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Hyperbolic linear Weingarten surfaces in $\mathbb{R}^3$

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Abstract. A hyperbolic linear Weingarten surface in $\mathbb{R}^3$ is a surface M whose mean and Gaussian curvatures satisfy the relationship $2aH + bK = c$ for real numbers a, b, c such that $a^2 + bc < 0$. In this work we obtain a representation for such a surface in terms of its Gauss map when, more generally, a, b, c are functions on M. We also study the completeness of such surfaces and describe a procedure to construct complete examples from solutions of the sine-Gordon equation.

Keywords: Linear Weingarten surfaces, Gauss map, sine-Gordon equation.

Mathematical subject classification: 53A05.

1 Introduction

Let $\psi : M \longrightarrow \mathbb{R}^3$ be an immersion from an orientable surface M in $\mathbb{R}^3$. As is well-known, ψ is said to be a linear Weingarten immersion if its mean curvature H and Gaussian curvature K satisfy the relationship

$$2aH + bK = c \tag{1}$$

on M for real numbers a, b, c.

Such a surface is said to be elliptic, hyperbolic or parabolic depending on whether the discriminant $a^2 + bc$ is positive, negative or zero. Those adjectives fit, actually, with the character of the equation (1) (see [3]). For instance, surfaces of constant mean curvature and surfaces of constant positive Gaussian curvature are elliptic, while surfaces of constant negative Gaussian curvature are hyperbolic.

We devote this paper to the study of hyperbolic linear Weingarten surfaces, in short, HLW-surfaces. Specifically, in Section 2 we obtain a representation

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for HLW-surfaces in terms of its Gauss map when, more generally, a, b, c are functions on M (Theorems 1 and 2). There exists a representation for elliptic linear Weingarten surfaces due to Gálvez, Martínez and Milán (see [2]).

We dedicate Section 3 to the study of complete HLW-surfaces. Thus, we see that a complete HLW-surface M endowed with the Lorentzian metric $aI + bII$ is conformally diffeomorphic to the Lorentz-Minkowski plane $\mathbb{L}^2$, and relate such a surface with the sine-Gordon equation (Theorem 5). Actually, it is already known the relationship between HLW-surfaces and the sine-Gordon equation (see, for instance, [7]), although we establish that relation in terms of global Tchebychev coordinates, which allows us to study the completeness of the surface. Conversely, we describe a procedure to construct complete HLW-surface from solutions of the sine-Gordon equation (Theorem 7) and give some examples from solutions of the pendulum equation. We would like to point out that the construction of complete HLW-surfaces in $\mathbb{R}^3$ has been studied recently in several works, as in [8] and [1] where some examples are obtained as an application of Ribaucour transformations.

2 A representation in terms of the Gauss map

Let $\psi : M \longrightarrow \mathbb{R}^3$ be an immersion satisfying (1) for functions a, b, c on M such that $a^2 + bc < 0$, and let us denote by N its Gauss map, and by I and II its first and second fundamental forms. Let us consider the symmetric tensor on M

$$\begin{aligned} \sigma(X, Y) &= aI(X, Y) + bII(X, Y) \\ &= a\langle X, Y\rangle - b\langle dN(X), Y\rangle, \qquad X, Y \in \mathfrak{X}(M), \end{aligned} \tag{2}$$

where $\langle , \rangle$ is the induced metric on M via ψ of the standard metric in $\mathbb{R}^3$. Since $a^2 + bc < 0$, σ becomes a lorentzian metric on M. In fact, let us take a local orthonormal frame $\{E_1, E_2\}$ for the induced metric $\langle , \rangle$ such that $-dN(E_i) = \lambda_i E_i$, $i = 1, 2$, where λ_1, λ_2 are the principal curvatures associated to N. Then

$$\sigma(E_i, E_j) = (a + b\lambda_i)\delta_{ij}, \qquad i, j = 1, 2,$$

δ_{ij} being the Kronecker delta, and so

$$det(\sigma) = (a + b\lambda_1)(a + b\lambda_2) = a^2 + b(2aH + bK) = a^2 + bc < 0.$$

