f_y\alpha(f_x)f_y^*)(y)=|f_y|^2(y)\,f_x(\sigma(y))=|f_y(y)|^2\,f_x(x)=1,$$

which shows that $y\in V$. □

This proposition shows that there exists a map

$$\Phi\colon\left\{\text{ideals of }\mathcal{O}(X,\alpha,L)\right\}\to\left\{\text{open }\sigma,\sigma^{-1}\text{-invariant sets of }X\right\}$$

given by $\Phi(I)=V$ where V is the open set of X such that $I\cap C(X)=C_0(V)$. The following proposition shows that Φ is surjective. To prove this proposition we need some lemmas.

Lemma 3.5. *Let V a σ-invariant set and $f_1,\cdots,f_n,g_1,\cdots,g_n\in C_c(U)$ such that $f_i\in C_0(V)$ or $g_i\in C_0(V)$ for some i. Then $\widetilde{f_n}^{\,*}\cdots\widetilde{f_1}^{\,*}\widetilde{g_1}\cdots\widetilde{g_n}\in C_0(V)$.*

Proof. Suppose $f_i\in C_0(V)$ and define $h_j=\widetilde{f_j}^{\,*}\cdots\widetilde{f_1}^{\,*}\widetilde{g_1}\cdots\widetilde{g_j}$ for $j\geq 1$ and $h_0=1$. Since $h_j\in C(X)$ for each j it follows that $f_i^*h_{i-1}g_i\in C_0(V)$. By 3.2 $C_0(V)$ is L-invariant, and so $h_i=\widetilde{f_i}^{\,*}h_{i-1}\widetilde{g_i}=L(f_i^*h_{i-1}g_i)\in C_0(V)$. By induction it may be showed that $h_n\in C_0(V)$. If $g_i\in C_0(V)$ the proof is analogous. □

To show that the map Φ is surjective we will show that if V is an open σ,σ^{-1}-invariant set then $\langle C_0(V)\rangle\cap C(X)=C_0(V)$. The following arguments are a preparation to prove this fact. Given $f\in\langle C_0(V)\rangle\cap C(X)$ and $\varepsilon>0$ then there are $a_i,b_i\in\mathcal{O}(X,\alpha,L)$, $h_i\in C_0(V)$ such that

$$\left\|f-\sum_{i=1}^{N}a_ih_ib_i\right\|\leq\varepsilon$$

where each a_i is of the form $a_i = m_1 \cdots m_{r_i} n_1^* \cdots n_{s_i}^*$ or $a_i \in C(X)$ and each b_i is of the form $b_i = p_1 \cdots p_{t_i} q_1^* \cdots q_{l_i}^*$ or $b_i \in C(X)$. Moreover we may suppose that $m_j = \widetilde{z_j}$, $n_j = \widetilde{w_j}$, $p_j = \widetilde{u_j}$, $q_j = \widetilde{v_j}$ for each m_j, n_j, p_j, and q_j. Considering the conditional expectation E induced by the gauge action and that

$$\left\| f - \sum_{i=1}^{N} E(a_i h_i b_i) \right\| = \left\| E\left(f - \sum_{i=1}^{N} a_i h_i b_i \right) \right\| \leq \varepsilon,$$

we may suppose that $r_i + t_i = s_i + l_i$, because

$$E(a_i h_i b_i) = \begin{cases} a_i h_i b_i & \text{if } r_i + t_i = s_i + l_i \\ 0 & \text{otherwise} \end{cases} .$$

Lemma 3.6. *Let V be an open σ, σ^{-1}-invariant set. Then for each i we have that $a_i h_i b_i \in C_0(V)$ or $a_i h_i b_i = \widetilde{f_1} \cdots \widetilde{f_n}\, \widetilde{g_n}^* \cdots \widetilde{g_1}^*$ where $f_j \in C_0(V)$ for some j or $g_j \in C_0(V)$ for some j.*

Proof. Recall that $a_i = \widetilde{z_1} \cdots \widetilde{z_{r_i}}\, \widetilde{w_1}^* \cdots \widetilde{w_{s_i}}^*$ or $a_i \in C(X)$, $b_i = \widetilde{u_1} \cdots \widetilde{u_{t_i}}\, \widetilde{v_1}^* \cdots \widetilde{v_{l_i}}^*$ or $b_i \in C(X)$ and $r_i + t_i = s_i + l_i$.

Suppose $s_i \leq t_i$. By 3.5 $w = \widetilde{w_1}^* \cdots \widetilde{w_{s_i}}^* h_i \widetilde{u_1} \cdots \widetilde{u_{s_i}} \in C_0(V)$ (if $s_i = 0$ then $w = h_i$). If $t_i \neq s_i$ then write $a_i h_i b_i = \widetilde{z_1} \cdots \widetilde{z_{r_i}}\, \widetilde{w u_{s_i+1}} \cdots \widetilde{u_{t_i}}\, \widetilde{v_1}^* \cdots \widetilde{v_{l_i}}^*$, and note that $w u_{s_i+1} \in C_0(V)$ and therefore $a_i h_i b_i$ is in the desired form. If $t_i = s_i$ then $r_i = l_i$. If $r_i = 0$ (and so $l_i = 0$) then $a_i h_i b_i = w \in C_0(V)$. If $r_i \neq 0$ write $a_i h_i b_i = \widetilde{z_1} \cdots \widetilde{z_{r_i} \alpha(w)}\, \widetilde{u_{s_i+1}} \cdots \widetilde{u_{t_i}}\, \widetilde{v_1}^* \cdots \widetilde{v_{l_i}}^*$, and in this case $z_{r_i} \alpha(w) \in C_0(V)$ by the fact that V is σ^{-1}-invariant, and so $a_i h_i b_i$ is in the desired form.

Supposing $s_i > t_i$ consider the element $(a_i h_i b_i)^*$, which is in the desired form of the lemma by the proof above, and therefore $a_i h_i b_i$ is also in the desired form. □

The following lemma is only a summary from 3.5 to 3.6.

Lemma 3.7. *If V is an open σ, σ^{-1}-invariant set then given $f \in \langle C_0(V) \rangle \cap C(X)$ and $\varepsilon > 0$, there exists $d_0 \in C_0(V)$ and $d_i = \widetilde{f_1^i} \cdots \widetilde{f_{n_i}^i}\, \widetilde{g_{n_i}^i}^* \cdots \widetilde{g_1^i}^*$, with $f_j^i \in C_0(V)$ or $g_j^i \in C_0(V)$ for some j, $i = 1, \cdots, N$, such that*

$$\left\| f - \left(d_0 + \sum_{i=1}^{N} d_i \right) \right\| \leq \varepsilon .$$

Now we prove the proposition which shows that the map Φ is surjective.

Proposition 3.8. *If $V \subseteq X$ is σ, σ^{-1}-invariant then $\langle C_0(V)\rangle \cap C(X) = C_0(V)$.*

Proof. It is clear that $C_0(V) \subseteq \langle C_0(V)\rangle \cap C(X)$. To show that $\langle C_0(V)\rangle \cap C(X) \subseteq C_0(V)$ we will show that given $f \in \langle C_0(V)\rangle \cap C(X)$, for every $\varepsilon > 0$ it holds that $|f(x)| \leq \varepsilon$ for each $x \notin V$.

