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On the dynamics of mechanical systems with homogeneous polynomial potentials of degree 4

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Abstract. In this work we study mechanical systems defined by homogeneous polynomial potentials of degree 4 on the plane, when the potential has a definite or semi-definite sign and the energy is non-negative. We get a global description of the flow for the non-negative potential case. Some partial results are obtained for the more complicated case of non-positive potentials. In contrast with the non-negative case, we prove that the flow is complete and we find special periodic solutions, whose stability is analyzed. By using results from Ziglin theory following Morales-Ruiz and Ramis we check the non-integrability of the Hamiltonian systems in terms of the potential parameters.

Keywords: Hamiltonian vector fields, homogeneous polynomial potentials, global flow.

Mathematical subject classification: Primary: 58F05; Secondary: 70H99.

1 Introduction

The homogeneous polynomial Hamiltonian systems are a recurrent research topic, since they are fundamental in order to understand the behavior of polynomial or analytical Hamiltonian systems. Indeed, in the search for more complete characterizations of integrable Hamiltonian systems one is often lead to the study of homogeneous systems, see for example, the papers by Yoshida ([20]) and Morales-Ruiz and Ramis ([17]), on necessary conditions for integrability of mechanical systems with homogeneous potentials. More recently, the work by Maciejewski and Przybylska ([16]) gives a complete characterization of meromorphically integrable homogeneous Hamiltonian systems of degree 3. Homogeneous potentials appear also in the modelling of natural phenomena or processes. Along this line we may mention Caranicolas et al. [1], [2] and

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Contopoulos [3] who have considered third or fourth degree homogeneous perturbations of homogeneous quadratic polynomials as models in the dynamics of galaxies. As it is commonplace in this area of research, they have applied numerical methods in order to explain interesting features of the concerned dynamical systems.

We studied in [7] some general properties of the flow of mechanical systems with polynomial potentials of degree lesser or equal than 4 in two variables. The main tool was the application of a McGehee blow up at infinity, which was systematically studied in [11]. From this blow up we get a two dimensional infinity manifold which allows us to study the flow in a neighborhood of infinity in the configuration space. In the negative energy case it is verified that the flow is gradient-like, so that any solution of the mechanical system has to be asymptotic to one of the equilibrium solutions on the collision manifold. Later on, in [8], it was analyzed the global flow for the case of homogeneous potentials of degree 3 and negative energy. This time, the flow extends the energy level to a compact 3-manifold whose boundary is the infinity manifold, which is invariant under the extended flow. Transversality properties of the flow were used in order to describe the global flow when the potential is separable, and the general case of negative energy was studied in [9], including a description of the global flow for the more complicated case of positive energy.

In this work we study the dynamics of planar mechanical systems whose associated potential V is a homogeneous polynomial of degree 4. The classification of the dynamics is a hard problem due the fact that the potential depends on 5 parameters. However, we obtain a very general description of the dynamics by restriction to *semidefinite* potentials. We will analyze the dynamics in the non-negative energy h levels, since we proved in [7] that when $h < 0$ the flow is gradient-like and every solution tends to an equilibrium on the infinity manifold.

The problem is stated in Section 2, where we describe the canonical form of a planar homogenous potential of degree 4. Then, in Section 3 we study the sign of the potentials in terms of their coefficients. Some particular solutions which are called *homothetic* are studied in Section 4. The flow of the mechanical system when the potential is positive definite or semidefinite is studied in Section 5. We prove that, generically, non-equilibrium solutions escape to infinity, and in particular, there are no periodic solutions. We also show that escape directions correspond to homothetic orbits. The negative definite and semidefinite potential case is studied in Section 6, proving the completeness of the flow and the boundedness of all solutions. The stability of certain periodic orbits is analyzed in terms of the parameters of the potential. Some examples are given in Section 7. Finally, in Section 8 we study non-integrability conditions of degree 4 homogeneous potentials as a function of the parameters.

2 Statement of the problem

We consider a planar mechanical system defined by

$$\ddot{\mathbf{q}} = \nabla V(\mathbf{q}), \quad \mathbf{q} = (x, y) \in \mathbb{R}^2, \tag{2.1}$$

where the associated potential V is a homogeneous polynomial of degree 4. The energy of this system is the Hamiltonian

$$H = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - V(x, y). \tag{2.2}$$

We denote by $\mathbf{p} = (p_1, p_2)$ the velocity vector $(\dot{x}, \dot{y})$, so that the corresponding Hamiltonian system equivalent to (2.1) is given by

$$\begin{aligned} \dot{x} &= p_1, \qquad \dot{p}_1 = \frac{\partial V}{\partial x}(\mathbf{q}) \\ \dot{y} &= p_2, \qquad \dot{p}_2 = \frac{\partial V}{\partial y}(\mathbf{q}). \end{aligned} \tag{2.3}$$

We will always consider in this paper that the energy level h is non negative. Recall that the Hill region R is the subset of the plane given by

$$R = \{\mathbf{q} = (x, y) \mid V(x, y) + h \geq 0\},$$

whose boundary is the zero velocity curve

$$Z = \{(x, y) \mid V(x, y) + h = 0\}.$$

Notice that if there is a point $\mathbf{q}^* \in R$ such that $V(\mathbf{q}^*) > 0$, then because of the homogeneity of the potential V, all the points in the line generated by $\mathbf{q}^*$ belong to the Hill region. Consequently, the Hill region is unbounded.

Without loss of generality, in any homogeneous potential of degree 4 in two variables

$$V(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4, \tag{2.4}$$

we can take $b = 0$. This is stated in the following theorem

Theorem 1. *Any mechanical system*

$$\begin{aligned} \dot{x} &= p_1, \qquad \dot{y} = p_2, \\ \dot{p}_1 &= \frac{\partial V}{\partial x}, \qquad \dot{p}_2 = \frac{\partial V}{\partial y}, \end{aligned} \tag{2.5}$$

where $V(x, y)$ is a potential as (2.4)*, becomes through a symplectic change of variables a mechanical system with potential*

$$V(x, y) = ax^4 + cx^2y^2 + dxy^3 + ey^4. \tag{2.6}$$

Indeed, it is enough to consider the symplectic transformation T given by

$$x = -sX + rY, \quad y = rX + sY,$$
$$p_1 = -\frac{1}{r^2+s^2}\,(sP_1 - rP_2), \quad p_2 = -\frac{1}{r^2+s^2}\,(-rP_1 - sP_2)\,.$$

We will assume from now on, that the mechanical system is given by (2.5), where the potential V is homogeneous of degree 4 and it has the normal form (2.6). Hence, the second order system of differential equations we have to study is

$$\begin{aligned} \ddot{x} &= 4ax^3 + 2cxy^2 + dy^3 \\ \ddot{y} &= 2cx^2y + 3dxy^2 + 4ey^3 = y\,[2cx^2 + 3dxy + 4ey^2]\,. \end{aligned} \tag{2.7}$$

We remark that V is invariant under the symmetry: $(x, y) \to (-x, -y)$.

3 On the characterization of definite and semidefinite potentials

In this section we analyze how the sign of homogeneous potentials is related to the values of the coefficients a, c, d, e. We define

Definition 1. *A real homogeneous function V, is positive (negative) definite if $V(x, y) > 0$ $(V(x, y) < 0)$, for any $(x, y) \neq 0$. V is positive (negative) semidefinite if $V(x, y) \geq 0$ $(V(x, y) \leq 0)$ and it is indefinite if it changes signs. We write $V > 0$ $(V < 0)$ when V is positive (negative) definite, and $V \geq 0$ $(V \leq 0)$ when V is positive (negative) definite or semidefinite.* Let Q be the quadratic form generated by the matrix

$$A = \begin{pmatrix} a & c/2 & 0 \\ c/2 & e & d/2 \\ 0 & d/2 & 0 \end{pmatrix}, \tag{3.1}$$

that is,

$$Q(z_1, z_2, z_3) = z_1(az_1 + cz_2) + z_2(dz_3 + ez_2). \tag{3.2}$$

We consider the function $f : \mathbb{R}^2 \to \mathbb{R}^3$, given by

$$f(x, y) = (x^2, y^2, xy). \tag{3.3}$$

It is easy to see that $V(x, y) = (Q \circ f)(x, y)$. Hence, $V > 0$ is equivalent to Q greater than zero on the image of f. So, V is definite if the image of f does not intersect the set of zeroes of Q, which is given by the equation

$$z_1(az_1 + cz_2) + z_2(dz_3 + ez_2) = 0. \tag{3.4}$$

This is the projective equation for the parabola

$$P_1 : \Psi = aX^2 + cX + dY + e = 0, \tag{3.5}$$

where $X = z_1/z_2$ and $Y = z_3/z_2$. If $z_1 = x^2$, $z_2 = y^2$ and $z_3 = xy$, the image of f in the variables X, Y becomes the parabola

$$P_0 : Y^2 = X. \tag{3.6}$$

It is clear that a necessary condition for V to be definite is that a and e have the same sign. Therefore, when $d = 0$, $a > 0$ and $e > 0$, the fact that $V > 0$ depends only on c. In particular, $c > 0$ implies $V > 0$. Equation (3.5) becomes $aX^2 + cX + e = 0$, whose roots are

$$X_1 = \frac{-c - \sqrt{\Delta}}{2a}, \qquad X_2 = \frac{-c + \sqrt{\Delta}}{2a}, \tag{3.7}$$

where $\Delta = c^2 - 4ae$. We have three cases, when $c < 0$: 1) if $\Delta < 0$, then $\Psi > 0$, for all X hence $V > 0$; 2) if $\Delta = 0$, then there is only one positive root, so that Ψ, annihilates in X_1 and it is positive for $X \neq X_1$. Consequently, $V \geq 0$ and it becomes zero in the lines determined by $X_1 y^2 = x^2$. Finally, 3) if $\Delta > 0$, both roots are positive and the four points $(X_1, \pm\sqrt{X_1})$, $(X_2, \pm\sqrt{X_2})$ belong to P_1; therefore $\Psi \leq 0$ in the interval $J : X_1 \leq X \leq X_2$. From this it follows that $V \leq 0$ in all the planar region determined by the interval J and it is positive in its complement.

