Bull Braz Math Soc, New Series 38(3), 397-425

Explicit examples of nonsolvable weakly hyperbolic operators with real coefficients

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Abstract. We give in this paper two explicit examples of nonsolvable weakly hyperbolic operators with real coefficients in two-space-dimensions.

Keywords: solvability, doubly characteristic operators, weakly hyperbolic operators.

Mathematical subject classification: 35A07.

1 Introduction

We provide here some explicit examples of nonsolvable weakly hyperbolic operators with real coefficients. These are, with $(t, x, y) \in \mathbb{R}^3$,

$$L_1 = \partial_t(\partial_t + y\partial_x) + \partial_y,$$
$$L_2 = \partial_t^2 - H(-y)|y|^k\partial_x^2 + \partial_y,\ k \in \mathbb{N}^*,\ H = 1_{\mathbb{R}_+},$$

where the notation $1_{\mathbb{R}_+}$ stands for the characteristic function of the set $\mathbb{R}_+$. Both examples are weakly hyperbolic operators in two-space-dimensions. The operator L_1 has affine coefficients and the operator L_2 has coefficients in C^{k-1}. Y.V. Egorov gave in [2] an example of a nonsolvable weakly hyperbolic operator in one-space-dimension with a quite complicated expression. Although our examples are 2-space-dimensional, we feel that their simple expression is worth noticing.

Let us begin by recalling some results about solvability for pseudo-differential operators with real principal symbols. Let L be a classical pseudo-differential operator on an open set Ω of $\mathbb{R}^n$ with a real principal symbol a_m. The double characteristic set is defined as

$$\Sigma_2 = \big\{(x, \xi) \in \dot{T}^*(\Omega) : a_m(x, \xi) = 0,\ d_\xi a_m(x, \xi) = 0\big\},$$

Received 19 October 2006.

where $\dot{T}^*(\Omega)$ is the cotangent bundle minus the zero section.

- If the set Σ_2 is empty, the operator L is of strong-real-principal-type and local solvability with a loss of one derivative holds according to the theorem 26.1.7 in [4].
- In the case where the operator L has a real principal symbol a_m such that its subprincipal symbol a^s_{m-1} satisfies

$$a_m(x,\xi)=0,\ d_\xi a_m(x,\xi)=0 \Rightarrow \operatorname{Im} a^s_{m-1}(x,\xi)\neq 0, \tag{1.0.1}$$

  if $(x,\xi)\in\dot{T}^*(\Omega)$, N. Lerner has proved in the theorem 1.1 of [5] that there is also local solvability with a loss of one derivative. For example, this is the case of most of the operators of the type

$$AB+C,$$

  where A, B, C are smooth real vector fields in $\mathbb{R}^3$ such that A, B and $[A,B]$ are linearly independent, for which F. Treves has shown in [8] that they are locally solvable.
- If we now assume that the set

$$\tilde{\Sigma}_2=\Big\{(x,\xi)\in\dot{T}^*(\Omega)\colon a_m(x,\xi)=0,\\ d_\xi a_m(x,\xi)=0,\ \operatorname{Im} a^s_{m-1}(x,\xi)=0\Big\},$$

  is non-empty, different situations can occur. For example, for the class of operators $AB+C$ studied by F. Treves in [8], the set $\tilde{\Sigma}_2$ can be non-empty, but the special structure of the principal symbol which appears as a product pq with $\{p,q\}\neq 0$ at $p=q=0$, allows this author to obtain a solvability result with a loss of derivatives. The set $\tilde{\Sigma}_2$ can also be non-empty in the cases studied by G.A. Mendoza and G.A. Uhlmann in [7], for which they introduced the additional assumption $\mathrm{Sub}(\mathcal{P})$, also with a product structure (of involutive type) for the principal symbol.

Let us mention that there is a nice example in [1] of an operator verifying (1.0.1), which is therefore locally solvable although a quasi–homogeneous version of condition (Ψ) is violated in that case. For the operators L_1 and L_2, the set $\tilde{\Sigma}_2$ is non-empty. The nonsolvability in any neighbourhood of 0 in $\mathbb{R}^3$ of the operator L_1 is a consequence of the result of nonsolvability proved by G.A. Mendoza and G.A. Uhlmann in the theorem 1.2 of [7]. We verify in this case that the operator L_1 violates the condition Sub($\mathcal{P}$) defined in [6] and [7]. To prove the nonsolvability in any neighbourhood of 0 for the operator with C^{k-1} coefficients L_2, we prove by building a quasimode that **no** a priori estimates of the following type could hold

$$\exists C_0 > 0,\ \exists N_0 \in \mathbb{N},\ \exists V_0 \text{ an open neighbourhood of 0 in } \mathbb{R}^3 \text{ such that}$$
$$\forall u \in C_0^\infty(V_0),\ C_0\|L_2^* u\|_{(k-3)} \geq \|u\|_{(-N_0)},$$

where the notation $\|\cdot\|_{(s)}$ stands for the $H^s(\mathbb{R}^3)$ Sobolev norm. This fact induces that there do **not** exist an integer $N_0 \in \mathbb{N}$ and an open neighbourhood V_0 of 0 in $\mathbb{R}^3$ such that for all $f \in H^{N_0}(V_0)$, there exists $u \in H^{-k+3}(\mathbb{R}^3)$ such that

$$L_2 u = f,$$

on V_0 (let us notice that the quantity $L_2 u$ is well defined for $u \in H^{-k+3}(\mathbb{R}^3)$). Indeed if it was the case, we would have using similar arguments to the ones given by L. Hörmander in the proof of Lemma 26.4.5 in [4] that for all $v \in C_0^\infty(V_0)$,

$$|(f, v)_{L^2(V_0)}| = |(L_2 u, v)| = |(u, L_2^* v)| \leq \|u\|_{(-k+3)}\|L_2^* v\|_{(k-3)}. \tag{1.0.2}$$