Given $f \in \langle C_0(V)\rangle \cap C(X)$ and $\varepsilon > 0$, by 3.7 we may consider $\| f - (d_0 + \sum_{i=1}^N d_i)\| \leq \varepsilon$ with $d_0 \in C_0(V)$, $d_i = \widetilde{f_1^i} \cdots \widetilde{f_{n_i}^i}\, \widetilde{g_{n_i}^i}^* \cdots \widetilde{g_1^i}^*$ where $f_j^i \in C_0(V)$ for some j or $g_j^i \in C_0(V)$ for some j. Define

$$K = \bigcup_{i=1}^{N} \bigcup_{j=1}^{n_i} \big(\operatorname{supp}(f_j^i) \cup \operatorname{supp}(g_j^i)\big)$$

which is a compact subset of U.

Claim 1. *If $x \notin V$ and $x \notin U$ then $|f(x)| \leq \varepsilon$*

If $x \notin U$, choose $h \in C(X)$, $0 \leq h \leq 1$, such that $h(x) = 1$ e $h_{|_K} = 0$. Then $hd_i = 0$ for $i \geq 1$ and so $\|h(f - d_0)\| = \|h(f - d_0 + \sum_{i=1}^N d_i)\| \leq \varepsilon$ from where $|f(x) - d_0(x)| = |(h(f - d_0))(x)| \leq \varepsilon$. Since $x \notin V$ it follows that $d_0(x) = 0$ and therefore $|f(x)| \leq \varepsilon$.

Now we study the case $x \notin V$ and $x \in U$. Let $N_0 = \max\{n_1, \ldots, n_N\}$. Supposing $N_0 = 0$, that is, $d_i = 0$ fore each $i \geq 1$, we have that $|f(x)| = |f(x) - d_0(x)| \leq \varepsilon$. Suppose therefore that $N_0 \geq 1$. Let us analyse the case $\sigma^{N_0-1}(x) \in U$. Define $x_j = \sigma^j(x)$ for $j \in \{0, \ldots, N_0\}$. For each $j \in \{0, \ldots, N_0 - 1\}$ take $h_j \in C_c(U)$ such that $h_j(x_j) = 1$, $0 \leq h_j \leq 1$ and $\sigma_{\operatorname{supp}(h_j)}$ is a homeomorphism.

Claim 2. *For each $i \in \{0, \cdots, N\}$, $h_i' = \widetilde{h_{N_0-1}}^* \cdots \widetilde{h_0}^* d_i \widetilde{h_0} \cdots \widetilde{h_{N_0-1}} \in C_0(V)$.*

For $i \geq 1$, since $f_j^i \in C_0(V)$ or $g_j^i \in C_0(V)$ for some j, by 3.5 we have that

$$u = \widetilde{h_{n_i-1}}^* \cdots \widetilde{h_0}^* \widetilde{f_1^i} \cdots \widetilde{f_{n_i}^i} \in C_0(V) \quad \text{or}$$
$$v = \widetilde{g_{n_i}^i}^* \cdots \widetilde{g_1^i}^* \widetilde{h_0} \cdots \widetilde{h_{n_i-1}} \in C_0(V).$$

Then $uv \in C_0(V)$ and again by 3.5 it follows that

$$\begin{aligned} h_i' &= \widetilde{h_{N_0-1}}^* \cdots \widetilde{h_0}^* d_i \widetilde{h_0} \cdots \widetilde{h_{N_0-1}} \\ &= \widetilde{h_{N_0-1}}^* \cdots \widetilde{h_{n_i}}^* \widetilde{uvh_{n_i}}\, \widetilde{h_{n_i+1}} \cdots \widetilde{h_{N_0-1}} \in C_0(V). \end{aligned}$$

For $i = 0$, since $d_0 h_0 \in C_0(V)$, again by 3.5 $h_0' = \widetilde{h_{N_0-1}}^* \cdots \widetilde{h_0}^* d_0 \widetilde{h_0} \cdots \widetilde{h_{N_0-1}} \in C_0(V)$. This shows the claim. □

Define $f' = \widetilde{h_{N_0-1}}^* \cdots \widetilde{h_0}^* f \widetilde{h_0} \cdots \widetilde{h_{N_0-1}}$. By the fact that $\sigma_{|\text{supp}(h_j)}$ is a homeomorphism it follows that $f(x_{N_0}) = f(x)$. Moreover, since $x_{N_0} \notin V$, by the fact that V is σ^{-1}-invariant and $x \notin V$, it follows that $h_i'(x_{N_0}) = 0$ for each i. Since $f', h_i' \in C(X)$ we have that

$$\begin{aligned} \left\| f' - \left(h_0' + \sum_{i=1}^{n} h_i' \right) \right\|_\infty &= \left\| \widetilde{h_{N_0-1}}^* \cdots \widetilde{h_0}^* \left(f - \left(d_0 + \sum_{i=1}^{N} d_i \right) \right) \widetilde{h_0} \cdots \widetilde{h_{N_0-1}} \right\| \\ &\le \left\| f - \left(d_0 + \sum_{i=1}^{N} d_i \right) \right\| < \varepsilon, \end{aligned}$$

from where $|f(x)| = |(f' - (h_0' + \sum_{i=1}^{n} h_i'))(x_{N_0})| < \varepsilon$.

It remains to analyze the case $x \notin V$, $x \in U$ but $\sigma^n(x) \notin U$ for some $n \le N_0 - 1$. For $i \in \{0, \cdots .n-2\}$ define h_j as above, that is, $h_j \in C_c(U)$ such that $h_j(x_1) = 1$, $0 \le h_j \le 1$ and $\sigma_{|\text{supp}(h_j)}$ is a homeomorphism. For x_{n-1} choose $h_{n-1} \in C_c(U)$ such that $0 \le h_{n-1} \le 1$, $h_{n-1}(x_{n-1}) = 1$, $\sigma_{|\text{supp}(h_{n-1})}$ is a homeomorphism and $\sigma(\text{supp}(h_{n-1})) \subseteq X \setminus K$. It is possible to choose such h_{n-1} because $\sigma(x_{n-1}) = \sigma^n(x) \in X \setminus U \subseteq X \setminus K$.