Proposition 1. *Let $V(x, y) = ax^4 + cx^2y^2 + ey^4$ with $a, e > 0$ and let X_1 and X_2 be given by* (3.7). *Then*

(1) *$V > 0$ if and only if $c \geq 0$ or $c < 0$ and $c^2 - 4ae < 0$,*

(2) *If $c < 0$ and $c^2 - 4ae = 0$, then $V \geq 0$ and it vanishes on the lines defined by $X_1\, y^2 = x^2$,*

(3) *If $c < 0$ and $c^2 - 4ae > 0$, then V is indefinite; it changes sign in the planar regions bounded by the lines $X_1\, y^2 = x^2$ and $X_2\, y^2 = x^2$.*

If $d \neq 0$, we may assume that $d = 1$, dividing V by d; From now on, we will take

$$V(x, y) = ax^4 + cx^2y^2 + xy^3 + ey^4.$$

It is not hard to see that the sign of Ψ and the shape of P_1 depend on the parameters a, c, as we show in Figure 1. Notice that if $P_1 \cap P_0 = \emptyset$, the sign of V is opposite to the sign of Ψ in the region bounded by P_1.

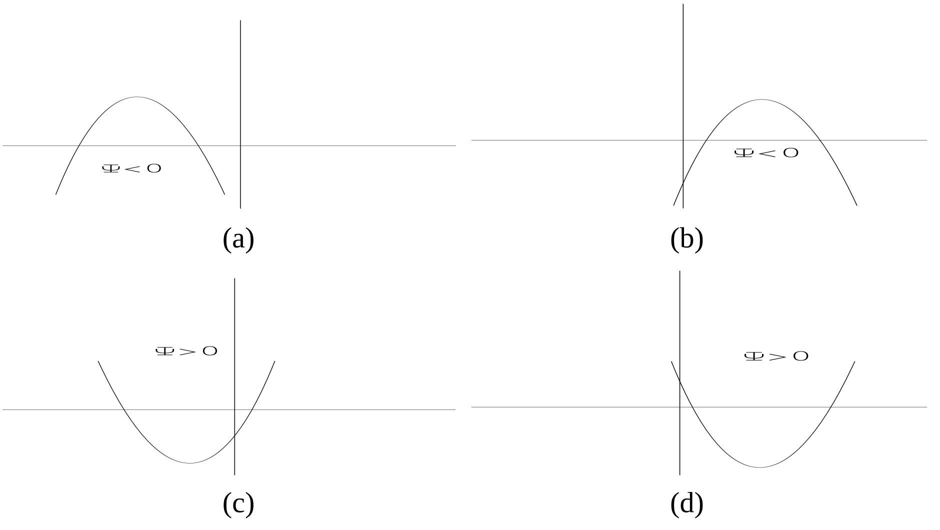


Figure 1: If $P_1 \cap P_0 = \emptyset$, then V is positive in the cases (a) and (b) and it is negative in the cases (c) and (d). (a) $a > 0, c > 0$. (b) $a > 0, c < 0$. (c) $a < 0, c < 0$. (d) $a < 0, c > 0$.

The complete classification of potentials according to their sign is a complicated task in the general case. Now, we give an algorithmic procedure to decide the sign of V. We introduce the notation $V[a, c, e]$ to take into account the dependence of the potential V whit respect to the parameters a, c and e, since we are taking $d = 1$. The parabola P_1 corresponding to $V[a, c, e]$, has a parallel tangent line to the one of the parabola P_0, at each point $X^* > 0$ which is a root of the equation

$$16a^2X^3 + 16acX^2 + 4c^2X - 1 = 0. \tag{3.8}$$

It is easy to verify that $27a^8 + 8a^7c^3$ is a positive multiple of the discriminant of the cubic. Then, $27a^8 + 8a^7c^3 > 0$ is a necessary and sufficient condition for a unique real root of the above equation. In this case, the root is positive and it is given by

$$X^* = \frac{-c}{3\,a} + \frac{c^2}{3\,2^{\frac{1}{3}}\left(27\,a^4 + 4\,a^3\,c^3 + 3\,\sqrt{3}\,\sqrt{27\,a^8 + 8\,a^7\,c^3}\right)^{\frac{1}{3}}} + \frac{\left(27\,a^4 + 4\,a^3\,c^3 + 3\,\sqrt{3}\,\sqrt{27\,a^8 + 8\,a^7\,c^3}\right)^{\frac{1}{3}}}{6\,2^{\frac{2}{3}}\,a^2}. \tag{3.9}$$

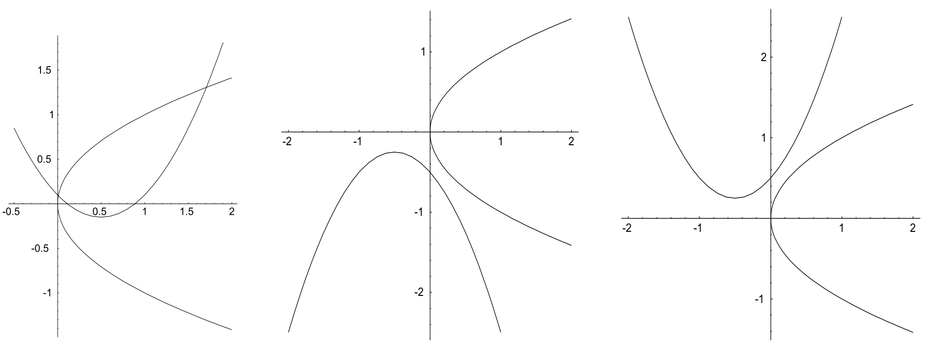
Let $Y_1^* = -aX^{*2} - cX^* - e$ and $Y_0^{*\pm} = \pm\sqrt{X^*}$. We have the following result.

Theorem 2. *Assume* $d = 1$ *and* $27a^8 + 8a^7c^3 > 0$.

(a) *If* $a > 0$, *then* $V[a, c, e] > 0$ *if and only if* $Y_1^* < Y_0^{*-}$,

(b) *If* $a < 0$, *then* $V[a, c, e] < 0$ *if and only if* $Y_1^* > Y_0^{*+}$.

In Figure 2, the relationship between the positions of parabolas P_0 and P_1 and the sign of V are shown. If the term $27a^8 + 8a^7c^3$ considered in the Theorem 2 is negative, the Equation (3.8) has more than one root and in this case, similar but more complicated conditions to determine the sign, can be found.



(a) (b) (c)

Figure 2: a) Indefinite Potential. (b) Positive Potential. (c) Negative Potential. In this figure, $d \neq 0$.

4 Homothetic solutions

In this section we study those solutions of the System (2.7), whose projections into the configuration space are straight lines.

Definition 2. *We say that a solution of the system* (2.7) *is homothetic if there is a differentiable real function* $\lambda(t)$ *(non constant) and an initial position* $\mathbf{q}_0$*, such that* $\mathbf{q}(t) = \lambda(t)\,\mathbf{q}_0$ *is a solution of System* (2.7).

In [18] these solutions are called gradient curves.

In order to characterize the homothetic solutions we may assume without loss of generality that the initial position $\mathbf{q}_0 = (x_0, y_0)$ is a unit vector. We see that $\mathbf{q}(t) = \lambda(t)\,\mathbf{q}_0$ is a solution of (2.7) if and only if,

$$\ddot{\lambda}\,\mathbf{q}_0 = \lambda^4\,\nabla V(\mathbf{q}_0).$$

From the homogeneity of V and the fact that $x_0^2 + y_0^2 = 1$, we get

$$\ddot{\lambda} = 4V(\mathbf{q}_0)\,\lambda^4,$$

and from the energy relation (2.2) we have

$$\frac{1}{2}\,\dot{\lambda}^2 - V(\mathbf{q}_0)\,\lambda^4 = h.$$

From this it follows that

Proposition 2. $\mathbf{q}(t) = \lambda(t)\,\mathbf{q}_0$ *is a homothetic solution of* (2.7) *if and only if*

$$\nabla V(\mathbf{q}_0) = 4V(\mathbf{q}_0)\,\mathbf{q}_0, \quad \frac{1}{2}\,\dot{\lambda}^2 - V(\mathbf{q}_0)\,\lambda^4 = h. \tag{4.1}$$

Now, we characterize the solutions of (4.1). We will begin by studying the solution λ of the above differential equation. Let us denote

$$\mu = V(\mathbf{q}_0). \tag{4.2}$$

To compute $\lambda(t)$ we have to distinguish 3 cases: $\mu > 0$, $\mu = 0$ and $\mu < 0$, for each sign of the energy h.