Since σ is a Lorentzian metric on M, we can take (locally) isothermal coordinates (u, v) such that

$$\sigma = \rho(du^2 - dv^2) \tag{3}$$

for a positive smooth function ρ. Let us put

$$\begin{aligned} \langle d\psi, d\psi\rangle &= E\,du^2 + 2F\,du dv + G\,dv^2, \\ \langle d\psi, -dN\rangle &= e\,du^2 + 2\,f\,du dv + g\,dv^2, \end{aligned} \tag{4}$$

where we are taking $N = (\psi_u \wedge \psi_v)/\sqrt{EG - F^2}$, $\wedge$ being the usual cross product in $\mathbb{R}^3$. Hence

$$\begin{aligned} N \wedge \psi_u &= \frac{-1}{\sqrt{EG - F^2}}(F\psi_u - E\psi_v), \\ N \wedge \psi_v &= \frac{-1}{\sqrt{EG - F^2}}(G\psi_u - F\psi_v). \end{aligned} \tag{5}$$

Using again (4), one gets

$$\begin{aligned} -N_u &= \frac{1}{EG - F^2}\left((eG - fF)\psi_u + (-eF + fE)\psi_v\right), \\ -N_v &= \frac{1}{EG - F^2}\left((fG - gF)\psi_u + (-fF + gE)\psi_v\right), \end{aligned}$$

and so, after a straightforward computation, it follows that

$$\begin{aligned} N \wedge N_u &= \frac{1}{\sqrt{EG - F^2}}(f\psi_u - e\psi_v), \\ N \wedge N_v &= \frac{1}{\sqrt{EG - F^2}}(g\psi_u - f\psi_v). \end{aligned} \tag{6}$$

On the other hand, observe that from (2), (3) and (4)

$$aE + be = \rho, \qquad aF + bf = 0, \qquad aG + bg = -\rho$$

whence, using (1),

$$\begin{aligned} -\rho^2 &= a^2(EG - F^2) + ab(eG - 2fF + gE) + b^2(eg - f^2) \\ &= (EG - F^2)(a^2 + bc). \end{aligned} \tag{7}$$

Then, from (5) and (6) we have

$$\begin{aligned} a(N \wedge \psi_u) - b(N \wedge N_u) &= \sqrt{|a^2 + bc|}\psi_v, \\ a(N \wedge \psi_v) - b(N \wedge N_v) &= \sqrt{|a^2 + bc|}\psi_u. \end{aligned} \tag{8}$$

Now, if we call $d = \sqrt{|a^2 + bc|}$, we get from (8)

$$N \wedge \psi_u = N \wedge \left(\frac{a}{d} N \wedge \psi_v - \frac{b}{d} N \wedge N_v \right) = -\frac{a}{d}\psi_v + \frac{b}{d} N_v,$$

$$N \wedge \psi_v = N \wedge \left(\frac{a}{d} N \wedge \psi_u - \frac{b}{d} N \wedge N_u \right) = -\frac{a}{d}\psi_u + \frac{b}{d} N_u,$$

which jointly with (8) allows us to obtain

$$\psi_u = -\frac{a}{c} N_u + \frac{d}{c} N \wedge N_v,$$

$$\psi_v = -\frac{a}{c} N_v + \frac{d}{c} N \wedge N_u.$$

Therefore we have the following:

Theorem 1. *Let $\psi : M \longrightarrow \mathbb{R}^3$ be an immersion satisfying $2aH + bK = c$ for functions a, b, c on M such that $a^2 + bc < 0$, with associated Gauss map N. If we consider on M local isothermal coordinates (u, v) for the Lorentzian metric $\sigma = aI + bII$, then ψ can be recovered in terms of N as*

$$\psi_u = -\frac{a}{c} N_u + \frac{d}{c} N \wedge N_v, \qquad \psi_v = -\frac{a}{c} N_v + \frac{d}{c} N \wedge N_u, \tag{9}$$

where $d = \sqrt{|a^2 + bc|}$.

Conversely, we have the following:

Theorem 2. *Let M be a simply connected Lorentz surface, $N : M \longrightarrow \mathbb{S}^2$ a differentiable map and $a, b, c : M \longrightarrow \mathbb{R}$ functions of class C^1 such that $a^2 + bc < 0$. Let (u, v) be isothermal coordinates for M and let us suppose that*

$$\begin{aligned} &-\left(\frac{a}{c}\right)_v N_u + \left(\frac{d}{c}\right)_v N \wedge N_v + \frac{d}{c} N \wedge N_{vv} \\ &= -\left(\frac{a}{c}\right)_u N_v + \left(\frac{d}{c}\right)_u N \wedge N_u + \frac{d}{c} N \wedge N_{uu}, \end{aligned} \tag{10}$$

where $d = \sqrt{|a^2 + bc|}$. Then (9) determines an immersion $\psi : M \longrightarrow \mathbb{R}^3$ (possibly degenerated at some points) with Gauss map N, satisfying $2aH + bK = c$, and such that the structure given by $\sigma = aI + bII$ is the one of M. Moreover $\psi : M \longrightarrow \mathbb{R}^3$ is unique up to similarity transformations of $\mathbb{R}^3$.