Let us consider

$$\begin{array}{rcl} T_v:\ H^{N_0}(V_0) & \rightarrow & \mathbb{C} \\ f & \mapsto & (f, v)_{L^2(V_0)}, \end{array}$$

for v in $C_0^\infty(V_0)$. We deduce from the previous estimate that for all f in $H^{N_0}(V_0)$, there exists $u \in H^{-k+3}(\mathbb{R}^3)$ such that

$$\sup_{v \in W} |T_v(f)| \leq \|u\|_{(-k+3)} < +\infty,$$

if $W = \{v \in C_0^\infty(V_0), \|L_2^* v\|_{(k-3)} \leq 1\}$. Since T_v is a bounded linear form for v in W, we deduce from the uniform boundedness principle that there exists a positive constant C_0 such that

$$\sup_{v \in W} \|T_v\| \leq C_0 < +\infty.$$

It follows that for all $f \in H^{N_0}(V_0)$ and $v \in C_0^\infty(V_0)$, $\|L_2^* v\|_{(k-3)} \leq 1$, we have

$$|(f, v)_{L^2(V_0)}| \leq C_0\|f\|_{(N_0)},$$

which induces by homogeneity that for all $f \in H^{N_0}(V_0)$ and $v \in C_0^\infty(V_0)$,

$$|(f, v)_{L^2(V_0)}| \leq C_0\|f\|_{(N_0)}\|L_2^* v\|_{(k-3)}, \tag{1.0.3}$$

if $\|L_2^*v\|_{(k-3)} \neq 0$. According to (1.0.2), we notice that this estimate (1.0.3) is also fulfilled if $\|L_2^*v\|_{(k-3)} = 0$. Using now that $\|T_v\| = \|v\|_{(-N_0)}$ for all v in $C_0^\infty(V_0)$, we obtain from (1.0.3) that the following estimate

$$\forall v \in C_0^\infty(V_0),\ C_0\|L_2^*v\|_{(k-3)} \geq \|v\|_{(-N_0)},$$

holds, which is not possible according to our result.

2 Nonsolvability of the operator L_1

The operator L_1 is defined in standard quantization (and also in Weyl quantization) by the symbol

$$p(t, x, y; \tau, \xi, \eta) = -\tau(\tau + y\xi) + i\eta.$$

We first notice that its principal symbol, $p_2 = -\tau(\tau + y\xi)$, is real and that the doubly characteristic set

$$\Sigma_2(L_1) = \big\{(t, x, y; \tau, \xi, \eta) \in \dot{T}^*(\mathbb{R}^3) : p_2 = 0,\ d_{\tau,\xi,\eta}p_2 = 0\big\},$$

where $\dot{T}^*(\mathbb{R}^3)$ stands for the cotangent bundle minus the zero section, is not empty since

$$\begin{aligned}\Sigma_2(L_1) \ &= \ \big\{(t, x, y; \tau, \xi, \eta) \in \dot{T}^*(\mathbb{R}^3) : y = \tau = 0,\ (\xi, \eta) \neq (0, 0)\big\}\\ &\qquad \cup \big\{(t, x, y; \tau, \xi, \eta) \in \dot{T}^*(\mathbb{R}^3) : \tau = \xi = 0,\ \eta \neq 0\big\}.\end{aligned}$$

Let us consider the two real-valued symbols $q = -\tau$ and $s = \tau + y\xi$, we have $p_2 = qs$. The set $\Sigma_2(L_1)$ is a submanifold of codimension 2 near the point $\nu_0 = (t_0, x_0, 0; 0, 1, 0) \in \Sigma_2(L_1)$ if $t_0, x_0 \in \mathbb{R}$, which is involutive since

$$\begin{aligned}\big(T_\nu \Sigma_2(L_1)\big)^\sigma \ &= \ \big\{(t, x, y; \tau, \xi, \eta) \in \mathbb{R}^6 : x = y = \tau = \xi = 0\big\}\\ &\qquad \subset T_\nu \Sigma_2(L_1) = \big\{(t, x, y; \tau, \xi, \eta) \in \mathbb{R}^6 : y = \tau = 0\big\},\end{aligned}$$

for all ν belonging to a neighbourhood of ν_0 in $\Sigma_2(L_1)$ if $T_\nu \Sigma_2(L_1)$ stands for the tangent plane of $\Sigma_2(L_1)$ in ν. We also notice that the Hamilton vector fields H_q, H_s and the radial vector field r, which are equal to

$$H_q = -\frac{\partial}{\partial t},\ H_s = \frac{\partial}{\partial t} - \xi\frac{\partial}{\partial \eta},\ r = \xi\frac{\partial}{\partial \xi} + \eta\frac{\partial}{\partial \eta},$$

at points in $\Sigma_2(L_1)$ near ν_0, are independent and that the imaginary part of the subprincipal symbol, $p_1^s = i\eta$, changes sign at the first order in 0 along the following bicharacteristic of the symbol s,

$$\begin{cases} \gamma'(t) = H_s\big(\gamma(t)\big) \\ \gamma(0) = \nu_0, \end{cases}$$

since $\mathrm{Im}\ p_1^s(\nu_0) = 0$ and

$$\begin{aligned} \frac{d}{dt}\big[\mathrm{Im}\ p_1^s\big(\gamma(t)\big)\big]\Big|_{t=0} &= d\,\mathrm{Im}\ p_1^s\big(\gamma(t)\big).H_s\big(\gamma(t)\big)\Big|_{t=0} \\ &= \sigma\Big(H_s(\gamma(t)),\, H_{\mathrm{Im}\ p_1^s}(\gamma(t))\Big)\Big|_{t=0} \\ &= \big\{s, \mathrm{Im}\ p_1^s\big\}\big(\gamma(t)\big)\Big|_{t=0} = -1 \neq 0. \end{aligned}$$

It follows that the condition $\mathrm{Sub}(\mathcal{P})$ defined by G.A. Mendoza and G.A. Uhlmann in [7] is violated and we deduce from Theorem 1.2 in [7] that the operator L_1 is not locally solvable at $\nu_0 \in \Sigma_2(L_1)$, which induces that the operator L_1 is nonsolvable in any neighbourhood of 0 in $\mathbb{R}^3$.