(1) For $h = 0$, there is no homothetic solution if $\mu \leq 0$ and for $\mu > 0$ we have $\lambda(t) = -\frac{1}{\sqrt{2\mu}(t+k)}$, k constant, an unbounded solution since it is not defined for any t.

(2) For $h > 0$ we rewrite the differential equation (4.1) as

$$\frac{1}{\sqrt{2h}}\,\dot{\lambda} = \pm\sqrt{1 + \frac{\mu}{h}\,\lambda^4}, \tag{4.3}$$

whose solution is analyzed below. For $\mu = 0$ and $\mu < 0$ we give explicit formulas for the solution. For $\mu > 0$ we only show that the solution is not defined for any t.

(2.a) If $\mu = 0$ the solution is given by

$$\lambda(t) = \pm\sqrt{2h}\, t + B. \tag{4.4}$$

We see that it is defined for any t and it is an escape solution.

(2.b) By integration of the energy relation (4.3) we obtain for $\mu < 0$

$$\begin{aligned} \lambda(t) &= \left(\frac{h}{|\mu|}\right)^{1/4} cn\left(2\,(|\mu| h)^{1/4}\, t, \frac{1}{\sqrt{2}}\right) \\ &= \lambda_0\, cn\left(2\,(|\mu| h)^{1/4}\, t, \frac{1}{\sqrt{2}}\right), \end{aligned} \tag{4.5}$$

where cn is the elliptic function of the first kind (see [13], p. 85 exercise (6), by taking $k = k' = \frac{1}{\sqrt{2}}$). It is known that cn is periodic, so that $\lambda(t)$ is periodic with period

$$\begin{aligned} T_\lambda &= \frac{2}{(|\mu| h)^{1/4}}\, cn^{-1}\left(0, \frac{1}{\sqrt{2}}\right) = \frac{2}{(|\mu| h)^{1/4}}\, \kappa \\ &= \frac{1}{(|\mu| h)^{1/4}}\, cn^{-1}\left(-1, \frac{1}{\sqrt{2}}\right), \end{aligned} \tag{4.6}$$

where $\kappa := cn^{-1}\left(0, \frac{1}{\sqrt{2}}\right)$. By using the result from page 89, exercise 29 in [13] we have

$$\kappa = \frac{[\Gamma(1/4)]^2}{4\sqrt{\pi}} \sim 1.854207\ldots \tag{4.7}$$

Hence, the bounded function $\lambda(t)$ is defined for any $t \in \mathbb{R}$ when $\mu < 0$.

(2.c) When $\mu > 0$ it is not easy to integrate the relation(4.3), but we get an estimate which follows from the energy relation. Indeed,

$$\arctan\left(\left(\frac{\mu}{h}\right)^{1/4} \lambda(t)\right) \le \frac{\sqrt{2}}{(\mu h)^{1/4}}\, t \le \sqrt{2}\, \arctan\left(\left(\frac{\mu}{h}\right)^{1/4} \lambda(t)\right). \tag{4.8}$$

In this way we see that $\lambda(t)$ is not defined for any $t > 0$ and it is unbounded.

Regarding the existence of the initial condition $\mathbf{q}_0 = (x_0, y_0)$ associated to the homothetic orbit, we need to solve the first equation in (4.1), which is equivalent to

$$\begin{aligned} V_x(x_0, y_0) = &\quad 4ax_0^3 + 2cx_0y_0^2 + dy_0^3 = 4V(x_0, y_0)\, x_0 \\ V_y(x_0, y_0) = &\quad 2cx_0^2y_0 + 3dx_0y_0^2 + 4ey_0^3 = 4V(x_0, y_0)\, y_0. \end{aligned} \tag{4.9}$$

We analyze the three possible cases:

1. We have that $(x_0, 0)$ with $x_0 \neq 0$ is a solution of (4.9), if and only if, $x_0 = \pm 1$. Here $\mu = a$.

2. We have that $(0, y_0)$ with $y_0 \neq 0$ is a solution of (4.9), if and only if, $d = 0$ and $y_0 = \pm 1$. Here $\mu = e$.

3. Consider the case (x_0, y_0) with $x_0 y_0 \neq 0$. We verify that the normalized initial condition (x_0, y_0) is given by

$$\begin{gathered} p(t) = dt^3 + 2(c - 2e)t^2 - 3dt + 2(2a - c) = 0, \\ x_0^2 + y_0^2 = 1, \quad t = \frac{y_0}{x_0}\,. \end{gathered} \tag{4.10}$$

The homothetic solutions are important for the description of escape motions in mechanical systems (see the following section).

5 The mechanical problem with $V \geq 0$

In this section we assume that the potential V is positive definite or semidefinite, and we analyze the asymptotic behavior and the escape directions for the solutions of the system (2.7). We recall that the energy level is non-negative.

We remark first that if there is $\mathbf{q}^* \neq (0, 0)$ such that $\nabla V(\mathbf{q}^*) = (0, 0)$, because of the homogeneity of V we have that $V(\mathbf{q}^*) = 0$. Hence, V is not positive definite. Again, from the homogeneity of V we have $\nabla V(\lambda\mathbf{q}^*) = (0, 0)$ for any $\lambda \in \mathbb{R}$. Therefore, the one dimensional set of points $\lambda(\mathbf{q}^*, 0, 0)$ consists of zero energy equilibrium points of the system.

Under the above conditions ($\nabla V(\mathbf{q}^*) = (0, 0)$, with $\mathbf{q}^* \neq (0, 0)$) it follows that if a zero energy solution is bounded, non periodic and is defined for all times, its α and ω limits must be equilibrium points.

5.1 Description of the flow

It is known that if a solution $\mathbf{q}(t)$ of (2.7) is not defined for all times, it is unbounded.

For the following theorem we need a preparation lemma whose proof is found in the appendix.

Lemma 1. *All the solutions of $\ddot{x} = 4ax^3$, escape to infinity in finite time if $a > 0$.*

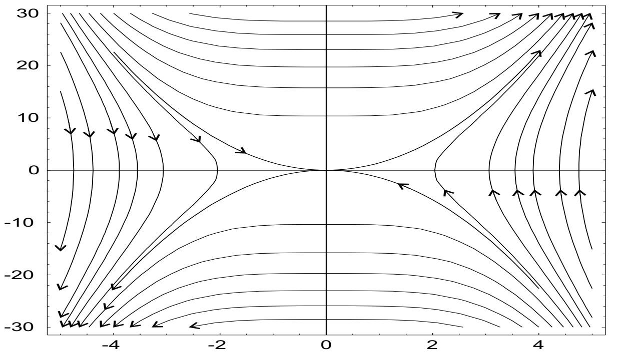


Figure 3: The phase portrait of $\ddot{x} = 4ax^3$.

In Figure 3 we show the phase portrait of the differential equation $\ddot{x} = 4ax^3$ with $a > 0$.

Theorem 3. *Let* $\mathbf{q}(t) = (x(t), y(t))$ *be a non equilibrium solution of* (2.7) *with* $h \geq 0$*, then*

1. *If* V *is positive definite or semidefinite then* $\mathbf{q}(t)$ *is unbounded.*
2. *If* V *is positive definite then* $\mathbf{q}(t)$ *is not defined for all times.*

Proof. To prove (1) we assume that $\mathbf{q}(t)$ is defined for all times and we will consider several cases. Consider first the case $h > 0$ and V positive semidefinite. For a fixed solution $\mathbf{q}(t) = (x(t), y(t))$ of the mechanical system (2.7), we define the function

$$g(t) = x^2(t) + y^2(t).$$

Differentiating twice respect to t and using the energy relation we get

$$\ddot{g}(t) = 2[2h + 6V(x(t), y(t))] \geq 4h. \tag{5.1}$$

By integration of this inequality we obtain

$$g(t) \geq 2h(t - t_0)^2 + 2\mathbf{q}(t_0) \cdot \mathbf{q}(t_0)\, t + g(t_0),$$

for any $t > t_0$. The conclusion follows by letting $t \to +\infty$.

Consider now the case V positive semidefinite and $h = 0$. By convenience, we denote $\mathbf{q}(t) \cdot \dot{\mathbf{q}}(t)$ by $f(t)$. It is clear that

$$\dot{g}(t) = 2f(t), \qquad \dot{f}(t) = 12V(\mathbf{q}(t)),$$

so that f is an increasing function. By taking into account our comments at the beginning of the section we conclude that $\mathbf{q}(t)$ is unbounded, since it is not an equilibrium solution.