Claim 3. *For $n_i \ge n+1$, $\widetilde{h_{n-1}}^* \cdots \widetilde{h_0}^* d_i \widetilde{h_0} \cdots \widetilde{h_{n-1}} = 0$.*

Denote by u the element $\widetilde{h_{n-2}}^* \cdots \widetilde{h_0}^* \widetilde{f_1^i} \cdots \widetilde{f_{n-1}^i}$ which is an element of $C(X)$. Then

$$\widetilde{h_{n-1}}^* \cdots \widetilde{h_0}^* \widetilde{f_1^i} \cdots \widetilde{f_{n+1}^i} = \widetilde{h_{n-1}}^* \widetilde{u f_n^i}\, \widetilde{f_{n+1}^i} = (L(h_{n-1}^* u f_n^i) f_{n+1}^i)^{\sim}.$$

We will show that $L(h_{n-1}^* u f_n^i) f_{n+1}^i = 0$. If $x \notin \text{supp}(f_{n+1}^i)$ or if $\sigma^{-1}(x) = \emptyset$ then $(L(h_{n-1}^* u f_n^i) f_{n+1}^i)(x) = 0$. Suppose therefore $x \in \text{supp}(f_{n+1})$ and $y \in \sigma^{-1}(x)$. Supposing that $y \in \sigma^{-1}(x) \cap \text{supp}(h_{n-1})$ we have that $x = \sigma(y) \in \sigma(\text{supp}(h_{n-1})) \subseteq X \setminus K$, which is an absurd because $x \in K$. Therefore if $y \in \sigma^{-1}(x)$ then $y \notin \text{supp}(h_{n-1})$, and by this way $L(h_{n-1}^* u f_n^i)(x) = \sum_{y \in \sigma^{-1}(x)} (h_{n-1}^* u f_n^i)(y) = 0$. So $L(h_{n-1}^* u f_n^i) f_{n+1}^i = 0$ and the claim is proved.

Claim 4. *For $n_i \leq n$, $h_i' = \widetilde{h_{n-1}}^* \cdots \widetilde{h_0}^* d_i \widetilde{h_0} \cdots \widetilde{h_{n-1}} \in C_0(V)$.*

The proof of this claim is analogous to the proof of claim 2.

Again $\widetilde{h_{n-1}}^* \cdots \widetilde{h_0}^* f \widetilde{h_0} \cdots \widetilde{h_{n-1}} = f'$ com $f'(x_n) = f(x)$. Moreover, by the fact that $x_n \notin V$ it follows that $h_i'(x_n) = 0$ for each i. Then

$$\left\| f' - h_0' - \sum_{n_i \leq n} h_i' \right\| = \left\| \widetilde{h_{n-1}}^* \cdots \widetilde{h_0}^* \left(f - (d_0 + \sum_{i=1}^{N} d_i) \right) \widetilde{h_0} \cdots \widetilde{h_{n-1}} \right\| < \varepsilon$$

from where $|f(x)| = |(f' - h_0' - \sum_{n_i \leq n} h_i')(x_n)| < \varepsilon$.

In this way, given $\varepsilon > 0$, for all $x \notin V$, we have that $|f(x)| \leq \varepsilon$. Therefore $f \in C_0(V)$. □

The following theorem is the main result of this section.

Theorem 3.9. *There exists a bijection between the gauge-invariant ideals of $\mathcal{O}(X, \alpha, L)$ and the open σ, σ^{-1}-invariant subsets of X.*

Proof. All what we have to do is to show that the map

$$\Phi\colon \{\text{gauge invariant ideals of } \mathcal{O}(X, \alpha, L)\} \to \{\text{open } \sigma, \sigma^{-1}\text{-invariant subsets of } X\},$$

given by $\Phi(I) = V$ where V is the open subset of X such that $I \cap C(X) = C_0(V)$, is bijective. By the previous proposition Φ is surjective. It remains to show that Φ is injective. For this, given $I \trianglelefteq \mathcal{O}(X, \alpha, L)$ gauge-invariant, let $V \subseteq X$ the open subset σ, σ^{-1}-invariant such that $I \cap C(X) = C_0(V)$. We will show that $\langle C_0(V) \rangle = I$. It is clear that $\langle C_0(V) \rangle \subseteq I$. By 3.3 there exists a *-isomorphism

$$\Psi\colon \frac{\mathcal{O}(X, \alpha, L)}{\langle C_0(V) \rangle} \to \mathcal{O}(X', \alpha', L')$$

where $X' = X \setminus V$. Let $\overline{\overline{I}}$ the image of I by the quotient map of $\mathcal{O}(X, \alpha, L)$ on $\mathcal{O}(X, \alpha, L)/\langle C_0(V) \rangle$. Since $\overline{\overline{I}}$ is gauge-invariant and Ψ is covariant by the gauge actions we have that $\Psi(\overline{\overline{I}})$ is gauge-invariant. Supposing $\overline{\overline{I}} \neq 0$, and so $\Psi(\overline{\overline{I}}) \neq 0$, it follows that $\Psi(\overline{\overline{I}}) \cap C(X') = C_0(V') \neq 0$ by 2.8. Let $0 \neq g \in C_0(V')$. Then $g = \Psi(\overline{\overline{f}})$ for some $f \in C(X)$ and $g = \Psi(\overline{\overline{a}})$ with $a \in I$. Therefore $\Psi(\overline{\overline{f}}) = g = \Psi(\overline{\overline{a}})$ from where $\overline{\overline{f}} = \overline{\overline{a}}$. In this way, $f - a \in \langle C_0(V) \rangle \subseteq I$ and so $f \in I$. It follows that $f \in I \cap C(X) = C_0(V)$, that is, $g = \Psi(\overline{\overline{f}}) = 0$, which is an absurd. Therefore $\overline{\overline{I}} = 0$ and this shows that $I = \langle C_0(V) \rangle$. □

Notice that we have showed that every gauge-invariant idel I of $\mathcal{O}(X, \alpha, L)$ is of the form $\langle C_0(V)\rangle$ where V is the σ, σ^{-1}-invariant open subset such that $I \cap C(X) = C_0(V)$. By this theorem we have the following non simplicity criteria of $\mathcal{O}(X, \alpha, L)$:

Corollary 3.10. *If U is nonempty and $U \cup \sigma(U)$ is not dense in X then $\mathcal{O}(X, \alpha, L)$ has at least one gauge-invariant nontrivial ideal.*

Proof. Note that $V = X \setminus \overline{U \cup \sigma(U)}$ is an open σ, σ^{-1}-invariant set. Since $U \cup \sigma(U)$ is not dense in X it follows that V is nonempty. Then $\langle C_0(V)\rangle$ is a nonzero gauge-invariant ideal of $\mathcal{O}(X, \alpha, L)$. By the previous theorem, supposing $\langle C_0(V)\rangle = \mathcal{O}(X, \alpha, L)$ we have that $C_0(V) = C(X)$, which is a contradiction, because $V \neq X$, by the fact that U is nonempty. □

4 Topologically free transformations

In this section we prove that under certain hypothesis about X, every ideal of $\mathcal{O}(X, \alpha, L)$ has nonzero intersection with $C(X)$ and based on this fact we show a relationship between the ideals of $\mathcal{O}(X, \alpha, L)$ and the σ, σ^{-1}-invariant open subsets of X. Also we show a simplicity criteria for the Cuntz-Krieger algebras for infinite matrices.