3 Nonsolvability of the operator L_2

The second operator L_2 that we study, is defined in standard quantization (and also in Weyl quantization) by the symbol

$$p = i\eta + \big(\theta_k(y)\xi^2 - \tau^2\big) = i\big(\eta + i(\tau^2 - \theta_k(y)\xi^2)\big),$$

where θ_k is the $C^{k-1}(\mathbb{R}, \mathbb{R})$ function defined for $k \in \mathbb{N}^*$ by

$$\theta_k(y) = (-1)^k y^k H(-y) \text{ if } H = 1_{\mathbb{R}_+},$$

where the notation 1_X stands for characteristic function of the set X. We notice that its principal symbol, $p_2 = \theta_k(y)\xi^2 - \tau^2$, is a real C^{k-1} symbol and that the doubly characteristic set

$$\begin{aligned} \Sigma_2(L_2) &= \big\{(t, x, y; \tau, \xi, \eta) \in \dot{T}^*(\mathbb{R}^3) \colon\ p_2 = 0,\ d_{\tau,\xi,\eta} p_2 = 0\big\} \\ &= \big\{(t, x, y; \tau, \xi, \eta) \in \dot{T}^*(\mathbb{R}^3) \colon\ \tau = 0,\ y \in \mathbb{R}_+\big\} \\ &\quad \cup \big\{(t, x, y; \tau, \xi, \eta) \in \dot{T}^*(\mathbb{R}^3) \colon\ \tau = \xi = 0\big\}, \end{aligned}$$

is not empty. This set contains some points, $(t, x, 0; 0, \pm 1, 0) \in \Sigma_2(L_2)$, where the imaginary part of the subprincipal symbol vanishes, $p_1^s = i\eta$. Then, we notice that since the function $y \mapsto \tau^2 - \theta_k(y)\xi^2$ changes sign from $-$ to $+$ whenever $\tau\xi \neq 0$ if y increases, the symbol p violates a quasi-homogeneous version of the condition (Ψ).

3.1 Construction of a quasimode

Let us consider $N_0 \in \mathbb{N}$,

$$\psi_1 \in C_0^\infty(\mathbb{R}^2, \mathbb{R}),\ \text{supp}\ \psi_1 \subset [1,4]^2,\ \psi_1 = 1 \text{ on } [2,3]^2, \tag{3.1.1}$$

$$\chi_0 \in C_0^\infty(\mathbb{R}, \mathbb{R}),\ \text{supp}\ \chi_0 \subset [-1,1],\ \chi_0 = 1 \text{ on } [-1/2, 1/2], \tag{3.1.2}$$

some positive parameters α and μ such that

$$\frac{1}{k} < \alpha < \frac{2}{k} \quad \text{and} \quad \frac{2}{k} < \mu < \alpha + \frac{1}{k}, \tag{3.1.3}$$

where k is the integer appearing in the definition of the operator L_2. We set for all $\lambda \geq 1$,

$$\psi_\lambda(\tau, \xi) = \lambda^{-\frac{1}{2}-\alpha}\psi_1(\lambda^{-\alpha}\tau, \lambda^{-1-\alpha}\xi). \tag{3.1.4}$$

Let us note supp $\chi_0\big(\lambda^\mu(\cdot + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}})\big)$ for the support of the function

$$y \mapsto \chi_0\big(\lambda^\mu(y + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}})\big).$$

Since using (3.1.2), we have for all $(\tau, \xi) \in [1,4]^2$ and $\lambda \geq 1$,

$$\begin{aligned}\text{supp}\ \chi_0\big(\lambda^\mu(\cdot + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}})\big) &\subset \big\{y \in \mathbb{R} : |y + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}| \leq \lambda^{-\mu}\big\} \\ &\subset \big\{y \in \mathbb{R} : -\lambda^{-\mu} - 4^{\frac{2}{k}}\lambda^{-\frac{2}{k}} \leq y \leq \lambda^{-\mu} - 4^{-\frac{2}{k}}\lambda^{-\frac{2}{k}}\big\},\end{aligned}$$

it follows from (3.1.3), $2/k < \mu$, that we can find a constant $\lambda_0 \geq 1$ and some positive constants c_1, c_2 such that $c_1 > c_2$ and for all $(\tau, \xi) \in [1,4]^2$, $\lambda \geq \lambda_0$,

$$\text{supp}\ \chi_0\big(\lambda^\mu(\cdot + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}})\big) \subset \big\{y \in \mathbb{R} : -c_1\lambda^{-\frac{2}{k}} \leq y \leq -c_2\lambda^{-\frac{2}{k}}\big\}. \tag{3.1.5}$$

Let us notice that since

$$L_2^* = -\partial_y + \theta_k(y)D_x^2 - D_t^2,\ \theta_k(y) = (-1)^k y^k H(-y),\ H = 1_{\mathbb{R}_+}, \tag{3.1.6}$$

with $D_x = i^{-1}\partial_x$, $D_t = i^{-1}\partial_t$, and since the function $y \mapsto \tau^2 - \theta_k(y)\xi^2$ changes sign from $-$ to $+$ at $y = -(\tau\xi^{-1})^{2/k}$ if $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, we can find a non-negative phase function Φ_1, which satisfies the equation

$$\begin{aligned}&\big(-\partial_y + \theta_k(y)\xi^2 - \tau^2\big)\big(e^{-\Phi_1(\tau,\xi,y)}\big) \\ = \ &\big(\partial_y\Phi_1(\tau,\xi,y) + \theta_k(y)\xi^2 - \tau^2\big)e^{-\Phi_1(\tau,\xi,y)} = 0,\end{aligned} \tag{3.1.7}$$

defined for all $y \in \mathbb{R}_-^*$ and $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ by

$$\Phi_1(\tau, \xi, y) = \int_{-(\tau\xi^{-1})^{\frac{2}{k}}}^{y} \big(\tau^2 - \theta_k(s)\xi^2\big)ds. \tag{3.1.8}$$