For the positive definite case, we introduce polar coordinates in the plane, i.e. $\mathbf{q} = (x, y) = (r\cos\theta, r\sin\theta)$. Then the System (2.7) takes the form

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 + 4r^3 V(\theta) \\ (r^2\dot{\theta})^{\cdot} &= -V_x r\sin\theta + V_y r\cos\theta \end{aligned} \tag{5.2}$$

where $V(\theta) = V(\cos\theta, \sin\theta)$. Since $V > 0$, there is $\delta > 0$ such that $V(\theta) \geq \delta$. Hence, from the first equation in (5.2) we get

$$\ddot{r} \geq 4\delta r^3.$$

From Lemma 1 we conclude that for $V > 0$ and $h = 0$, the solution $\mathbf{q}(t)$ is unbounded. This proves item (2) also. So, the theorem is proved. □

Remark 1. From the proof of the above theorem it follows that

- If $h = 0$ and $f(t_0) = \mathbf{q}(t_0)\cdot\dot{\mathbf{q}}(t_0) > 0$, the solution $\mathbf{q}(t)$ escapes to infinity.
- The system (2.5) does not have periodic solutions when V is positive definite or semidefinite.
- In the particular case where $c = -2a = -2e = -1$, $d = 0$, the potential $V \geq 0$ takes the form $1/2(x^2 - y^2)^2$ and the corresponding system has the particular solution $x(t) = y(t) = \alpha t + \beta$ defined for any $t \in \mathbb{R}$ and its energy is given by $h = \alpha^2$. Hence, if $\alpha \neq 0$ the solution escape to infinity as stated in the above theorem. Moreover, in this example $f(t) = 2\alpha(\alpha t + \beta)$ takes positive and negative values.
- Suppose that $\mathbf{q}(t)$ is a zero energy solution of (2.7). If $f(t) = 0$ for any t then $g(t) = \mathit{constant}$ for any t, i.e., $x^2(t) + y^2(t) = \mathit{constant}$. From (5.1) we have $V(\mathbf{q}(t)) = 0$ and from the energy relation it follows that $\dot{x}(t) = \dot{y}(t) = 0$ for any t, and we have an equilibrium solution. Hence, if a solution $\mathbf{q}(t) = (x(t), y(t))$ of (2.7) with zero energy satisfies $x(t)\dot{x}(t) + y(t)\dot{y}(t) = 0$ for any t with V positive definite or semidefinite, then it is an equilibrium solution.
- When $h = 0$, $V(\theta(t))$ does not converge to 0 in finite time. Indeed, assuming that there exists $t^* \in \mathbb{R}$ such that $V(\theta(t)) \to 0$ when $t \to t^*$, we

get from the energy relation $0 = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - r^4 V(\theta)$ that $\dot{r}(t) \to 0$, $\dot{\theta}(t) \to 0$ when $t \to t^*$, so that the solution $\mathbf{q}(t)$ tends to the equilibrium $(0, 0)$, in finite time, which is a contradiction.

- It is easy to verify that the only potential of degree four such that $V(\theta) =$ constant is the potential $V = \alpha(x^2 + y^2)^2$ with α constant.

- From the above items we verify that there is no solution of the form $\mathbf{q}(t) = r(\cos\theta(t), \sin\theta(t))$ with r a positive constant for the case V positive definite or semidefinite and with energy $h \geq 0$.

5.2 Escape directions

To determine the escape directions we use a McGehee type blow up at the infinity which was analyzed by Lacomba and Ibort [11], for two degrees of freedom Hamiltonian systems of the form

$$H(x, y, p_1, p_2) = \frac{1}{2}\left(p_1^2 + p_2^2\right) - V(x, y),$$

where V is a homogeneous function of degree $\alpha > 0$. To apply a blow up at the infinity, we transform the configuration space coordinates to polar coordinates, but where the radial coordinate is replaced by its reciprocal. The new position coordinates ρ, θ, satisfy

$$\rho = \frac{1}{\sqrt{x^2 + y^2}};\ x = \frac{1}{\rho}\cos\theta;\ y = \frac{1}{\rho}\sin\theta.$$

In the new coordinates the behavior at infinity is determined by the so called *Infinity surface*, defined by

$$N_\infty = \left\{(\rho, \theta, v, u) \mid \rho = 0, \frac{1}{2}\left(u^2 + v^2\right) = V(\theta)\right\},$$

which is independent of h and invariant under the extended flow. Any solution escaping to infinity must tend to a hyperbolic equilibrium point of the flow on the infinity surface (see details in [8] and [9]).

The θ coordinate at an equilibrium point in N_∞ satisfies $V'(\theta) = 0$. In our case, we have

$$\begin{aligned} V'(\theta) \ =\ & \sin(\theta)\,[-4a\cos^3(\theta) - 2c\cos(\theta)\sin^2(\theta) + 2c\cos^3(\theta) \\ & - d\sin^3(\theta) + 3d\cos^2(\theta)\sin(\theta) + 4e\sin^2(\theta)\cos(\theta)]. \end{aligned} \tag{5.3}$$

To compute the number of roots of $V'(\theta) = 0$ we separate into two cases: $d \neq 0$ and $d = 0$. For the first case we have $V'(\pi/2) = -d \neq 0$, so that

$$V'(\theta) = \sin(\theta)\cos^3(\theta)[-dt^3 + (-2c+4e)t^2 + 3dt + (-4a+2c)], \quad t = \tan(\theta).$$

Remark 2. Notice that the cubic polynomial between brackets is minus the polynomial defined by equation (4.10) in Section 4. Hence, the homothetic solutions run along the escape directions.

From this remark we see that for $d \neq 0$, there are at most 3 real roots for the cubic polynomial. Hence, for any root t^* we have $\tan(\theta^*) = t^*$. Consequently, there are at most six values of θ (between 0 and 2π).

When $d = 0$ and $\theta \neq \pi/2$ we have

$$V'(\theta) = 2\sin(\theta)\cos^3(\theta)\,[(c-2a) + (2e-c)t^2], \quad t = \tan(\theta).$$

Summarizing

Proposition 3. *Let $V \geq 0$, then there are at most 8 escape directions. If $d \neq 0$, two of them correspond to $\theta = 0$ and $\theta = \pi$ (x axis). If $d = 0$ four of them correspond to $\theta = 0$, $\theta = \pi$ (x axis), $\theta = \pi/2$ and $\theta = 3\pi/2$ (y axis).*

6 The mechanical problem for $V \leq 0$

In this section we will assume that the potential V is negative definite or semidefinite, which implies that $a \leq 0$ and $e \leq 0$. If $V < 0$ the level surfaces are compact manifolds for each $h \geq 0$, since they are given by

$$H = \frac{1}{2}\|\dot{\mathbf{q}}\| - V(\mathbf{q}) = h.$$

So, we get the following result

Proposition 4. *If $V < 0$ and $h \geq 0$, then the solutions $\mathbf{q}(t)$ of the system (2.7) are defined for any $t \in \mathbb{R}$ and they are bounded.*

This is in contrast with the positive semidefinite case, (see Theorem 3). If $V(x, y) = 0$ for some $(x, y) \neq (0, 0)$ and V is negative semidefinite, the Hill region is unbounded. Indeed, if $V = 0$ along the line $y = kx$, the cylinder $\{1/2(\dot{x}^2 + \dot{y}^2) = h, y = kx\}$ is contained in the h energy level. When $h = 0$ we get a line which can be considered as a degenerate cylinder.

6.1 The periodic solution $y = \dot{y} = 0$

The system (2.7) always admits the periodic solution

$$y = \dot{y} = 0,$$

which we denote by Γ, and it satisfies

$$\ddot{x} = 4ax^3,$$

and from the energy relation we have

$$\frac{1}{2}\dot{x}^2 - ax^4 = h.$$

We see that the initial position x_0, determines from the energy relation the initial velocity (except by the sign) which is given by

$$v_0 = \dot{x}_0 = \pm\sqrt{2h + ax_0^4}. \tag{6.1}$$

We remark that the subset $\{x = \dot{x} = 0\}$ is invariant under the system (2.7), if and only if $d = 0$.