4.1 The theorem of intersection of ideals of $\mathcal{O}(X, \alpha, L)$ with $C(X)$

Let us begin with the lemma:

Lemma 4.1.

a) *For each $f \in C_c(U)$,* $\operatorname{supp}(L(f)) \subseteq \sigma(\operatorname{supp}(f))$.

b) *Let $h, f_1, \ldots, f_n, g_1, \ldots, g_n$ be elements of $C_c(U)$ such that σ^{n-1}* $(\operatorname{supp}(h)) \subseteq U$. *Then* $\operatorname{supp}(\widetilde{f_k}^* \cdots \widetilde{f_1}^* h \widetilde{g_1} \cdots \widetilde{g_k}) \subseteq \sigma^k(\operatorname{supp}(h))$ *for each $k \in \{0, \ldots, n\}$.*

Proof.

a) The proof of this fact is similar to the proof given in [6, 8.7], although our context is a little different. Let $x \in X$ with $L(f)(x) \neq 0$. Suppose $x \notin \sigma(\operatorname{supp}(f))$. Choose $g \in C(X)$ such that $g(x) = 1$ and $g_{|\sigma(\operatorname{supp}(f))} = 0$. If

$y \in \text{supp}\,(f)$ then $\alpha(g)(y) = g(\sigma(y)) = 0$ because $\sigma(y) \in \sigma(\text{supp}\,(f))$. This shows that $f\alpha(g) = 0$. So we have

$$0 \neq L(f)(x) = L(f)(x)g(x) = (L(f)g)(x) = L(f\alpha(g)) = 0,$$

which is an absurd. Therefore $x \in \sigma(\text{supp}\,(f))$.

b) By a) we have that

$$\text{supp}\,(\widetilde{f_1}^{\,*} h \widetilde{g_1}) = \text{supp}\,(L(f_1^* h g_1)) \subseteq \sigma(\text{supp}\,(f_1^* h g_1)),$$

and it is clear that $\sigma(\text{supp}\,(f_1^* h g_1)) \subseteq \sigma(\text{supp}\,(h))$. Suppose that

$$\text{supp}\,(\widetilde{f_{k-1}}^{\,*} \cdots \widetilde{f_1}^{\,*} h \widetilde{g_1} \cdots \widetilde{g_{k-1}}) \subseteq \sigma^{k-1}(\text{supp}\,(h)) \text{ for } 2 \leq k \leq n.$$

Then, by placing $g = \widetilde{f_{k-1}}^{\,*} \cdots \widetilde{f_1}^{\,*} h \widetilde{g_1} \cdots \widetilde{g_{k-1}}$, by a) we have that

$$\text{supp}\,(\widetilde{f_k}^{\,*} g \widetilde{g_k}) = \text{supp}\,(L(f_k^* g g_k)) \subseteq \sigma(\text{supp}\,(f_k^* g g_k)).$$

Since $\text{supp}\,(f_k^* g g_k) \subseteq \text{supp}\,(g)$, and by the induction hypothesis $\text{supp}\,(g) \subseteq \sigma^{k-1}(\text{supp}\,(h))$, it follows that $\text{supp}\,(f_k^* g g_k) \subseteq \sigma^{k-1}(\text{supp}\,(h))$. By hypothesis we have that $\sigma^{k-1}(\text{supp}\,(h)) \subseteq U$ and so $\sigma(\text{supp}\,(f_k^* g g_k)) \subseteq \sigma^k(\text{supp}\,(h))$. $\square$

For each $i \neq j$ in $\mathbb{N}$ define

$$V^{i,j} = \{x \in X : \sigma^i(x) = \sigma^j(x)\}.$$

Note that for $x \in X$ to be an element of $V^{i,j}$ it is necessary that x lies in $\text{dom}(\sigma^i) \cap \text{dom}(\sigma^j)$.

Lemma 4.2. *If $f_1, \cdots f_i, g_1, \cdots, g_j \in C_c(U)$ with $i \neq j$ then for each $x \notin V^{i,j}$ there exists $h \in C(X)$ such that $0 \leq h \leq 1$, $h(x) = 1$, and $h\widetilde{f_1} \cdots \widetilde{f_i}\, \widetilde{g_j}^{\,*} .. \widetilde{g_1}^{\,*} h = 0$.*

Proof. By taking adjoints we may suppose that $i > j$, and so $i > 0$. Define the set

$$K = \left(\bigcup_{r=1}^{i} \text{supp}\,(f_r) \right) \left(\bigcup_{s=1}^{j} \text{supp}\,(g_s) \right)$$

which is a compact subset of U. If $x \notin U$, take $h \in C(X)$, $0 \leq h \leq 1$, $h(x) = 1$ and $h_{|K} = 0$. Then $hf_1 = 0$, which proves the lemma in this case. So we may suppose that $x \in U$. We will consider two cases: the first when

$x \notin \mathrm{dom}(\sigma^i)$ and the second when $x \in \mathrm{dom}(\sigma^i)$. Suppose $x \notin \mathrm{dom}(\sigma^i)$. Then there exists $1 \leq k \leq i-1$ such that $\sigma^k(x) \notin U$ (note that $i \geq 2$ because $x \in U = \mathrm{dom}(\sigma)$). So $\sigma^k(x) \notin K$. Take $V_0 \subseteq X$ an open subset with $\sigma^k(x) \in V_0$ and $V_0 \cap K = \emptyset$. Then $V = \sigma^{-k}(V_0) \ni x$ is an open subset in U. Choose $h \in C_c(U)$ with $\mathrm{supp}\,(h) \subseteq V$, $0 \leq h \leq 1$ and $h(x) = 1$. Then, since $\sigma^{k-1}(\mathrm{supp}\,(h^2)) \subseteq \sigma^{k-1}(V) \subseteq U$, by 4.1 b),

$$\mathrm{supp}\,(\widetilde{f_k}^{\,*} \cdots \widetilde{f_1}^{\,*} h^2 \widetilde{f_1} \cdots \widetilde{f_k}) \subseteq \sigma^k(\mathrm{supp}\,(h^2)) \subseteq \sigma^k(V) \subseteq V_0.$$

Since $V_0 \cap K = \emptyset$ and $\mathrm{supp}\,(f_{k+1}) \subseteq K$ we have that

$$(\widetilde{f_k}^{\,*} \cdots \widetilde{f_1}^{\,*} h^2 \widetilde{f_1} \cdots \widetilde{f_k})\, \widetilde{f_{k+1}} = 0$$

from where $h \widetilde{f_1} \cdots \widetilde{f_{k+1}} \cdots \widetilde{f_i} = 0$. Therefore $h \widetilde{f_1} \cdots \widetilde{f_i}\, \widetilde{g_j}^{\,*} \cdots \widetilde{g_1}^{\,*} h = 0$. It remains to show the case $x \in \mathrm{dom}(\sigma^i)$. By the fact that $i > j$ it follows that $x \in \mathrm{dom}(\sigma^j)$. Therefore, since $x \notin V^{i,j}$ we have that $\sigma^i(x) \neq \sigma^j(x)$. Let $V_i \ni \sigma^i(x)$ and $V_j \ni \sigma^j(x)$ open subsets such that $V_i \cap V_j = \emptyset$. Let $V = \sigma^{-i}(V_i) \cap \sigma^{-j}(V_j)$ and note that V is an open subset which contains x. Take $h \in C_c(U)$ with $0 \leq h \leq 1$, $h(x) = 1$ and $\mathrm{supp}\,(h) \subseteq V$. Then, since $\sigma^{i-1}(V) \subseteq U$ and $\sigma^{j-1}(V) \subseteq U$, by 4.1 b) we have that

$$\mathrm{supp}\,(\widetilde{f_i}^{\,*} \cdots \widetilde{f_1}^{\,*} h^2 \widetilde{f_1} \cdots \widetilde{f_i}) \subseteq \sigma^i(\mathrm{supp}\,(h^2)) \subseteq V_i$$

and

$$\mathrm{supp}\,(\widetilde{g_j}^{\,*} \cdots \widetilde{g_1}^{\,*} h^2 \widetilde{g_1} \cdots \widetilde{g_j}) \subseteq \sigma^j(\mathrm{supp}\,(h^2)) \subseteq V_j.$$