Indeed, since from (3.1.6) and (3.1.8),

$$\frac{\partial^2 \Phi_1}{\partial y^2}(\tau, \xi, y) = k(-y)^{k-1}\xi^2 \geq 0,$$

if $y \in \mathbb{R}_-^*$ and $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, the function $y \mapsto \Phi_1(\tau, \xi, y)$ is convex on $\mathbb{R}_-^*$ and we deduce from the fact

$$\Phi_1\Big(\tau, \xi, -(\tau\xi^{-1})^{\frac{2}{k}}\Big) = 0, \ \frac{\partial \Phi_1}{\partial y}\Big(\tau, \xi, -(\tau\xi^{-1})^{\frac{2}{k}}\Big) = 0 \tag{3.1.9}$$

and from the Taylor formula that

$$\begin{aligned} &\Phi_1(\tau, \xi, y) \\ &= \big(y + (\tau\xi^{-1})^{\frac{2}{k}}\big)^2 k\xi^2 \int_0^1 (1-\theta)\big((\tau\xi^{-1})^{\frac{2}{k}}(1-\theta) - \theta y\big)^{k-1} d\theta, \end{aligned} \tag{3.1.10}$$

if $y \in \mathbb{R}_-^*$ and $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. The property of non-negativity of the function Φ_1 on $\mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_-^*$ is clear on the formula (3.1.10). We also set for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\begin{aligned} \Phi_\lambda(\tau, \xi, y) &= \Phi_1(\lambda^\alpha \tau, \lambda^{1+\alpha}\xi, y) \\ &= \int_{-(\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}}^{y} \big(\lambda^{2\alpha}\tau^2 - \theta_k(s)\lambda^{2+2\alpha}\xi^2\big)ds, \end{aligned} \tag{3.1.11}$$

which is also a non-negative function. A direct computation shows from (3.1.6) and (3.1.11) that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\begin{aligned} \Phi_\lambda(\tau, \xi, y) &= \lambda^{2\alpha}\tau^2 y + \frac{k}{k+1}\lambda^{2\alpha - \frac{2}{k}}\tau^{2+\frac{2}{k}}\xi^{-\frac{2}{k}} \\ &\quad + \frac{(-1)^{k+1}}{k+1}\lambda^{2+2\alpha}\xi^2 y^{k+1}. \end{aligned} \tag{3.1.12}$$

We can now define for all $\lambda \geq \lambda_0$ the function u_λ defined by

$$\begin{aligned} &u_\lambda(t, x, y) \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i(x\xi + t\tau)}\psi_\lambda(\tau, \xi)\chi_0\big(\lambda^\mu(y + (\tau\xi^{-1})^{\frac{2}{k}})\big)e^{-\Phi_1(\tau,\xi,y)}d\tau d\xi. \end{aligned} \tag{3.1.13}$$

If we note $\mathcal{F}_{t,x}$ the Fourier transform in the variables t, x, it follows from (3.1.13) that

$$\begin{aligned} U_\lambda(\tau,\xi,y) &= \big(\mathcal{F}_{t,x}u_\lambda\big)(\tau,\xi,y) \\ &= \psi_\lambda(\tau,\xi)\chi_0\big(\lambda^\mu(y+(\tau\xi^{-1})^{\frac{2}{k}})\big)e^{-\Phi_1(\tau,\xi,y)} \end{aligned} \tag{3.1.14}$$

and we can notice from (3.1.1), (3.1.4), (3.1.5) and the change of variables $(\tilde{\tau},\tilde{\xi}) = (\lambda^{-\alpha}\tau, \lambda^{-1-\alpha}\xi)$ that for all $(\tau,\xi) \in \operatorname{supp}\psi_\lambda$ and $\lambda \geq \lambda_0$,

$$\operatorname{supp} U_\lambda(\tau,\xi,\cdot) \subset \mathbb{R}_-^*. \tag{3.1.15}$$

In view of (3.1.10), (3.1.14) and (3.1.15), it follows that the family $(u_\lambda)_{\lambda\geq\lambda_0}$ belongs to the space $C^\infty\big(\mathbb{R}_y, S(\mathbb{R}^2_{t,x})\big)$ (because (3.1.1) and (3.1.4) imply that the function ψ_λ has a compact support in $\mathbb{R}_+^* \times \mathbb{R}_+^*$) and has its support included in the set $\mathbb{R}^2_{t,x} \times (\mathbb{R}_y)_-$. We deduce from this fact and (3.1.6) that

$$L_2^* u_\lambda \in C^\infty(\mathbb{R}^3_{t,x,y})3 \tag{3.1.16}$$

and a direct computation using (3.1.6), (3.1.7) and (3.1.14) gives that

$$\begin{aligned} \mathcal{F}_{t,x}(L_2^*u_\lambda) &= -\partial_y U_\lambda(\tau,\xi,y) + \big(\theta_k(y)\xi^2-\tau^2\big)U_\lambda(\tau,\xi,y) \\ &= -\lambda^\mu\psi_\lambda(\tau,\xi)\chi_0'\big(\lambda^\mu(y+(\tau\xi^{-1})^{\frac{2}{k}})\big)e^{-\Phi_1(\tau,\xi,y)}. \end{aligned} \tag{3.1.17}$$

3.2 Upper bound for $\|L_2^*u_\lambda\|_{(N_0)}$

From (3.1.17) and Parseval's formula, we notice that to obtain an upper bound for the quantity $\|L_2^*u_\lambda\|_{(N_0)}$, it is enough to get an upper bound for the quantities

$$\begin{aligned} A_{j_1,j_2,j_3,j_4}(\lambda) = \|&\lambda^{\mu(j_1+1)}\chi_0^{(j_1+1)}\big(\lambda^\mu(y+(\tau\xi^{-1})^{\frac{2}{k}})\big) \\ &\tau^{j_2}\xi^{j_3}\psi_\lambda(\tau,\xi)\partial_y^{j_4}\big(e^{-\Phi_1(\tau,\xi,y)}\big)\|_{L^2(\mathbb{R}^3)}, \end{aligned} \tag{3.2.1}$$

where $(j_1,j_2,j_3,j_4) \in \mathbb{N}^4$ are some integers such that $j_1+j_2+j_3+j_4 = N_0$. Using a change of variables, (3.1.4) and (3.1.11), we obtain that

$$\begin{aligned} A_{j_1,j_2,j_3,j_4}(\lambda)^2 = \; &\lambda^{2\mu(j_1+1)+2\alpha j_2+2j_3(1+\alpha)} \\ &\int_{\mathbb{R}^3}\Big[\chi_0^{(j_1+1)}\Big(\lambda^\mu(y+(\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}})\Big)^2 \\ &\times\; \tau^{2j_2}\xi^{2j_3}\psi_1(\tau,\xi)^2\big|\partial_y^{j_4}\big(e^{-\Phi_\lambda(\tau,\xi,y)}\big)\big|^2\Big]dy\,d\tau\,d\xi. \end{aligned} \tag{3.2.2}$$

Let us stress the fact that from (3.1.1), (3.1.5), (3.1.10) and (3.1.11), if $(\tau, \xi) \in$ supp ψ_1 and $\lambda \geq \lambda_0$, the function $\Phi_\lambda(\tau, \xi, \cdot)$ is C^∞ on

$$\text{supp}\ \chi_0^{(j_1+1)}\big(\lambda^\mu(\cdot + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}})\big) \subset \mathbb{R}_-^*.$$

Thus, the expression (3.2.2) is well-defined. We need now the following lemma.