Replacing λ by x and μ by a as in Section 4, we have

$$\begin{aligned} x(t) &= \left(\frac{h}{|a|}\right)^{1/4} cn\left(2\,(|a|h)^{1/4}\,t,\, \frac{1}{\sqrt{2}}\right) \\ &= x_0\, cn\left(2\,(|a|h)^{1/4}\,t,\, \frac{1}{\sqrt{2}}\right), \end{aligned} \tag{6.2}$$

$$\begin{aligned} \dot{x}(t) &= -2\,(|a|h)^{1/4}\, x_0\, sn\left(2\,(|a|h)^{1/4}\,t,\, \frac{1}{\sqrt{2}}\right)\, dn\left(2\,(|a|h)^{1/4}\,t,\, \frac{1}{\sqrt{2}}\right), \\ \dot{x}(0) &= 0, \end{aligned} \tag{6.3}$$

for the initial condition

$$x_0 = \left[\frac{h}{|a|}\right]^{1/4} > 0, \quad v_0 = \dot{x}_0 = 0. \tag{6.4}$$

The period is given by

$$T_x = \frac{2}{(|a|h)^{1/4}}\,\kappa, \tag{6.5}$$

with κ as in (4.7). Notice that

- T_x is increasing if and only if h is decreasing. In particular, $T_x \to +\infty$ if and only if $h \to 0$.
- $x(t) = 0$, if and only if, $t = \frac{1}{4}(1+2k)\ T_x$, $k \in \mathbb{Z}$. Therefore, the first positive time where $x(t) = 0$ is $t_1 = \frac{T_x}{4}$.
- Since the function $dn(,) > 0$, it follows that $\dot{x}(t) = 0$, if and only if,

  $$sn\left(\sqrt{2}\,(|a|h)^{1/4}\,t, \frac{1}{\sqrt{2}}\right) = 0\,.$$

  This happens when

  $$t = \frac{1}{2}(1+k)T_x, \quad k \in \mathbb{Z}.$$

  Hence the first positive time where $\dot{x}(t) = 0$ is $t_3 = \frac{T_x}{2}$.

6.1.1 Stability of Γ

To study the stability of any T-periodic solution $\gamma(t) = (x(t), y(t), \dot{x}(t), \dot{y}(t))$ we use Floquet theory. To apply this we write the variational equations for (2.7) associated to the solution γ; that is

$$\dot{X} = A(t)X, \quad X = (u_1, u_2, v_1, v_2), \tag{6.6}$$

where

$$A(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 12ax^2(t) + 2cy^2(t) & [4cx(t) + 3dy(t)]y(t) & 0 & 0 \\ [4cx(t) + 3dy(t)]y(t) & 2cx^2(t) + 12ey^2(t) + 6dx(t)y(t) & 0 & 0 \end{pmatrix}, \tag{6.7}$$

with $A(t+T) = A(t)$.

The (linear) stability of γ is equivalent to the stability of the zero solution of system (6.6). We recall some important properties for Hill's equations (see for example, [10]).

Consider the Hill equation

$$\ddot{u} + p(t)\,u = 0, \quad u, t \in \mathbb{R},$$

where $p(t+T) = p(t)$ is of class C^0 in $\mathbb{R}$, which can be rewritten as

$$\dot{u} = v, \quad \dot{v} = -p(t)u. \tag{6.8}$$

So, we have the following remark.

Remark 3.

(1) Any solution of (6.8) is defined for all $t \in \mathbb{R}$.

(2) If $p(-t) = p(t)$ then the solution of (6.8) with initial conditions $u(0) = 1$ and $\dot{u}(0) = 0$ is even, and it is odd if the initial conditions are $u(0) = 0$ and $\dot{u}(0) = 1$.

(3) If $u(t)$ is a solution of (6.8) then $\tilde{u}(t) = u(t)k(t)$ with $k(t) = \int_0^t 1/u^2(s)\,ds$ is also a solution of (6.8) which is linearly independent with u. Hence, $\{u, \tilde{u}\}$ is a basis of the space of solutions for (6.8).

(4) If the T-periodic function $p(t)$ is non positive but it is not identically zero, then the zero solution of the Hill equation(6.8) is unstable.

(5) Under the conditions of the last item, we see that a characteristic multiplier has module greater than one and the other multiplier has module lesser than one.

The matrix A corresponding to the variational equations for our periodic solution Γ in Subsection 6.1, is

$$A(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 12ax^2(t) & 0 & 0 & 0 \\ 0 & 2cx^2(t) & 0 & 0 \end{pmatrix}. \tag{6.9}$$

Equation (6.6) becomes

$$\ddot{u}_1 - 12ax^2(t)u_1 = 0, \tag{6.10}$$

and

$$\ddot{u}_2 - 2cx^2(t)u_2 = 0, \tag{6.11}$$

i.e., it corresponds to a pair of Hill equations. From point 4 in Remark 6.1.1 (by taking $p(t) = -2cx^2(t)$ and $c > 0$) we get

Proposition 5. *If* $c > 0$ *then the zero solution of* (6.11) *is unstable.*

Since the variational equations associated to the periodic solution $(x(t), 0, \dot{x}(t), 0)$ are uncoupled, from this proposition and item 5 in Remark 6.1.1 we get

Theorem 4. *If* $c > 0$*,* $a < 0$*,* $e < 0$*,* $d \in \mathbb{R}$*, the periodic solution* $\Gamma : (x(t), 0, \dot{x}(t), 0)$ *of system* (2.7) *is Liapunov unstable.*

We consider now the case $c \leq 0$. We first see that

$$p_1(t) = 12|a|x^2(t) = 12|a| \left(\frac{h}{|a|}\right)^{1/2} cn^2\left(2\,(|a|h)^{1/4}\,t, \frac{1}{\sqrt{2}}\right) \geq 0$$

where $p_1(t)$ is a $T = \frac{T_x}{2}$ -periodic function. We now verify that the Hill equation (6.10) is the variational equation for the periodic solution $(x(t), \dot{x}(t))$ of the Hamiltonian system

$$\dot{x} = y, \quad \dot{y} = 4ax^3,$$

with Hamiltonian function

$$H = \frac{1}{2}y^2 - ax^4.$$

Since the system is time-independent with a first integral it is known that the characteristic multiplier 1 of the monodromy matrix $X(T)$ for the system (6.10), has multiplicity 2. From Floquet theory the system (6.10) has at least one T-periodic solution which we denote by $u_1(t)$. The proof of the following result is contained in the appendix.

Lemma 2. *Assume that the Hill equation* $\ddot{u} + p(t)u = 0$ *(*$p(t+T) = p(t)$*, of class* C^1 *in* $\mathbb{R}$*) has the characteristic multiplier 1 with multiplicity 2, then the monodromy matrix is not diagonalizable.*

From this lemma, we see that solution $u_2(t)$ is unbounded, so that the zero solution of Hill equation is unstable. This proves the following

Theorem 5. *The periodic solution* $(x(t), 0, \dot{x}(t), 0)$ *of system* (2.7) *is linearly unstable for any* $a < 0$*,* $c \leq 0$*,* $e < 0$ *and* $d \in \mathbb{R}$*.*

Remark 4. Recall that when $c > 0$ and $a < 0$, $e < 0$, $d \in \mathbb{R}$ the periodic solution $(x(t), 0, \dot{x}(t), 0)$ is Liapunov unstable according to Theorem 4. However, for the case $c \leq 0$ we have only proved in Theorem 5 the linear instability.

7 Some particular cases

The case $V < 0$ is complicated due to the great variety of dynamical behavior, as shown by the following examples.

7.1 The case V negative definite with $d = 0$

In this case the equations of motion (2.7) take the form

$$\begin{aligned} \ddot{x} &= 2x\,[2ax^2 + cy^2] \\ \ddot{y} &= 2y\,[cx^2 + 2ey^2]. \end{aligned} \tag{7.1}$$

The invariance of the potential V under the reflections with respect to the coordinate axes implies the same for the system (7.1) as well as the symmetry of the Hill region R with respect to the same axes. Hence, it is enough to study the Hill region in the first quadrant. In this case the Hill region is compact and we will compute the curvature of its zero velocity curve which is defined by

$$V(x, y) = ax^4 + cx^2y^2 + ey^4 = -h. \tag{7.2}$$

If $x = 0$, this relation implies (for $e \neq 0$) that $y = \pm(-h/e)^{1/4}$. Hence the tangent vector to the zero velocity curve at the point $(0, (-h/e)^{1/4})$ is parallel to the x axis. In the same way, the tangent vector is parallel to the y axis at a point of the form $((-h/e)^{1/4}, 0)$.

The oriented curvature for a curve defined by $y = y(x)$ is given by

$$C = \frac{y''}{[1 + y'^2]^{3/2}}, \qquad \text{where} \quad {}' = \frac{d}{dx}.$$

By implicit differentiation on (7.2) we get

$$\begin{aligned} C &= \frac{-V_x^2 V_{yy} - V_y^2 V_{xx} + 2V_x V_y V_{xy}}{[V_x^2 + V_y^2]^{3/2}} \\ &= x^4\, V(x, y)\, \frac{[-2ec\, t^4 + (c^2 - 12ae)\, t^2 - 2ac]}{[4\,(4a^2x^6 + 4e^2y^6 + \{(c^2 + 4ce)y^2 + (c^2 + 4ae)x^2\}x^2y^2)]^{3/2}}, \\ t &= \frac{y}{x}. \end{aligned} \tag{7.3}$$

The sign of the curvature depends on the zeroes of the polynomial

$$q(u) = 2ec\, u^2 - (c^2 - 12ae)\, u + 2ac \quad u = t^2,$$

which are given by

$$u_\pm = \frac{1}{2ec}\left[(c^2 - 12ae) \pm \sqrt{(c^2 - 12ae)^2 - 16aec^2}\right],$$

but only positive roots are admissible. Then we get

Proposition 6. *If the concavity of the zero velocity curve* $V = -h$ *changes, then* $[c^2 - 12ae]^2 \geq 16aec^2 \geq 0$ *and* $c^2 - 12ae > 0$*. The inflection points are given by* $y = t_\pm x$*, with* $t_\pm = \pm\sqrt{u_\pm}$*.*

On the other hand, if we assume that $a < 0$, $c < 0$ and $e < 0$ the expressions

$$2ax^2 + cy^2, \quad cx^2 + 2ey^2$$

are negative definite, so that

$$\operatorname{sgn}(\ddot{x}) = -\operatorname{sgn}(x) \quad \text{and} \quad \operatorname{sgn}(\ddot{y}) = -\operatorname{sgn}(y). \tag{7.4}$$

We now state some properties and introduce some notation for system (7.1), when $a < 0$, $c < 0$ and $e < 0$.