Since V_i and V_j are disjoints it follows that

$$(\widetilde{f_i}^{\,*} \cdots \widetilde{f_1}^{\,*} h^2 \widetilde{f_1} \cdots \widetilde{f_i})(\widetilde{g_j}^{\,*} \cdots \widetilde{g_1}^{\,*} h^2 \widetilde{g_1} \cdots \widetilde{g_j}) = 0,$$

from where $h \widetilde{f_1} \cdots \widetilde{f_i}\, \widetilde{g_j}^{\,*} \cdots \widetilde{g_1}^{\,*} h = 0$. □

Definition 4.3. *We say that the pair (X, σ) is topologically free if for each $V^{i,j}$, the closure $\overline{V^{i,j}}$ in X has empty interior.*

By the Baire's theorem, X is topologically free if $\bigcup_{i,j \in \mathbb{N}} \overline{V^{i,j}}$ has empty interior. In this way, $Y = X \setminus \bigcup_{i,j \in \mathbb{N}} \overline{V^{i,j}}$ is dense in X.

Let S be the set of positive linear functionals of $\mathcal{O}(X, \alpha, L)$ given by

$$S = \{\varphi\colon\ \varphi \text{ is a positive linear functional and } \varphi_{|C(X)} = \delta_y \text{ for some } y \in Y\}$$

where $\delta_y(f) = f(y)$ for each $f \in C(X)$. We don't know the characteristic of these functionals, nevertheless for $a \in \mathcal{O}(X, \alpha, L)$ and $f \in C(X)$ it holds the following relation:

Lemma 4.4. *If φ is a positive linear functional of $\mathcal{O}(X, \alpha, L)$ such that $\varphi_{|C(X)} = \delta_x$ for some $x \in X$ then for each $f \in C(X)$ and $a \in \mathcal{O}(X, \alpha, L)$ we have that $\varphi(fa) = \varphi(f)\varphi(a)$ and $\varphi(af) = \varphi(a)\varphi(f)$.*

Proof. By taking adjoints it suffices to prove the case $\varphi(af) = \varphi(a)\varphi(f)$. For each $b \in \mathcal{O}(X, \alpha, L)$ we have that $(b - \varphi(b))^*(b - \varphi(b)) \geq 0$. Therefore if φ is a positive functional then $\varphi(b^*b) - \varphi(b^*)\varphi(b) = \varphi((b - \varphi(b))^*(b - \varphi(b))) \geq 0$, from where $\varphi(b)^*\varphi(b) \leq \varphi(b^*b)$. Since $f^*a^*af \leq f^*f\|a\|^2$ it follows that $\varphi(f^*a^*af) \leq \varphi(f^*f)\|a\|^2$. Put $b = af$, and so $0 \leq \varphi(af)^*\varphi(af) \leq \varphi(f^*a^*af) \leq \varphi(f^*f)\|a\|^2 = \|a\|^2|f(x)|^2$, where x is such that $\varphi_{|C(X)} = \delta_x$. This shows that if $f(x) = 0$ then $\varphi(af) = 0$. Define $g = f - f(x)$. Then $g(x) = 0$ and so $\varphi(ag) = 0$. By this way

$$\begin{aligned}\varphi(af) - \varphi(a)\varphi(f) &= \varphi(af) - \varphi(a)f(x) = \varphi(af) - \varphi(af(x)) \\ &= \varphi(a(f - f(x))) = \varphi(ag) = 0\end{aligned}$$

and the lemma is proved. □

For each $a \in \mathcal{O}(X, \alpha, L)$ define

$$|||a||| = \sup\{|\varphi(a)| : \varphi \in S\}$$

which is a seminorm for $\mathcal{O}(X, \alpha, L)$.

We are not able to show that $|||\ |||$ is nondegenerated in $\mathcal{O}(X, \alpha, L)$, but in L_n $|||\ |||$ has the property, given by the following lemma, that $|||r||| \neq 0$ for every positive nonzero element of L_n, remembering that $L_n = C(X) + K_1 + \cdots + K_n$ for each $n \geq 1$ and $L_0 = C(X)$.

Lemma 4.5. *Let (X, σ) be topologically free. For each $r \in L_n$ with $r \geq 0$ and $r \neq 0$ it holds that $|||r||| \neq 0$.*

Proof.

Claim 1. *If $0 \neq r \in L_n$, r positive and $r \notin C(X)$ then there exists $g \in C_c(U)$ with $\sigma_{|\mathrm{supp}(g)}$ a homeomorphism and $\widetilde{g}^*r\widetilde{g} \neq 0$.*

Since $r \geq 0$ we may write $r = b^*b$ with $b \in L_n$. Suppose that for each $g \in C_c(U)$ with $\sigma_{|\mathrm{supp}(g)}$ homeomorphism, it holds that $\widetilde{g}^*r\widetilde{g} = 0$, and so

$\widetilde{g}^* b^* = 0$. Then (making use of partition of unity we may write each $f \in C_c(U)$ as a sum of g as above) we have that $\widetilde{f}^* b^* = 0$ for each $f \in C_c(U)$ and so $M^* b^* = 0$. It follows that $K_1 b^* = 0$, and since $C_0(U) \subseteq K_1$ by 2.3 b) we have that $C_0(U) b^* = 0$ and by 2.4 b) it follows that $b^* \in C(X)$. In this way $r = b^* b \in C(X)$, which contradicts the hypothesis and the claim is proved.

Claim 2. *If* $0 \neq r \in L_n$, $r \geq 0$ *and* $r \notin C(X)$ *then there exists* $g_1, \ldots,$ $g_i \in C_c(U)$ *such that* $\sigma_{|\mathrm{supp}\,(g_j)}$ *is a homeomorphism for each* j *and* $0 \neq \widetilde{g_i}^* \cdots \widetilde{g_1}^* r \widetilde{g_1} \cdots \widetilde{g_i} \in C(X)$.