Lemma 3.2.1. *For all $\nu \in \mathbb{N}^3$, there exist some functions a_l, $l = 0, \ldots, |\nu|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants β_l, $l = 0, \ldots, |\nu|(k+1)$ verifying*

$$\beta_l \leq 2|\nu|(\alpha + 1),$$

such that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\partial^\nu_{\tau,\xi,y}\big(e^{-\Phi_\lambda}\big) = e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} a_l\left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right) y^l \lambda^{\beta_l}. \tag{3.2.3}$$

Proof. We prove this lemma by induction on $|\nu|$. If $|\nu| = 0$, the expression (3.2.3) holds with $a_0 = 1$ and $\beta_0 = 0$. Let us assume now that for $\nu \in \mathbb{N}^3$, there exist some functions a_l, $l = 0, \ldots, |\nu|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants β_l, $l = 0, \ldots, |\nu|(k+1)$ verifying $\beta_l \leq 2|\nu|(\alpha + 1)$ such that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\lambda \geq 1$, the expression (3.2.3) holds. Since from (3.1.12), we have for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\partial_y \Phi_\lambda(\tau, \xi, y) = \lambda^{2\alpha}\tau^2 + (-1)^{k+1}\lambda^{2+2\alpha}\xi^2 y^k, \tag{3.2.4}$$

$$\partial_\tau \Phi_\lambda(\tau, \xi, y) = 2\lambda^{2\alpha}\tau y + 2\lambda^{2\alpha - \frac{2}{k}}\xi^{-\frac{2}{k}}\tau^{1+\frac{2}{k}}, \tag{3.2.5}$$

$$\partial_\xi \Phi_\lambda(\tau, \xi, y) = \frac{2}{k+1}\Big[-\lambda^{2\alpha-\frac{2}{k}}\tau^{2+\frac{2}{k}}\xi^{-\frac{2}{k}-1} + (-1)^{k+1}\lambda^{2+2\alpha}\xi y^{k+1}\Big]. \tag{3.2.6}$$

We have also for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\begin{aligned}
&\partial_y\left(e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} a_l\left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right) y^l \lambda^{\beta_l}\right) \\
&= e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} a_l\left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right)\left[-(\partial_y\Phi_\lambda)y^l + l y^{l-1}\right]\lambda^{\beta_l},
\end{aligned} \tag{3.2.7}$$

$$\begin{aligned}
&\partial_\tau \left(e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} a_l \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right) y^l \lambda^{\beta_l} \right) \\
&\quad = e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} \lambda^{\beta_l} y^l \Big[-(\partial_\tau \Phi_\lambda)\, a_l \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right) \\
&\qquad + \partial_\tau \left(a_l \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right)\right) \Big]
\end{aligned} \tag{3.2.8}$$

and

$$\begin{aligned}
&\partial_\xi \left(e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} a_l \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right) y^l \lambda^{\beta_l} \right) \\
&\quad = e^{-\Phi_\lambda} \sum_{l=0}^{|\nu|(k+1)} \lambda^{\beta_l} y^l \Big[-(\partial_\xi \Phi_\lambda) a_l \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right) \\
&\qquad + \partial_\xi \left(a_l \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right)\right) \Big].
\end{aligned} \tag{3.2.9}$$

We deduce from (3.2.4), (3.2.5), (3.2.6), (3.2.7), (3.2.8) and (3.2.9) that if

$$\tilde{\nu} \in \{\nu + (1,0,0), \nu + (0,1,0), \nu + (0,0,1)\},$$

there exist some functions $\tilde{a}_l$, $l = 0, \dots, |\tilde{\nu}|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants $\tilde{\beta}_l$, $l = 0, \dots, |\tilde{\nu}|(k+1)$ verifying $\tilde{\beta}_l \leq 2|\tilde{\nu}|(\alpha+1)$ such that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\partial_{\tau,\xi,y}^{\tilde{\nu}} \big(e^{-\Phi_\lambda}\big) = e^{-\Phi_\lambda} \sum_{l=0}^{|\tilde{\nu}|(k+1)} \tilde{a}_l \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right) y^l \lambda^{\tilde{\beta}_l}. \tag{3.2.10}$$

Indeed, let us consider for example the case where $\tilde{\nu} = \nu + (0,0,1)$. We obtain from (3.2.3), (3.2.4) and (3.2.7) the expression

$$\begin{aligned}
\partial_{\tau,\xi,y}^{\tilde{\nu}} \big(e^{-\Phi_\lambda}\big) \;=\;& e^{-\Phi_\lambda} \sum_{l=0}^{|\tilde{\nu}|(k+1)-(k+2)} a_{l+1} \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right) (l+1) y^l \lambda^{\beta_{l+1}} \\
&- e^{-\Phi_\lambda} \sum_{l=0}^{|\tilde{\nu}|(k+1)-(k+1)} a_l \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right) \left(\tau^{\frac{1}{k}}\right)^{2k} y^l \lambda^{\beta_l + 2\alpha} \\
&- e^{-\Phi_\lambda} \sum_{l=k}^{|\tilde{\nu}|(k+1)-1} a_{l-k} \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right) (-1)^{k+1} \left(\xi^{\frac{1}{k}}\right)^{2k} y^l \lambda^{\beta_{l-k} + 2 + 2\alpha},
\end{aligned}$$

which can be written in the form (3.2.10). Since the power of λ is less or equal than $2+2\alpha$ in every term of the right-hand-side of (3.2.5) and (3.2.6), and that these terms are polynomial functions in the variables