1. Since any solution of (7.1) is analytic and it is defined for all $t \in \mathbb{R}$, is not true that $x(t)$ or $y(t)$ have a fixed sign for all $t \in \mathbb{R}$. In fact, we see that if $x(t) > 0$ for every $t \in \mathbb{R}$, then from (7.1), $\ddot{x}(t) < 0$ holds for every $t \in \mathbb{R}$, which is a contradiction. The same argument works in any case.

2. From the above item it follows that there exist first positive times $t_1 \in \mathbb{R}^+$ and $t_2 \in \mathbb{R}^+$ such that $x(t_1) = 0$ and $y(t_2) = 0$.

3. From relations (7.4) we have that $\ddot{x}(t_1) = 0$ and $\ddot{y}(t_2) = 0$. This means that x and y change concavity at times t_1 and t_2, respectively .

4. It is not possible that $\dot{x}(t) > 0$ (resp. $\dot{y}(t) > 0$) for each $t \in \mathbb{R}$. Indeed, if $\dot{x}(t) > 0$ for $t \in \mathbb{R}$, then $x(t)$ is a strictly increasing function and therefore $x(t) > x(t_1)$. Let us fix $t^* > t_1$ such that $x(t^*) = \delta > 0$. Then $2ax^2(t) + cy^2(t) < 2a\delta^2 < 0$. Hence, $2x(t)\,[2ax^2(t) + cy^2(t)] < 4a\delta^3$. So,
$$\ddot{x}(t) < 4a\delta^3, \quad \forall t > t^*.$$

   By integrating twice between t^* and t we get

$$x(t) < x(t^*) + \dot{x}(t^*)(t - t^*) + 2a\delta^3(t - t^*)^2.$$

   Letting $t \to +\infty$ we obtain

$$x(t) \to -\infty,$$

   since $a < 0$, which is a contradiction. The proof for $y(t)$ is similar.

5. From the above item, there are positive times t_3 and t_4 such that $\dot{x}(t_3) = 0$ and $\dot{y}(t_4) = 0$.

In the following propositions t_1, t_2, t_3 and t_4 are as defined above.

Proposition 7. *Let* $(x(t), y(t))$ *be a solution of* (7.1), *with initial condition* $(x(0), y(0)) = (x_0, 0)$, $(\dot{x}(0), \dot{y}(0)) = (0, v_0)$ *and* x_0, v_0 *are greater than zero, such that* $t_1 = t_2$. *Then,* $(x(t), y(t))$ *is* $T = 4t_1$*-periodic and its orbit is symmetrical with respect to the coordinate axes.*

Proof. Since system (7.1) is invariant under reflections $(x, y) \to (-x, y)$ and $(x, y) \to (x, -y)$, and the orbit of $(x(t), y(t))$ is perpendicular to the x axis in $t = 0$, it will be enough to trace a quarter of orbit. Due to $x(t_1) = y(t_1) = 0$, we only need to prove that at $t = t_1$ the orbit is not tangent to the y axis. We have that $\dot{x}(t)$ will be decreasing in the interval $(0, t_1)$, since $x(t) > 0$ for $t \in (0, t_1)$. Hence

$$\dot{x}(t) < \dot{x}(t_*) := \lambda < \dot{x}(0) = 0,$$

where $t_* \in (0, t_1)$ is a fixed but arbitrary time. By continuity we have

$$\dot{x}(t_1) \leq \lambda < 0,$$

which concludes the proof. □

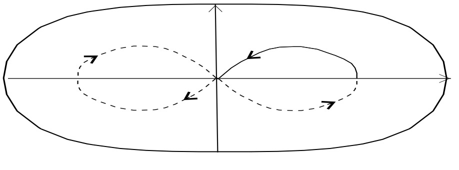


Figure 4: Symmetrical periodic solution passing through the $(0, 0)$.

Proposition 8. *Let* $(x(t), y(t))$ *be a solution of* (7.1), *with initial condition* $(x(0), y(0)) = (x_0, 0)$, $(\dot{x}(0), \dot{y}(0)) = (0, v_0)$, *where* x_0, v_0 *are greater than zero, such that* $t_3 = t_4$. *Then* $(x(t), y(t))$ *is* $T = 4t_3$ *periodic and its orbit is symmetrical with respect to the* x *axis.*

Proof. We see that the orbit of $x(t), y(t))$ touches the zero velocity curve since $\dot{x}(t_3) = \dot{y}(t_3) = 0$; its periodicity and symmetry with respect to the x axis follows from the initial condition. □

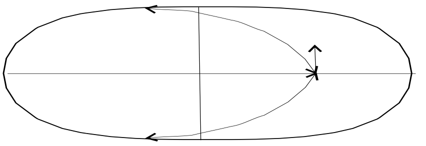


Figure 5: A periodic solution touching the zero velocity curve.

In this case ($d = 0$), we have two particular periodic solutions, one of them is of the form $(x(t), 0, \dot{x}(t), 0)$ where $x(t)$ is given by (6.2) whose stability is described by Theorems 4-5. The other one is given by $x = \dot{x} = 0$, which is written in terms of an elliptic function

$$\begin{aligned} y(t) &= \sqrt[4]{\frac{h}{|e|}}\, cn\left(2\sqrt[4]{|e|h}\, t, \frac{1}{\sqrt{2}}\right) \\ &= y_0\, cn\left(2\sqrt[4]{|e|h}\, t, \frac{1}{\sqrt{2}}\right), \quad y_0 = \sqrt[4]{\frac{h}{|e|}}, \end{aligned} \tag{7.5}$$

whose period is

$$T_y = \frac{2}{(|e|h)^{1/4}}\,\kappa. \tag{7.6}$$

We summarize some properties of this solution.

- $y(t) = 0$, if and only if, $t = \frac{1+2k}{4} T_y$, $k \in \mathbb{Z}$.
- $\dot{y}(t) = 0$, if and only if, $sn(\sqrt{4|e|}\; y_0 t, \sqrt{2}/2) = 0$. This holds when $t = \frac{1}{2}(1+k)T_y, \quad k \in \mathbb{Z}$.
- It is possible to construct a periodic solution satisfying $\tilde{y}(0) = 0$ and $\dot{\tilde{y}} > 0$. Indeed, we define

$$\tilde{y}(t) = y(t + \frac{3}{4}T_y). \tag{7.7}$$

  We see that

$$\tilde{y}(0) = y\left(\frac{3}{4}T_y\right) = 0, \;\; \dot{\tilde{y}}(0) = \sqrt{4|e|}y_0^2\, dn\left(\frac{3}{4}\, T_y\sqrt{4|e|}y_0, \sqrt{2}/2\right) > 0.$$

- Notice that the first positive time such that $\tilde{y}(t) = 0$ is given by $t_2 = \frac{1}{2}T_y$.
- The first positive time such that $\dot{\tilde{y}}(t) = 0$ is $t_4 = \frac{T_y}{4}$.

The analysis of the stability of the periodic solution $(0, y(t), 0, \dot{y}(t))$ of system (7.1) is similar to the one for the solution $(x(t), 0, \dot{x}(t), 0)$ in 6.1.1, replacing a by e. In this case, the matrix A in (6.7), reduces to

$$A(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2cy^2(t) & 3dy^2(t) & 0 & 0 \\ 3dy^2(t) & 12ey^2(t) & 0 & 0 \end{pmatrix}. \tag{7.8}$$

Hence the corresponding variational equation (6.6) is

$$\ddot{u}_1 - 2cy^2(t)u_1 = 0, \quad \ddot{u}_2 - 12ey^2(t)u_2 = 0. \tag{7.9}$$

As a corollary to Theorems 4-5 applied to the case $d = 0$, we have

Theorem 6. *If $a < 0$, $c > 0$, $e < 0$ and $d = 0$ then the periodic solution $(0, y(t), 0, \dot{y}(t))$ of system (7.1) is Liapunov unstable.*

If $a < 0$, $c \leq 0$, $e < 0$, $d = 0$ then the periodic solution $(0, y(t), 0, \dot{y}(t))$ of system (7.1) is linearly unstable.