By Claim 1 there exists $g_1 \in C_c(U)$ such that $\sigma_{|\mathrm{supp}(g_1)}$ is homeomorphism and $0 \neq \widetilde{g_1}^* r \widetilde{g_1}$. Note that $\widetilde{g_1}^* r \widetilde{g_1} \in L_{n-1}$. By induction suppose $0 \neq \widetilde{g_l}^* \cdots \widetilde{g_1}^* r \widetilde{g_1} \cdots \widetilde{g_l} \in L_1$ where $g_j \in C_c(U)$ and $\sigma_{|\mathrm{supp}(g_j)}$ is a homeomorphism for each j. Then, by Claim 1, or $\widetilde{g_l}^* \cdots \widetilde{g_1}^* r \widetilde{g_1} \cdots \widetilde{g_l} \in C(X)$ or there exists $g_{l+1} \in C_c(U)$ with $\sigma_{|\mathrm{supp}(g_{l+1})}$ homeomorphims and $0 \neq \widetilde{g_{l+1}}^* \widetilde{g_l}^* \cdots \widetilde{g_1}^* r \widetilde{g_1} \cdots \widetilde{g_l}\, \widetilde{g_{l+1}}$. Since $\widetilde{g_{l+1}}\, \widetilde{g_l}^* \cdots \widetilde{g_1}^* r \widetilde{g_1} \cdots \widetilde{g_l}\, \widetilde{g_{l+1}} \in C(X)$ the claim is proved.

We will now show the lemma. Let $r \in L_n$, r positive and no null. It is enough to show that there exists $\varphi \in S$ such that $\varphi(r) \neq 0$. Since (X, σ) is topologically free then

$$Y = \left(X \setminus \bigcup_{i,j} \overline{V^{i,j}} \right)$$

is dense in X. So, if $r \in C(X)$ then there exists $y \in Y$ such that $r(y) > 0$. Take φ which extends δ_y, and therefore $\varphi(r) \neq 0$. Suppose $r \notin C(X)$. Choose $f_{x_1}, \cdots, f_{x_i} \in C_c(U)$ as in Claim 2. Then $0 \neq h = \widetilde{f_{x_i}}^* \cdots \widetilde{f_{x_1}}^* r \widetilde{f_{x_1}} \cdots \widetilde{f_{x_i}} \in C(X)$. So

$$h^* h h^* = \widetilde{f_{x_i}}^* \cdots \widetilde{f_{x_1}}^* r \widetilde{f_{x_1}} \cdots \widetilde{f_{x_i}}\, h \widetilde{f_{x_i}}^* \cdots \widetilde{f_{x_1}}^* r \widetilde{f_{x_1}} \cdots \widetilde{f_{x_i}} \neq 0$$

from where $g = \widetilde{f_{x_1}} \cdots \widetilde{f_{x_i}}\, h \widetilde{f_{x_i}}^* \cdots \widetilde{f_{x_1}}^* \neq 0$. How $\sigma_{|\mathrm{supp}\,(f_{x_i})}$ is homeomorphism it follows by 2.3 a) that $\widetilde{f_{x_i}}\, h \widetilde{f_{x_i}}^* \in C(X)$. Applying these arguments sucessively it may be proved that $g = \widetilde{f_{x_1}} \cdots \widetilde{f_{x_i}}\, h \widetilde{f_{x_i}}^* \cdots \widetilde{f_{x_1}}^* \in C(X)$. By the the same argments it follows that $u = \widetilde{f_{x_1}} \cdots \widetilde{f_{x_i}}\, \widetilde{f_{x_i}}^* \cdots \widetilde{f_{x_1}}^* \in C(X)$. Since $g \neq 0$ there exists $y \in Y$ such that $g(y) \neq 0$. Take $\varphi \in S$ which extends δ_y. Then we have that $\varphi(g) = g(y) \neq 0$. By 4.4, since $g = uru$, $\varphi(g) = \varphi(uru) = \varphi(u)\varphi(r)\varphi(u)$ and therefore $\varphi(r) \neq 0$. □

Now we are able to prove the main result of this section.

Theorem 4.6. *If* (X, σ) *is topologically free then each nonzero ideal of* $\mathcal{O}(X, \alpha, L)$ *has nonzero intersection with* $C(X)$.

Proof. By 2.7 it suffices to prove that every nonzero ideal of $\mathcal{O}(X, \alpha, L)$ has nonzero intersection with K. Let $0 \neq I \trianglelefteq \mathcal{O}(X, \alpha, L)$. Suppose $I \cap K = 0$. Then the quotient *-homomorphism $\pi : \mathcal{O}(X, \alpha, L) \to \mathcal{O}(X, \alpha, L)/I$ is such that $\pi_{|_K}$ is an isometry.

Claim. *For each* $b \in \mathcal{O}(X, \alpha, L)$ *it holds that* $|||E(b)||| \leq \|\pi(b)\|$ *where* E *is the conditional expectation defined in section* 1.2.

Let a be of the form

$$a = \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} a_{i,j}$$

with $a_{0,0} \in C(X)$ and $a_{i,j} \in M^i M^{j*}$ for $i \neq 0$ or $j \neq 0$, $a_{i,j} = \sum_{1 \leq k \leq n_{i,j}} a^k_{i,j}$, $a^k_{i,j} = \widetilde{f^k_{i,j,1}} \cdots \widetilde{f^k_{i,j,i}}\, \widetilde{g^k_{i,j,1}}^* \cdots \widetilde{g^k_{i,j,j}}^*$ where $f^k_{i,j,l}, g^k_{i,j,t} \in C_c(U)$ for each i, j, k, l and t. Given $\varepsilon > 0$ there exists $\varphi \in S$ which extends δ_y for some $y \in Y$ such that $|||E(a)||| - \varepsilon \leq |\varphi(E(a))|$. Note that $y \notin V^{i,j}$ for $i \neq j$. Then, for every $a^k_{i,j}$ with $i \neq j$, by 4.2 there exists $h^k_{i,j} \in C(X)$, $0 \leq h^k_{i,j} \leq 1$, such that $h^k_{i,j}(y) = 1$ and $ha^k_{i,j}h = 0$. Define

$$h = \prod_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} \prod_{1 \leq k \leq n_{i,j}} h^k_{i,j}\,.$$

Then $ha_{i,j}h = 0$ for each $i \neq j$ from where $hah = hE(a)h$, and moreover $h(y) = 1$. By 4.4 $\varphi(hE(a)h) = \varphi(h)\varphi(E(a))\varphi(h) = h(y)\varphi(E(a))h(y) = \varphi(E(a))$, and so $\varphi(E(a)) = \varphi(hE(a)h) = \varphi(hah)$. Since $hah = hE(a)h \in K$ e $\pi_{|_K}$ is an isometry it follows that $\|hah\| = \|\pi(aha)\|$. Then

$$|||E(a)||| - \varepsilon \leq |\varphi(E(a))| = |\varphi(hah)| \leq \|hah\| = \|\pi(hah)\| \leq \|\pi(a)\|.$$

Since ε is arbitrary it follows that $|||E(a)||| \leq \|\pi(a)\|$ for a in this form. Given $b \in \mathcal{O}(X, \alpha, L)$, for each $\varepsilon > 0$ choose $a \in \mathcal{O}(X, \alpha, L)$ as above such that $\|a - b\| \leq \varepsilon$. Then

$$\begin{aligned} |||E(b)||| &\leq |||E(b-a)||| + |||E(a)||| \leq |||E(a)||| + \varepsilon \leq \|\pi(a)\| + \varepsilon \\ &\leq \|\pi(a-b)\| + \|\pi(b)\| + \varepsilon \leq \|\pi(b)\| + 2\varepsilon. \end{aligned}$$