$$\tau^{\frac{1}{k}},\quad \tau^{-\frac{1}{k}},\quad \xi^{\frac{1}{k}},\quad \xi^{-\frac{1}{k}}\quad \text{and}\quad y$$

with a degree in y lower than $k+1$, we have only to use (3.2.5), (3.2.6), (3.2.8), (3.2.9) and the fact that the quantities

$$\begin{aligned}\partial_\tau\left(a_l\left(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}\right)\right) &= \frac{1}{k}\tau^{\frac{1}{k}}\left(\tau^{-\frac{1}{k}}\right)^k(\partial_1 a_l)\left(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}\right)\\ &\quad-\frac{1}{k}\left(\tau^{-\frac{1}{k}}\right)^{k+1}(\partial_2 a_l)\left(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}\right)\end{aligned}$$

and

$$\begin{aligned}\partial_\xi\left(a_l\left(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}\right)\right) &= \frac{1}{k}\xi^{\frac{1}{k}}\left(\xi^{-\frac{1}{k}}\right)^k(\partial_3 a_l)\left(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}\right)\\ &\quad-\frac{1}{k}\left(\xi^{-\frac{1}{k}}\right)^{k+1}(\partial_4 a_l)\left(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}\right),\end{aligned}$$

are some polynomial functions in the variables $\tau^{\frac{1}{k}}$, $\tau^{-\frac{1}{k}}$, $\xi^{\frac{1}{k}}$ and $\xi^{-\frac{1}{k}}$, to obtain (3.2.10) when

$$\tilde{\nu}=\nu+(1,0,0)\quad\text{or}\quad\tilde{\nu}=\nu+(0,1,0).$$

This proves the induction property at the rank $|\nu|+1$ and ends the proof of the lemma 3.2.1. □

We deduce from (3.1.5) and the lemma 3.2.1 that there exists a positive constant C_{j_4} such that for all $(\tau,\xi)\in[1,4]^2$, $\lambda\geq\lambda_0$ and

$$y\in\operatorname{supp}\chi_0\left(\lambda^\mu\left(\cdot+\left(\lambda^{-1}\xi^{-1}\tau\right)^{\frac{2}{k}}\right)\right),$$

$$\left|\partial_y^{j_4}\left(e^{-\Phi_\lambda(\tau,\xi,y)}\right)\right|\leq C_{j_4}\lambda^{2j_4(1+\alpha)}e^{-\Phi_\lambda(\tau,\xi,y)}. \tag{3.2.11}$$

Moreover, we obtain from (3.1.2) and (3.1.5) that for all $(\tau,\xi)\in[1,4]^2$ and $\lambda\geq\lambda_0$,

$$\operatorname{supp}\chi_0^{(j_1+1)}\left(\lambda^\mu\left(\cdot+\left(\lambda^{-1}\xi^{-1}\tau\right)^{\frac{2}{k}}\right)\right)\subset\Omega_{\lambda,\tau,\xi}, \tag{3.2.12}$$

if we note

$$\Omega_{\lambda,\tau,\xi}=\left\{y\in\mathbb{R}_-^*:2^{-1}\lambda^{-\mu}\leq\left|y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}}\right|\leq\lambda^{-\mu}\right\}. \tag{3.2.13}$$

Then, we deduce from (3.1.1), (3.2.2), (3.2.11), (3.2.12) and (3.2.13) that for all $\lambda \geq \lambda_0$,

$$\begin{aligned} A_{j_1,j_2,j_3,j_4}(\lambda)^2 \; &\leq \; C_{j_4}^2 \lambda^{2\mu(j_1+1)+2j_2\alpha+2j_3(1+\alpha)+4j_4(1+\alpha)} \|\chi_0^{(j_1+1)}\|_{L^\infty(\mathbb{R})}^2 \\ &\quad \times \int_{\mathbb{R}^2} \tau^{2j_2}\xi^{2j_3}\psi_1(\tau,\xi)^2 \Big(\int_{\Omega_{\lambda,\tau,\xi}} e^{-2\Phi_\lambda(\tau,\xi,y)} dy \Big) d\tau d\xi \\ &\leq \; C_{j_4}^2 \lambda^{2\mu(j_1+1)+2j_2\alpha+2j_3(1+\alpha)+4j_4(1+\alpha)-\mu} \|\chi_0^{(j_1+1)}\|_{L^\infty(\mathbb{R})}^2 \\ &\quad \times \int_{\mathbb{R}^2} \tau^{2j_2}\xi^{2j_3}\psi_1(\tau,\xi)^2 \Big(\sup_{y \in \Omega_{\lambda,\tau,\xi}} e^{-2\Phi_\lambda(\tau,\xi,y)} \Big) d\tau d\xi. \end{aligned} \tag{3.2.14}$$

We obtain from (3.1.10) that for all $y \in \mathbb{R}_-^*$ and $(\tau,\xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$,

$$\begin{aligned} \Phi_1(\tau,\xi,y) \; &= \; \big(y + (\tau\xi^{-1})^{\frac{2}{k}}\big)^2 k\xi^2 \int_0^1 (1-\theta)\big((\tau\xi^{-1})^{\frac{2}{k}}(1-\theta) - \theta y\big)^{k-1} d\theta \\ &\geq \; \big(y + (\tau\xi^{-1})^{\frac{2}{k}}\big)^2 k\xi^2 \int_0^1 (1-\theta)^k (\tau\xi^{-1})^{2-\frac{2}{k}} d\theta, \end{aligned}$$

which induces that

$$\Phi_1(\tau,\xi,y) \geq \frac{k}{k+1}\xi^2(\tau\xi^{-1})^{2-\frac{2}{k}}\big(y + (\tau\xi^{-1})^{\frac{2}{k}}\big)^2. \tag{3.2.15}$$