7.1.1 The particular case $V = -x^4 - 2\alpha x^2y^2 - y^4 < 0$, $\alpha \geq 0$

In this case $a = -1$, $c = -2\alpha$, $d = 0$ and $e = -1$. According to Theorem 5, we know that the periodic solution $(x(t), 0, \dot{x}(t), 0)$ is linearly unstable. The stability analysis of this periodic solution was made in [1] for α close to 3. In the integrable case $\alpha = 3$ the variational equation (6.6) has two equal blocks, which means that the associated Hill equations (6.10) and (6.11) coincide. Using the Krein-Lynbarskii Theorem (see [19]) one can prove that for values of α of the form $\alpha = 3 + \epsilon$ with ϵ in one of the following intervals $0 < \epsilon < \delta$ or $-\delta < \epsilon < 0$, (δ is a small positive number) the characteristic multipliers of the second block in the monodromy matrix are outside the unit circle. Then, the corresponding periodic solution $(x(t), 0, \dot{x}(t), 0)$ for that α is Liapunov unstable since at least one characteristic multiplier has module greater than 1. We remark that this argument can not be applied to potentials in Subsection 7.1, since the periodic solution $x(t)$ there, does depend on a.

Consider now $\alpha = 1$, that is, $V = -(x^2 + y^2)^2$. From the energy relation we have

$$h = \frac{1}{2}(p_1^2 + p_2^2) + (x^2 + y^2)^2.$$

The equations of motion (2.7) take the form

$$\begin{aligned} \ddot{x} &= -4x\ (x^2 + y^2), \\ \ddot{y} &= -4y\ (x^2 + y^2). \end{aligned} \tag{7.10}$$

In this case

$$\nabla V(x, y) = -4(x^2 + y^2)\ (x, y),$$

i.e., $\nabla V(x, y)$ is parallel to (x, y). Notice that system (7.10) is integrable; the first integrals are the energy and the angular momentum $C = \mathbf{q} \times \dot{\mathbf{q}}$. We see that the lines $y = mx$ ($m \in \mathbb{R}$), are invariant for the system (7.10), which are the homothetic solutions. These solutions can be found explicitly by solving the system

$$\begin{aligned} \ddot{x} &= -4(1+m^2)\ x^3, \\ \frac{\dot{x}^2}{2} &= \frac{h}{1+m^2} - (1+m^2)x^4. \end{aligned} \tag{7.11}$$

Integrating the energy relation as we did in previous sections we see that the solutions of the system (7.11) are periodic and can be written as

$$x(t) = x_0\ cn(\sqrt{4\ (1+m^2)}\ x_0\ t, \sqrt{2}/2) \tag{7.12}$$

with initial condition

$$x_0 = \left(\frac{h}{1+m^2}\right)^{1/4},$$

and period

$$T = \frac{2}{h^{1/4}(1+m^2)^{1/4}}\ \kappa.$$

Since we have $a = -1$, $e = -1$, $c = -2$, $d = 0$ in system (7.10), we can apply Theorem 5. Hence, this periodic solution is linearly unstable.

7.1.2 The particular case $V(x, y) = ax^4 + ey^4 < 0$

We now consider that $a < 0$ and $e < 0$. The equations of motion (2.7) become the uncoupled system

$$\ddot{x} = 4ax^3, \quad \ddot{y} = 4ey^3. \tag{7.13}$$

The energy of the system is the Hamiltonian function

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax^4 - ey^4. \tag{7.14}$$

We split any fixed energy level as

$$H = h = h_1 + h_2 > 0,$$

where

$$h_1 = \frac{1}{2}\dot{x}^2 - ax^4, \tag{7.15}$$

and

$$h_2 = \frac{1}{2}\dot{y}^2 - ey^4. \tag{7.16}$$

Hence, the System (7.13) is integrable. Notice that $h_1,\ h_2$ must be non negative, and generically the corresponding invariant submanifold is a torus.

By integration, we can express the solution by elliptic functions of the first kind

$$x(t) = x_0\ cn(2\sqrt{|a|}\,x_0 t, \sqrt{2}/2), \quad x(0) = x_0 = \left(\frac{h_1}{|a|}\right)^{1/4} > 0,$$

and

$$y(t) = y_0\ cn(2\sqrt{|e|}\,y_0 t, \sqrt{2}/2), \quad y(0) = y_0 = \left(\frac{h_2}{|e|}\right)^{1/4} > 0.$$

The following properties hold for $x(t)$ and $y(t)$,

1. $x(t)$ is T_1 periodic, where $T_1 = \frac{2}{[h_1|a|]^{1/4}}\,\kappa = \frac{2\kappa}{\sqrt{h_1}}\,x_0 = \frac{2\kappa}{\sqrt{|a|}}\,\frac{1}{x_0}$.
2. $y(t)$ is T_2 periodic, where $T_2 = \frac{2}{[h_2|e|]^{1/4}}\,\kappa = \frac{2\kappa}{\sqrt{h_2}}\,y_0 = \frac{2\kappa}{\sqrt{|e|}}\,\frac{1}{y_0}$.
3. $\dot{x}(t) = -2\sqrt{|a|}\,x_0^2\ sn(\sqrt{4|a|}\,x_0 t, \sqrt{2}/2)\ dn(2\sqrt{|a|}\,x_0 t, \sqrt{2}/2)$, $\dot{x}(0) = 0$.
4. $\dot{y}(t) = -\sqrt{4|e|}\,y_0^2\ sn(\sqrt{4|e|}\,y_0 t, \sqrt{2}/2)\ dn(\sqrt{4|e|}\,y_0 t, \sqrt{2}/2)$, $\dot{y}(0) = 0$.

Then the solution $(x(t), y(t), \dot{x}(t), \dot{y}(t))$ is periodic, if and only if,

$$\frac{T_1}{T_2} = \frac{[h_2|e|]^{1/4}}{[h_1|a|]^{1/4}} \in \mathbb{Q}. \tag{7.17}$$

Equation (6.6) gives rise to the uncoupled system

$$\ddot{u}_1 - 12ax^2(t)u_1 = 0, \quad \ddot{u}_2 - 12ex^2(t)u_2 = 0. \tag{7.18}$$

Since the System (7.13) is uncoupled, its stability depends on the one of any of the components of the solution, say $(x(t), \dot{x}(t))$. From the above properties 1, 2, we get the following result.

Theorem 7. *The solutions of* (7.13) *are orbitally stable, although they are Liapunov linearly unstable.*

Proof. Liapunov instability follows as in Theorem 4, because of the uncoupling of the variational system. $\square$

8 Non integrability conditions

In this section we apply some results on non integrability of Hamiltonian polynomial systems proved in [17]. According to this, we solve first the equation

$$\nabla V(q) = -q,$$

where V is given by (2.6). We get the system

$$x(4ax^2 + 2cy^2) + dy^3 = -x$$

$$y(2cx^2 + 3dxy + 4ey^2) = -y.$$

We have to consider 3 cases:

(1) If $x = 0$ and $d \neq 0$, then we get the trivial solution $y = 0$.

(2) If $x = 0$ and $d = 0$, then $y = \pm\frac{1}{2\sqrt{-e}}$ with $e < 0$, obtaining the solutions $p_1 = \left(0, \pm\frac{1}{2\sqrt{-e}}\right)$, where $e < 0$ and $d = 0$.

(3) If $y = 0$ and $x \neq 0$, we get $x = \pm\frac{1}{2\sqrt{-a}}$ with $a < 0$, giving the solutions $p_2 = \left(\pm\frac{1}{2\sqrt{-a}}, 0\right)$, where $a < 0$.

The Hessian Matrix of $-V(x, y)$ is in general

$$Hess(-V) = \begin{pmatrix} -(12ax^2 + 2cy^2) & -(4cxy + 3dy^2) \\ -(4cxy + 3dy^2) & -(2cx^2 + 6dxy + 12ey^2) \end{pmatrix}.$$

Replacing the above solution points we get for the case (2)

$$A(p_1) = \begin{pmatrix} \frac{c}{2e} & 0 \\ 0 & 3 \end{pmatrix},$$

with $e < 0$ and $d = 0$, whose eigenvalues are $\lambda_1 = \frac{c}{2e}$ and $\lambda_2 = 3$.

For case (3) we have

$$A(p_2) = \begin{pmatrix} 3 & 0 \\ 0 & \frac{c}{2a} \end{pmatrix},$$

with $a < 0$. In this case, the eigenvalues are $\lambda_1 = 3$, and $\lambda_2 = \frac{c}{2a}$.

For these non trivial cases and using Theorem 3 in [17], we see that a necessary condition for a Hamiltonian system with homogeneous potential of degree 4 to be completely integrable is that the eigenvalues λ_i of $Hess(-V)$ can be written as

(i) $p(2p-1)$;

(ii) $-\dfrac{1}{8} + \dfrac{1}{8}\left(\dfrac{4}{3} + 4p\right)^2$;

(iii) $\dfrac{1}{2}\left[\dfrac{3}{4} + 4p(p+1)\right]$, where $p \in \mathbb{Z}$.

The common eigenvalue 3 satisfies condition (i). Then, the non integrability depends on whether the remaining eigenvalue does not satisfy any of the above conditions.