Proof. First, observe that from (9) we have that $\langle N, \psi_u \rangle = \langle N, \psi_v \rangle = 0$. Note also that the condition (10) says that $\psi_{uv} = \psi_{vu}$, which means that ψ is integrable.

On the other hand, if we put I and II as in (4), we easily obtain that

$$aE + be = -(aG + bg) =: \rho, \qquad aF + bf = 0,$$

and so $\sigma = \rho(du^2 - dv^2)$, that is, the structure given by σ is the one of M.

Let us see, to finish, that $2aH + bK = c$. It is a straightforward computation to see that

$$Eg - 2Ff + Ge = \frac{1}{b}\big(- \rho(E - G) - 2a\big(EG - F^2\big)\big)$$

and

$$eg - f^2 = \frac{1}{b^2}\big(- \rho^2 + a\rho(E - G) + a^2\big(EG - F^2\big)\big),$$

whence

$$a(Eg - 2Ff + Ge) + b\big(eg - f^2\big) = \frac{1}{b}\big(- \rho^2 - a^2\big(EG - F^2\big)\big). \qquad (11)$$

Now, since $\det(N, N_u, N_v)^2 + \langle N_u, N_v\rangle^2 - \langle N_u, N_u\rangle\langle N_v, N_v\rangle = 0$, one gets

$$d^2\big(EG - F^2\big) - \rho^2 = d^2\big(EG - F^2\big) - \langle a\psi_u - bN_u, \psi_u\rangle^2 = 0,$$

which jointly with (11) says that $2aH + bK = c$ as we wanted to show. $\square$

Remark 3. In the case where a, b, c are constant, the condition (10) can be written as $N \wedge (N_{uu} - N_{vv}) = 0$. Conversely, if $\psi : M \longrightarrow \mathbb{R}^3$ is a HLW immersion satisfying (1) for constants a, b, c, then the Gauss map N is harmonic for the Lorentzian metric σ.

Remark 4. (9) determines, actually, an immersion (that is, without singular points), if we ask N to satisfy the additional condition

$$a^2 N_u \wedge N_v + ad(\langle N_v, N_v\rangle - \langle N_u, N_u\rangle)N - d^2 \det(N, N_u, N_v)N \neq 0$$

This is nothing but the condition $\psi_u \wedge \psi_v \neq 0$ written in terms of a, b, c and N.

3 Complete HLW-surfaces

We devote this section to the study of complete HLW-surfaces and their relationship with the sine-Gordon equation. For our study we will assume that $a \neq 0$, so we exclude the case of K-surfaces. This does not mean a problem because, from Hilbert Theorem, there do not exist complete surfaces in $\mathbb{R}^3$ with constant negative Gaussian curvature. We will take, without loss of generality, $a > 0$.

We have the following:

Theorem 5. *Let $\psi : M \longrightarrow \mathbb{R}^3$ be a complete HLW immersion satisfying $a^2 + bc < 0$ for constants $a, b, c \in \mathbb{R}$ such that $a > 0$. Then (M, σ) is conformally diffeomorphic to the Lorentz-Minkowski plane $\mathbb{L}^2$ and its first and second fundamental forms can be written as*

$$\begin{aligned} I &= du^2 + 2\cos\omega du dv + dv^2, \\ II &= -\frac{a}{b}\, du^2 + 2\frac{-a\cos\omega - \sqrt{-a^2 - bc}\,\sin\omega}{b}\, du dv - \frac{a}{b}\, dv^2, \end{aligned} \tag{12}$$

where $\omega(u, v)$ is a differentiable function on $\mathbb{L}^2$. Moreover, the function $\varphi(u, v) = \omega(u, v) + \omega_0$ is a solution of the equation