It follows from (3.1.11) and (3.2.15) that for all $y \in \mathbb{R}_-^*$, $(\tau,\xi) \in [1,4]^2$ and $\lambda \geq 1$,

$$\begin{aligned} \Phi_\lambda(\tau,\xi,y) \; &\geq \; \frac{k}{k+1}\xi^2(\tau\xi^{-1})^{2-\frac{2}{k}}\lambda^{2\alpha+\frac{2}{k}}\big(y + (\tau\lambda^{-1}\xi^{-1})^{\frac{2}{k}}\big)^2 \\ &\geq \; c_3\lambda^{2\alpha+\frac{2}{k}}\big(y + (\tau\lambda^{-1}\xi^{-1})^{\frac{2}{k}}\big)^2, \end{aligned} \tag{3.2.16}$$

with $c_3 = 4^{\frac{2}{k}-2}k/(k+1) > 0$. Thus, we obtain using (3.2.13) and (3.2.16) that for all $(\tau,\xi) \in [1,4]^2$ and $\lambda \geq \lambda_0$,

$$\sup_{y \in \Omega_{\lambda,\tau,\xi}} e^{-2\Phi_\lambda(\tau,\xi,y)} \leq e^{-\frac{c_3}{2}\lambda^{2(\alpha+\frac{1}{k}-\mu)}}. \tag{3.2.17}$$

Getting back to (3.2.14), the next proposition follows from (3.1.1), (3.2.1), (3.2.14), (3.2.17) and the fact that from (3.1.3),

$$\mu < \alpha + \frac{1}{k}.$$

Proposition 3.2.1. *We have*

$$\|L_2^* u_\lambda\|_{(N_0)} = O\left(e^{-\frac{c_3}{8}\lambda^{2(\alpha+\frac{1}{k}-\mu)}}\right) \text{ when } \lambda \to +\infty. \tag{3.2.18}$$

3.3 Lower bound for the quantity $\|u_\lambda\|_{(-N_0)}$

It follows from (3.1.4), (3.1.11), (3.1.14) and a change of variables that

$$\begin{aligned}\|u_\lambda\|^2_{(-N_0)} &= \int_{\mathbb{R}^3} |\psi_\lambda(\tau,\xi)|^2 \\ &\quad \Big|\int_{\mathbb{R}} e^{-iy\eta}\chi_0\big(\lambda^\mu(y+(\xi^{-1}\tau)^{\frac{2}{k}})\big)e^{-\Phi_1(\tau,\xi,y)}dy\Big|^2 \\ &\quad \times (1+\eta^2+\xi^2+\tau^2)^{-N_0}\frac{d\eta d\tau d\xi}{(2\pi)^3} \\ &= \int_{\mathbb{R}^3} |\psi_1(\tau,\xi)|^2 \\ &\quad \Big|\int_{\mathbb{R}} e^{-iy\eta}\chi_0\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)e^{-\Phi_\lambda(\tau,\xi,y)}dy\Big|^2 \\ &\quad \times (1+\eta^2+\lambda^{2+2\alpha}\xi^2+\lambda^{2\alpha}\tau^2)^{-N_0}\frac{d\eta d\tau d\xi}{(2\pi)^3}.\end{aligned} \tag{3.3.1}$$

By using the following estimates, for all $(\tau,\xi) \in [1,4]^2$ and $\lambda \geq \lambda_0 \geq 1$,

$$1+\eta^2+\lambda^{2+2\alpha}\xi^2+\lambda^{2\alpha}\tau^2 \leq c_4\lambda^{2+2\alpha}(1+\eta^2),$$
$$(1+\eta^2+\lambda^{2+2\alpha}\xi^2+\lambda^{2\alpha}\tau^2)^{-N_0} \geq c_4^{-N_0}\lambda^{-2(1+\alpha)N_0}(1+\eta^2)^{-N_0},$$

where $c_4 = 33$ and from (3.1.1), supp $\psi_1 \subset [1,4]^2$ and $\psi_1 = 1$ on $[2,3]^2$, we deduce from (3.3.1) that for all $\lambda \geq \lambda_0$,

$$\|u_\lambda\|^2_{(-N_0)} \geq \frac{c_4^{-N_0}}{(2\pi)^2}\lambda^{-2(1+\alpha)N_0}\int_{[2,3]^2} \|g_{\lambda,\tau,\xi}\|^2_{H^{-N_0}(\mathbb{R}_y)} d\tau d\xi, \tag{3.3.2}$$

if we note

$$g_{\lambda,\tau,\xi}(y) = \chi_0\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)e^{-\Phi_\lambda(\tau,\xi,y)}. \tag{3.3.3}$$

Then, we use that

$$\|g_{\lambda,\tau,\xi}\|^2_{L^2(\mathbb{R}_y)} \leq \|g_{\lambda,\tau,\xi}\|_{H^{N_0}(\mathbb{R}_y)}\|g_{\lambda,\tau,\xi}\|_{H^{-N_0}(\mathbb{R}_y)}. \tag{3.3.4}$$

The following lemma allows us to get an uniform lower bound, respectively to get an uniform upper bound with respect to the variables (τ,ξ) in $[2,3]^2$ for the quantities $\|g_{\lambda,\tau,\xi}\|_{L^2(\mathbb{R}_y)}$ and $\|g_{\lambda,\tau,\xi}\|_{H^{N_0}(\mathbb{R}_y)}$.

Lemma 3.3.1. *We can find some positive constants* c_5 *and* c_6 *such that for all* $(\tau,\xi)\in[2,3]^2$ *and* $\lambda\geq\lambda_0$,

$$\|g_{\lambda,\tau,\xi}\|_{H^{N_0}(\mathbb{R}_y)}\leq c_5\lambda^{(\mu+2+2\alpha)N_0-\frac{\mu}{2}} \text{ and } \|g_{\lambda,\tau,\xi}\|_{L^2(\mathbb{R}_y)}\geq c_6\lambda^{-\frac{1+\alpha}{2}}. \tag{3.3.5}$$