Again, since ε is arbitrary it follows that $|||E(b)||| \leq \|\pi(b)\|$, and the claim is proved. □

Observe that $\overline{E(I)}$ is a closed ideal of K. Also, $\overline{E(I)}$ is nonzero, because $0 \neq I$ and E is faithful. Then $\overline{E(I)} \cap L_n \neq 0$ for some n (see [2, III.4.1]). Let $0 \neq c \in \overline{E(I)} \cap L_n$. Then, since $c^*c \in L_n$ and c^*c is positive and nonzero it follows by 4.5 that $|||c^*c||| \neq 0$. We shall prove that $|||c^*c||| = 0$, and this will be an absurd. For each $a = E(b) \in E(I)$ with $b \in I$ we have that

$$|||a^*a||| = |||E(b^*)E(b)||| = |||E(b^*E(b))||| \leq \|\pi(b^*E(b))\|.$$

By the fact that $b^*E(b) \in I$ it follows that $\pi(b^*(E(b))) = 0$ and so $|||a^*a||| = 0$. This shows that $|||a^*a||| = 0$ for each $a \in E(I)$. Given $\varepsilon > 0$, take $a \in E(I)$ such that $\|a^*a - c^*c\| \leq \varepsilon$. Then

$$|||c^*c||| \leq |||c^*c - a^*a||| + |||a^*a||| = |||c^*c - a^*a||| \leq \|c^*c - a^*a\| \leq \varepsilon.$$

So $|||c^*c||| \leq \varepsilon$ for each $\varepsilon > 0$ from where $|||c^*c||| = 0$, and that is an absurd. Therefore $I \cap K \neq 0$, and the theorem is proved. □

4.2 Relationship between the ideals of $\mathcal{O}(X, \alpha, L)$ and the σ, σ^{-1}-invariant open subsets of X

We obtain here a relationship between the ideals of $\mathcal{O}(X, \alpha, L)$ and the σ, σ^{-1}-invariant open subsets of X under an additional hypothesis about (X, σ), which is that for every closed σ, σ^{-1}-invariant subset X' of X, $(X', \sigma_{|_{X'}})$ is topologically free.

Proposition 4.7. *Let I be an ideal of $\mathcal{O}(X, \alpha, L)$ and $V \subseteq X$ the open subset such that $I \cap C(X) = C_0(V)$. If $(X', \sigma_{|_{X'}})$ is topologically free (where $X' = X \setminus V$) then $I = \langle C_0(V) \rangle$.*

Proof. By 3.4 V is σ, σ^{-1}-invariant, from where X' is also σ, σ^{-1}-invariant. By 3.3 there exists a *-isomorphism

$$\Psi\colon \frac{\mathcal{O}(X, \alpha, L)}{\langle C_0(V) \rangle} \to \mathcal{O}(X', \alpha', L').$$

Obviously $\langle C_0(V) \rangle \subseteq I$. Suppose $I \neq \langle C_0(V) \rangle$. Then $\overline{\overline{I}} \neq 0$ and so $\Psi(\overline{\overline{I}}) \neq 0$. By 4.6, $\Psi(\overline{\overline{I}}) \cap C(X') \neq 0$. Let $0 \neq g \in \Psi(\overline{\overline{I}}) \cap C(X')$. Then $g = \Psi(\overline{\overline{a}})$ for some $a \in I$ and also $g = \Psi(\overline{\overline{f}})$, because $\Psi(\overline{\overline{C(X)}}) = C(X')$. Therefore $\Psi(\overline{\overline{a}}) = \Psi(\overline{\overline{f}})$ from where $\overline{\overline{a}} = \overline{\overline{f}}$ and so $f - a \in \langle C_0(V) \rangle \subseteq I$, in other words, $f \in I$. In this way $f \in I \cap C(X) = C_0(V)$ and so $\overline{\overline{f}} = 0$ from where $g = \Psi(\overline{\overline{f}}) = 0$, which is a absurd. So we conclude that $I = \langle C_0(V) \rangle$. □

Theorem 4.8. *If (X, σ) is such that $(X', \sigma_{|_{X'}})$ is topologically free for every closed subset σ, σ^{-1}-invariant X' of X then every ideal of $\mathcal{O}(X, \alpha, L)$ is of the form $\langle C_0(V)\rangle$ for some open subset $V \subseteq X$. Moreover, the map $V \longrightarrow \langle C_0(V)\rangle$ is a bijection between the open σ, σ^{-1}-invariante subsets of X and the ideals of $\mathcal{O}(X, \alpha, L)$.*

Proof. Let $I \trianglelefteq \mathcal{O}(X, \alpha, L)$, and $C_0(V) = I \cap C(X)$. By 3.4 V is σ, σ^{-1}-invariant, from where $X' = X \setminus V$ is also σ, σ^{-1}-invariant. By hypothesis $(X', \sigma_{|_{X'}})$ is topologically free. By 4.7, $I = \langle C_0(V)\rangle$. In particular, note that every ideal of $\mathcal{O}(X, \alpha, L)$ is gauge-invariant. So, by 3.9 the map a $V \longrightarrow \langle C_0(C)\rangle$ is a bijection. □

4.3 A simplicity criteria for the Cuntz-Krieger algebras for infinite matrices

Recall that $G_R(A)$ is the oriented graph whose vertex are the elements of G such that given $x, y \in G$ there exists an oriented edge from x to y if $A(x, y) = 1$. An path from x to y is a finite sequence $x_1 \cdots x_n$ such that $x_1 = x$, $x_n = y$ and $A(x_i, x_{i+1}) = 1$ for each i. We will say that $G_R(A)$ é transitive if for each $x, y \in G$ there exists a path from x to y.

The main result of this section is that if $Gr(A)$ is transitive then the Cuntz-Krieger algebra O_A is simple. This result is essentially Theorem [4, 14.1].

The following proposition singles out the σ, σ^{-1}-invariant open subsets of $\widetilde{\Omega_A}$.