$$\varphi_{uv} = r \sin\varphi, \tag{13}$$

where $r = |c/b|$ and $\omega_0 \in \mathbb{R}$ is such that $\omega_0 \in (-\pi, 0)$ and

$$\sin\omega_0 = \frac{2a\sqrt{-a^2 - bc}}{bc}.$$

Proof. Let $\psi : M \longrightarrow \mathbb{R}^3$ be as in the statement, and let us take coordenates (x, y) such that $\sigma = \theta dx dy$ for a function θ. First, let us check that the identity map $Id\colon (M, \sigma) \longrightarrow (M, I)$ is harmonic, which is equivalent to see that $\langle\psi_x, \psi_x\rangle$ is a function of x and $\langle\psi_y, \psi_y\rangle$ a function of y (see [9]). In fact, since the third fundamental form is given by $III = -KI + 2HII$, we have that

$$\langle N_x, N_x\rangle = -K\langle\psi_x, \psi_x\rangle + 2H\langle -N_x, \psi_x\rangle.$$

On the other hand, we also have that

$$0 = \sigma(\psi_x, \psi_x) = a\langle\psi_x, \psi_x\rangle + b\langle -N_x, \psi_x\rangle$$

and so, since $2aH + bK = c$,

$$\langle \psi_x, \psi_x \rangle = \frac{-b}{c} \langle N_x, N_x \rangle$$

which is a function of x because N is harmonic for σ (see Remark 3). We can reason analogously to see that $\langle \psi_y, \psi_y \rangle$ is a function of y.

Now, since ψ is complete, the Hilbert-Holmgren Theorem assures that (M, σ) is conformally diffeomorphic to the Lorentz-Minkowski plane $\mathbb{L}^2$ (see, for instance, [9]). Therefore we can assert that there exist global Tchebychev coordinates (u, v) on $M \equiv \mathbb{L}^2$ such that

$$I = du^2 + 2\cos\omega\, du dv + dv^2,$$

$$\sigma = -2\sqrt{-K(I,\sigma)}\sin\omega\, du dv,$$

where $\omega(u, v)$ is a differentiable function on $\mathbb{L}^2$ and

$$K(I,\sigma) = \frac{-\rho^2}{EG - F^2}$$

is the extrinsic curvature of the pair (I, σ) (see [4]). Note that, using (7),

$$K(I,\sigma) = a^2 + bc < 0.$$

Moreover, it is easy to see that the intrinsic curvature of I is given by

$$K = \frac{\omega_{uv}}{\sin\omega}.$$

But, since

$$II = \frac{1}{b}\sigma - \frac{a}{b} I = -\frac{a}{b} du^2 + 2\frac{-a\cos\omega - \sqrt{-a^2 - bc}\sin\omega}{b} du dv - \frac{a}{b} dv^2,$$

we also have that

$$K = K(I, II) = \frac{2a^2 + bc}{b^2} - 2\frac{a\sqrt{-a^2 - bc}}{b^2} \text{ cotan } \omega,$$

where $K(I, II)$ is the extrinsic curvature of ψ, and then

$$\omega_{uv} = \alpha \sin\omega + \beta \cos\omega \tag{14}$$

for

$$\alpha = \frac{2a^2 + bc}{b^2}, \qquad \beta = \frac{-2a\sqrt{-a^2 - bc}}{b^2}. \tag{15}$$

Observe that $r = \sqrt{\alpha^2 + \beta^2}$ and $\beta \neq 0$, so we can take $\omega_0 \in (-\pi, 0)$ such that $\cos\omega_0 = \alpha/r$ and $\sin\omega_0 = \beta/r$ as we said in the statement of the Theorem. Now, using (14) it is a simple computation to check that $\varphi(u, v) = \omega(u, v) + \omega_0$ satisfies the equation (13). □

Remark 6. Note that (13) is, in essence, the sine-Gordon equation. In fact, if we rewrite (13) with respect to the coordinates $x = \sqrt{r}(u+v)$, $y = \sqrt{r}(u-v)$, one gets

$$\varphi_{xx} - \varphi_{yy} = \sin\varphi. \tag{16}$$

In addition, we would like to point out that equations (12) correspond essentially to equations (2.9) and (2.13) in [7] up to a change of coordinates.