Proposition 9. *For the solutions* $p_2 = \left(\pm \frac{1}{2\sqrt{-a}}, 0\right)$ *in case* **(3)**, *we see that the Hamiltonian system* (2.2) *with* V *given by* (2.6), *is not integrable if any of the following relations is satisfied*

(1) $\alpha = \dfrac{c}{4a} \leq \dfrac{-1}{16}$,

(2) α *is not a solution of any of the following equations*

$$\begin{gathered}(p-1/4)^2 = \alpha + 1/16, \quad (p+1/3)^2 = \alpha + 1/16, \\ (p+1/2)^2 = \alpha + 1/16,\end{gathered} \tag{8.1}$$

for $p \in \mathbb{Z}$. *A similar result holds for the solutions* $p_1 = \left(0, \pm\frac{1}{2\sqrt{-e}}\right)$ *in case* **(2)** *with* $e < 0$ *and* $d = 0$, *replacing* a *by* e *in* **(1)** *and* **(2)**.

Proof. By equating the eigenvalue 2α with the above expressions (i)-(iii), we get equations (8.1). Since the left hand side of these equations is a non negative number, α must be greater than $-1/16$. On the other hand, (8.1) does not have an integer solution when $\alpha = -1/16$. □

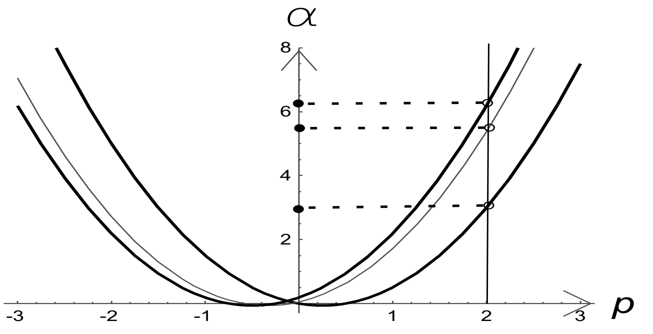


Figure 6: The α coordinate of the intersections points of the line $p = n,\ n \in \mathbb{Z}$ with the parabolas are values for which integrability is undecidable (see (8.1)).

The Figure 6 describes how to get the values of α, for which integrability is undecidable. Notice that case (2) with $c = 0$ is integrable.

The analysis of solutions where $x \neq 0$ and $y \neq 0$ is harder. When $d = 0$, we get

$$\begin{aligned} &-4ax^2 - 2cy^2 = 1 \\ &-2cx^2 - 4cy^2 = 1, \end{aligned} \tag{8.2}$$

where we obtain the additional cases

(4) If in addition $c = 0$, we get an integrable system since it is separable.

(5) If $a = 0$ and $c \neq 0$, we obtain $y^2 = -\frac{1}{2c}$, $x^2 = -\frac{1}{2c} + \frac{e}{c^2}$.

(6) If $e = 0$ and $c \neq 0$, we have $x^2 = -\frac{1}{2c}$, $y^2 = -\frac{1}{2c} + \frac{a}{c^2}$.

We analyze now case (6) which is similar to (5). We must have $c < 0$ and since $0 \le y^2 = \frac{2a-c}{2c^2}$ it is necessary that $a \ge \frac{c}{2}$ holds. This gives the solution points

$$p_3 = \left(\delta_1/\sqrt{-2c}, \delta_2\sqrt{2a-c}/\sqrt{2}|c|\right),$$

where $\delta_j = \pm 1$. The characteristic polynomial for the Hessian matrix at these points is

$$\lambda^2 + (\Delta + 1)\lambda + 3(\Delta - 2) = 0,$$

where $\Delta = 1 + 4c/a$. Its roots are

$$\lambda_1 = \Delta - 2 = -1 + 4c/a := \beta, \quad \lambda_2 = 3.$$

From necessary conditions (i-ii-iii) for integrability we have

$$(p-1/4)^2 = \beta/2 + 1/16, \quad (p+1/3)^2 = \beta/2 + 1/16, \\ (p+1/2)^2 = \beta/2 + 1/16, \tag{8.3}$$

where $p \in \mathbb{Z}$. Replacing α by $\beta/2$ in the above figure, we get the values of β for which the integrability is undecidable. Even for the case where $d = 0$ it is hard to obtain generic results since the solutions of System (8.2) correspond to the intersection of two conic curves, which depends on the parameters a, c and e.

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9 Appendix

Lemma 1. *If $a > 0$, all the solutions of $\ddot{x} = 4ax^3$, escape to infinity in finite time.*

Proof. The energy relation is given by

$$h = \frac{1}{2}\dot{x}^2 - ax^4.$$

Hence,

$$\frac{dx}{\sqrt{h + ax^4}} = \pm\sqrt{2}dt.$$

On the other hand,

$$\sqrt{1 + ax^4} \leq 1 + \sqrt{a}\, x^2 \leq \sqrt{2}\sqrt{1 + ax^4},$$

if $h = 1$. Then,

$$\frac{dx}{1 + \sqrt{a}x^2} \leq \pm\sqrt{2}dt \leq \sqrt{2}\, \frac{dx}{1 + \sqrt{a}x^2},$$

and upon integration

$$\frac{1}{a^{1/4}}\, \tan(\pm a^{1/4}\, t + \arctan(a^{1/4}x_0)) \leq x(t) \\ \leq \frac{1}{a^{1/4}}\, \tan(\pm\sqrt{2}a^{1/4}\, t + \arctan(a^{1/4}x_0)),$$

for each t where the solution $x(t)$ is defined.

When $h = 0$ by direct integration we get

$$x(t) = \frac{x_0}{\mp x_0\sqrt{2a}t + 1}.$$

For $h < 0$, and due to $x > a^{-1/4}$ we obtain

$$(a^{1/4}x - 1)^{3/2} < \sqrt{-1 + ax^4} < 1 + \sqrt{a}x^2,$$

taking $h = -1$. Therefore,

$$\frac{dx}{1 + \sqrt{a}x^2} \leq \pm\sqrt{2}dt \leq \frac{dx}{(a^{1/4}x - 1)^{3/2}},$$

and by integrating again

$$\frac{1}{\sqrt[4]{a}}\left[1 + \frac{1}{\left(\mp\frac{\sqrt{2}}{2}\sqrt[4]{a}\, t + \frac{1}{\sqrt{\sqrt[4]{a}x_0 - 1}}\right)^2}\right] \leq x(t)$$

$$\leq \frac{1}{\sqrt[4]{a}}\tan(\pm\sqrt[4]{a}\sqrt{2}t + \arctan(\sqrt[4]{a}x_0)),$$

from where the proof follows. □

Lemma 2. *Assume that Hill equation $\ddot{u} + p(t)u = 0$, where $p(t + T) = p(t)$ is of class C^1 in $\mathbb{R}$ has 1 as a characteristic multiplier of multiplicity two, then the monodromy matrix is not diagonalizable.*

Proof. Let $u_1(t)$ be the T-periodic solution of Hill equation with initial conditions $u_1(0) = u_1^0$ and $\dot{u}_1(0) = \dot{u}_1^0$. From property (3) in Subsection 6.1.1 we know that a solution linearly independent with $u_1(t)$ is $u_2(t) = u_1(t)k(t)$ where $k(t) = \int_0^t 1/u_1^2(s)\, ds$. Hence, the monodromy matrix becomes

$$X(T) = \begin{pmatrix} u_1(T) & u_2(T) \\ \dot{u}_1(T) & \dot{u}_2(T) \end{pmatrix}.$$

Since $\lambda = 1$ is an eigenvalue of multiplicity two, the following relation

$$\lambda^2 - 2\lambda + 1 = \lambda^2 - Tr(X(T))\,\lambda + det(X(T))$$

holds. Therefore,

$$\begin{aligned} &\text{(i)} \quad u_1(T) + \dot{u}_2(T) = 2, \\ &\text{(ii)} \quad u_1(T)\dot{u}_2(T) - u_2(T)\dot{u}_1(T) = 1. \end{aligned} \tag{9.1}$$

We have two cases: (1) $u_1^0 \neq 0$, and (2) $u_1^0 = 0$.

In the first case we assume without loss of generality that $u_1^0 = 1$, so that $u_1(T) = 1$. This fact together with definition of u_2, implies that $u_2(T) = k(T) > 0$. From equation (i) we obtain $\dot{u}_2(T) = 1$. By replacing the values of $u_1(T)$ and $\dot{u}_2(T)$ in equation (ii), we conclude that $\dot{u}_1(T) = 0$. Then we have

$$X(T) = \begin{pmatrix} 1 & k(T) \\ 0 & 1 \end{pmatrix}.$$

In the second case, we assume without loss of generality that $\dot{u}_1(0) = 1$ and then $\dot{u}_1(T) = 1$. Since $u_1(T) = u_1^0 = 0$, it follows from (i) that $\dot{u}_2(T) = 2$ and from (ii) we obtain $u_2(T) = -1$. Therefore, we get

$$X(T) = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}.$$

Summarizing, the monodromy matrix $X(T)$ in both cases is not diagonalizable. □

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