Proposition 4.9. *If $G_R(A)$ is transitive, the unique σ-invariants nonempty open subsets of $\widetilde{\Omega_A}$ are $\widetilde{\Omega_A} \setminus \emptyset$ and $\widetilde{\Omega_A}$.*

Proof. Let V be a σ-invariant open subset of $\widetilde{\Omega_A}$. Let $\xi \in V$ an element whose stem is infinite. (such elements form a dense subset in $\widetilde{\Omega_A}$). Choose V_n neighbourhood of ξ in V,

$$V_n = \{\nu \in \widetilde{\Omega_A};\, w(\nu)_{|_n} = w(\xi)_{|_n}\}$$

where $w(\nu)$ is the stem of ν. Let $\mu \in \widetilde{\Omega_A}$ such that $|w(\mu)| \geq 1$ and let $x \in G$, with $x \in \mu$. Since $G_R(A)$ is transitive there exists a path $x_1 \cdots x_m$ from $w(\xi)_n$ to x, and by this way $w(\xi)_{|_n} x_2 \cdots x_{m-1}\mu \in V_n \subseteq V$. Since V is σ-invariant it follows that $\mu \in V$ because $\mu = \sigma^{n+m-2}(w(\xi)_{|_n} x_2 \cdots x_{m-1}\mu)$. So $U \subseteq V$. If $\emptyset \neq \xi \in \widetilde{\Omega_A} \setminus U$ then there exists $x \in G$ such that $x^{-1} \in \xi$. Since $x\xi \in U \subseteq V$ and $\sigma(x\xi) = \xi$ it follows that $\xi \in V$. This shows that $\widetilde{\Omega_A} \setminus \emptyset \subseteq V$, from where the result follows. □

Since $\widetilde{\Omega_A}$ and $\widetilde{\Omega_A} \setminus \emptyset$ are σ^{-1}-invariant it follows by the previous proposition that the unique σ, σ^{-1}-invariant open nonempty subsets of $\widetilde{\Omega_A}$ are $\widetilde{\Omega_A}$ and $\widetilde{\Omega_A} \setminus \emptyset$.

Given $\xi \in \mathrm{dom}(\sigma^i)$ with $w(\xi) = x_1 x_2 \cdots$ we have that $w(\sigma^i(\xi)) = x_{i+1}x_{i+2}\ldots$ This shows that if $\xi \in V^{i,j}$ then $w(\xi)$ is infinite, because if we suppose that $|w(\xi)| = n$, then we have that $n - i = |w(\sigma^i(\xi))| = |w(\sigma^j(\xi))| = n - j$ from where $i = j$, which is an absurd.

The following proposition shows a relationship between $Gr(A)$ and $\widetilde{\Omega_A}$.

Proposition 4.10. *If $Gr(A)$ is transitive then $\widetilde{\Omega_A}$ is topologically free.*

Proof. Suppose $i > j$, $i = j + k$ and that $\overline{V^{i,j}}$ has nonempty interior. Let ν be an interior point of $\overline{V^{i,j}}$ and $V_\nu \subseteq \overline{V^{i,j}}$ an open subset which contains ν. Then there exists an element $\xi \in V_\nu \cap V^{i,j}$. Since $\sigma^i(\xi) = \sigma^j(\xi)$ we have that

$$x_{i+1}x_{i+2}\cdots = w(\sigma^i(\xi)) = w(\sigma^j(\xi)) = x_{j+1}x_{j+2}\cdots,$$

from where $x_{i+r} = x_{j+r}$ for $r \geq 1$. Since $i = j + k$ it follows that $x_{i+k} = x_{j+k} = x_i$, and also that $x_{i+(k+r)} = x_{j+(k+r)} = x_{(j+k)+r} = x_{i+r}$ for each $r \geq 1$. Applying the last equality repeatedly it follows that $x_{i+nk+r} = x_{i+r}$ for each $n \in \mathbb{N}$ and $r \geq 1$. This shows that $w(\xi) = x_1 \cdots x_{i-1} sss \cdots$, where $s = x_i x_{i+1} \cdots x_{i+(k-1)}$. Since $w(\xi)$ is infinite, there exists $n \geq i$ such that $V_n = \{\eta \in \widetilde{\Omega_A} : w(\eta)_{|_n} = x_1 \cdots x_n = w(\xi)_{|_n}\} \subseteq V_\nu$.

Claim. $V_n = \{\xi\}$.

Supposing $\eta \in V_n \cap V^{i,j}$, with the same arguments as above it may be proved that $w(\eta) = x_1 x_2 \cdots x_{i-1} sss \cdots$, from where $w(\eta) = w(\xi)$, and since η, ξ have infinite stems it follows that $\eta = \xi$. Let $\nu \in V_n$. Then, since $V_n \subseteq \overline{V^{i,j}}$ there exists a net $(\nu_l)_l \subseteq V^{i,j}$ such that $\nu_l \to \nu$. Since $\nu \in V_n$ and V_n is open we may suppose that $(\nu_l)_l \subseteq V_n$. Therefore $\nu_l = \xi$ for each l and so $\nu = \xi$. This proves the claim. □

Let $y \in G \setminus \{x_i, x_{i+1}, \cdots, x_{i+(k-1)}\}$. By the fact that $Gr(A)$ is transitive there exists a path $y_1 \cdots y_r$ where $y_1 = x_{n+1}$ and $y_r = y$ and an other path $z_1 \cdots z_t$ such that $z_1 = y$ e $z_t = x_1$. In this way we may consider the infinite admissible word $x_1 \cdots x_n y_1 \cdots y_r z_2 \cdots z_{t-1} w(\xi)$ which is the stem of some element $\mu \in \widetilde{\Omega_A}$. Notice that $\mu \in V_n$ by the definition of V_n and that $\mu \neq \xi$, because its stems are distinct. This contradicts the claim. Therefore, $\overline{V^{i,j}}$ has empty interior, and so $\widetilde{\Omega_A}$ is topologically free. □

We will prove now the main result of this section.

Proposition 4.11. *If $Gr(A)$ is transitive the unique ideals of $\widetilde{O_A}$ are the null ideal, O_A and $\widetilde{O_A}$.*

Proof. By 4.9 the unique closed σ, σ^{-1}-invariants subsets of $\widetilde{\Omega_A}$ are $\widetilde{\Omega_A}$, the set $\{\emptyset\}$ (if $\emptyset \in \widetilde{\Omega_A}$, that is, if $O_A \neq \widetilde{O_A}$ by [4, 8.5]) and the empty set. Since these subsets are topologically free, by 4.8 the ideals of $\mathcal{O}(\widetilde{\Omega_A}, \alpha, L)$ are precisely 0, $\langle C_0(\widetilde{\Omega_A} \setminus \emptyset)\rangle$ and $\mathcal{O}(\widetilde{\Omega_A}, \alpha, L)$. Therefore if $\emptyset \notin \widetilde{\Omega_A}$ (that is, if $O_A = \widetilde{O_A}$) then $\mathcal{O}(\widetilde{\Omega_A}, \alpha, L)$ has no nontrivial ideals and the proposition is proved in this case. If $\emptyset \in \widetilde{\Omega_A}$ then by 4.8 $\mathcal{O}(\widetilde{\Omega_A}, \alpha, L)$ has exactly one nontrivial ideal, which is $\langle C_0(\widetilde{\Omega_A} \setminus \emptyset)\rangle$. Therefore $\widetilde{O_A}$ has also exactly one nontrivial ideal. By [4, 8.5] $O_A \neq \widetilde{O_A}$ and since $0 \neq O_A \trianglelefteq \widetilde{O_A}$ it follows that O_A is a nontrivial ideal of $\widetilde{O_A}$, and so is unique. □

A direct consequence of this proposition is that if $G_R(A)$ is transitive then O_A is simple.

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Ruy Exel and **Danilo Royer**
Departamento de Matemática
Universidade Federal de Santa Catarina
88040-900 Florianópolis, SC
BRAZIL

E-mails: exel@mtm.ufsc.br and royer@mtm.ufsc.br