Conversely we have:

Theorem 7. *Let $r > 0$ be a positive constant, $\varphi(u, v)$ a solution of (13) defined on the whole plane $\mathbb{R}^2$ and let us suppose that there exists a real constant $\omega_0 \in (-\pi, 0)$ such that for all $(u, v) \in \mathbb{R}^2$, $\varphi(u, v) \subseteq C \subseteq (\omega_0, \omega_0 + \pi)$ for a certain compact C. Let a, b, c be real constants, $a^2 + bc < 0$, $a > 0$, such that α, β given as in (15) satisfy that $r = \sqrt{\alpha^2 + \beta^2}$ and $\sin\omega_0 = \beta/r$. Then there exists a complete surface in $\mathbb{R}^3$ such that $2aH + bK = c$, whose first and second fundamental forms are given by (12) for global coordinates $(u, v) \in \mathbb{R}^2$, where $\omega(u, v) = \varphi(u, v) - \omega_0$.*

Proof. Let r, φ, ω_0, a, b, c and ω as in the statement. Recall that given a Riemannian metric I and a symmetric (2,0)-tensor II on a simply-connected 2-dimensional manifold M, if I and II satisfy the Gauss and Codazzi equations of the Euclidean space $\mathbb{R}^3$, then there exists a unique immersion (up to an isometry) $\psi : M \longrightarrow \mathbb{R}^3$ such that I and II are its first and second fundamental forms, respectively.

In fact, it is easy to check that I and II satisfy the Codazzi equation. On the other hand, the Gauss equation holds because $\omega_{uv} = \alpha \sin\omega + \beta\cos\omega$ and so the intrinsic curvature of I and $K(I, II)$ coincide.

Now, it is a straightforward computation to see that $2aH + bK = c$.

To finish, observe that I is a complete metric because, since $\varphi(u, v) \subseteq C \subseteq (\omega_0, \omega_0 + \pi)$, then $\omega(u, v) \subseteq C' \subseteq (0, \pi)$ for a compact C'. □

After this result, it seems natural to ask oneself if there exist, actually, solutions of the equation (13) verifying the suitable assumptions to construct complete HLW-surfaces following the described procedure. Let us see that they do exist by giving some easy examples.

Recall that the pendulum equation is given by

$$\varphi''(t) \;=\; \sin\varphi(t). \tag{17}$$

Note that the solutions of the sine-Gordon equation (16) depending on one variable are solutions of the pendulum equation, and viceversa.

As is well-known, for each initial data $\varphi(0) := \varphi_0 \in (-\pi/2, \pi/2)$, $\varphi_0 \neq 0$, there is a unique non zero solution $\varphi(t)$ of the equation (17) defined on $\mathbb{R}$. Physically, φ_0 is nothing but the angle of the pendulum with respect to the vertical axes for the time $t = 0$ and it is, in magnitude, the maximum angle which attains the pendulum in its oscillatory motion.

Let us take constants $a, b, c \in \mathbb{R}$ such that $a^2 + bc < 0$, $a > 0$. Our aim is to construct a complete HLW-surface such that $2aH + bK = c$. Let us define α and β as in (15). Since $a > 0$ and so $\beta < 0$, we can take $\omega_0 \in (-\pi, 0)$ such that $\sin\omega_0 = \beta/r$, where $r = \sqrt{\alpha^2 + \beta^2}$.

Now, let us take a constant φ_0 such that $0 < \varphi_0 < \min\{|\omega_0|, \pi + \omega_0\}$ and let φ be a solution of the pendulum equation (17) with initial data $\varphi(0) = \varphi_0$. Observe that $\varphi(t)$ satisfies that $\omega_0 + \varepsilon < \varphi < \pi + \omega_0 - \varepsilon$ for $\varepsilon > 0$ small enough.

If we put $x = t$, then $\varphi(x, y) := \varphi(x)$ is a solution of the equation (16). Thus, by considering the parameters $u = (x + y)/(2\sqrt{r})$, $v = (x - y)/(2\sqrt{r})$, $\varphi(u, v)$ is a solution of the equation (13) such that $\varepsilon < \omega < \pi - \varepsilon$, from which we can construct a complete HLW-surface following the procedure described above.

Finally, we would like to point out that there exist infinitely many bounded solutions $\varphi(u, v)$ of (13) verifying the conditions required in Theorem 7; among them, a wide family of doubly periodic ones (see, for instance, [5], [6]). These solutions provide, thus, infinitely many complete HLW-surfaces.

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