Proof. To obtain the first estimate, it is enough to get a bound for $\lambda\geq\lambda_0$ of the new quantities

$$A_{j_1,j_2}(\lambda,\tau,\xi)=\|\lambda^{j_1\mu}\chi_0^{(j_1)}\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\partial_y^{j_2}\big(e^{-\Phi_\lambda(\tau,\xi,y)}\big)\|_{L^2(\mathbb{R}_y)},$$

uniformly with respect to the variables $(\tau,\xi)\in[2,3]^2$ where $(j_1,j_2)\in\mathbb{N}^2$ are some integers verifying $j_1+j_2=N_0$. We obtain using (3.2.11) and the non-negativity of the function Φ_λ that for all $(\tau,\xi)\in[2,3]^2$ and $\lambda\geq\lambda_0$,

$$\begin{aligned} A_{j_1,j_2}(\lambda,\tau,\xi)^2 &= \lambda^{2j_1\mu}\int_{\mathbb{R}}\chi_0^{(j_1)}\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)^2\big|\partial_y^{j_2}\big(e^{-\Phi_\lambda(\tau,\xi,y)}\big)\big|^2dy\\ &\leq C_{j_2}^2\lambda^{2j_1\mu+4j_2(1+\alpha)}\int_{\mathbb{R}}\chi_0^{(j_1)}\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)^2dy\\ &= C_{j_2}^2\|\chi_0^{(j_1)}\|_{L^2(\mathbb{R})}^2\lambda^{2j_1\mu+4j_2(1+\alpha)-\mu}. \end{aligned}$$

We deduce from this last estimate that there exists a positive constant c_5 such that for all $(\tau,\xi)\in[2,3]^2$ and $\lambda\geq\lambda_0$,

$$\|g_{\lambda,\tau,\xi}\|_{H^{N_0}(\mathbb{R}_y)}\ \leq\ c_5\lambda^{(\mu+2+2\alpha)N_0-\frac{\mu}{2}},$$

which shows the first estimate of (3.3.5). We want now to get an uniform lower bound for the quantity $\|g_{\lambda,\tau,\xi}\|_{L^2(\mathbb{R}_y)}$ with respect to the variables $(\tau,\xi)\in[2,3]^2$ for $\lambda\geq\lambda_0$. Using (3.1.10), we obtain that for all $y\in\mathbb{R}_-^*$ and $(\tau,\xi)\in\mathbb{R}_+^*\times\mathbb{R}_+^*$,

$$\begin{aligned} \Phi_1(\tau,\xi,y) &= \Big(y+(\tau\xi^{-1})^{\frac{2}{k}}\Big)^2k\xi^2\int_0^1(1-\theta)\Big((\tau\xi^{-1})^{\frac{2}{k}}(1-\theta)-\theta y\Big)^{k-1}d\theta\\ &\leq k\xi^2\Big(y+(\tau\xi^{-1})^{\frac{2}{k}}\Big)^2\Big((\tau\xi^{-1})^{\frac{2}{k}}+|y|\Big)^{k-1}. \end{aligned} \tag{3.3.6}$$

We deduce from (3.1.11) and (3.3.6) that for all $y\in\mathbb{R}_-^*$, $(\tau,\xi)\in[2,3]^2$ and $\lambda\geq1$,

$$\Phi_\lambda(\tau,\xi,y)\leq 9k\lambda^{2+2\alpha}\left(y+\Big(\tau\xi^{-1}\lambda^{-1}\Big)^{\frac{2}{k}}\right)^2\Big(3^{\frac{2}{k}}2^{-\frac{2}{k}}+|y|\Big)^{k-1}. \tag{3.3.7}$$

We obtain using (3.3.7) and the change of variables, $u = y + (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}$, that for all $(\tau, \xi) \in [2,3]^2$ and $\lambda \geq 1$,

$$\begin{aligned}
&\|e^{-\Phi_\lambda(\tau,\xi,y)} 1_{\mathbb{R}_-^*}(y)\|^2_{L^2(\mathbb{R}_y)} \\
&\geq \int_{-\infty}^{0} e^{-18k\lambda^{2+2\alpha}(y+(\tau\lambda^{-1}\xi^{-1})^{\frac{2}{k}})^2(3^{\frac{2}{k}}2^{-\frac{2}{k}}+|y|)^{k-1}} dy \\
&\geq \int_{-\infty}^{-(\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}} e^{-18k\lambda^{2+2\alpha}(y+(\tau\lambda^{-1}\xi^{-1})^{\frac{2}{k}})^2(3^{\frac{2}{k}}2^{-\frac{2}{k}}+|y|)^{k-1}} dy \qquad (3.3.8)\\
&\geq \int_{-\infty}^{0} e^{-18k\lambda^{2+2\alpha}u^2(3^{\frac{2}{k}}2^{-\frac{2}{k}}+|u-(\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}|)^{k-1}} du.
\end{aligned}$$

Since we can find a positive constant c_7 such that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in [2,3]^2$ and $\lambda \geq 1$,

$$18k\left(3^{\frac{2}{k}}2^{-\frac{2}{k}} + |y - (\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}}|\right)^{k-1} \leq c_7(1 + |y|^{k-1}), \qquad (3.3.9)$$

it follows from (3.3.8) and (3.3.9),

$$\|e^{-\Phi_\lambda(\tau,\xi,y)} 1_{\mathbb{R}_-^*}(y)\|^2_{L^2(\mathbb{R}_y)} \geq \int_{-\infty}^{0} e^{-c_7\lambda^{2+2\alpha}u^2(1+|u|^{k-1})} du. \qquad (3.3.10)$$

Then using some changes of variables, we deduce from (3.3.10) that for all $(\tau, \xi) \in [2,3]^2$ and $\lambda \geq 1$,

$$\begin{aligned}
&\|e^{-\Phi_\lambda(\tau,\xi,y)} 1_{\mathbb{R}_-^*}(y)\|^2_{L^2(\mathbb{R}_y)} \\
&\geq \int_{0}^{+\infty} e^{-c_7\lambda^{2+2\alpha}u^2(1+u^{k-1})} du \\
&= \lambda^{-1-\alpha}\int_{0}^{+\infty} e^{-c_7v^2(1+v^{k-1}\lambda^{-(1+\alpha)(k-1)})} dv \qquad (3.3.11)\\
&\geq \lambda^{-1-\alpha}\int_{0}^{+\infty} e^{-c_7v^2(1+v^{k-1})} dv = c_8\lambda^{-1-\alpha}.
\end{aligned}$$

Next, if we note

$$\tilde{\Omega}_{\lambda,\tau,\xi} = \left\{y \in \mathbb{R}_-^* : |y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}}| \geq 2^{-1}\lambda^{-\mu}\right\}, \qquad (3.3.12)$$

using from (3.1.2) that $\chi_0 = 1$ on $[-1/2, 1/2]$ and (3.2.16), we obtain that for all $(\tau,\xi) \in [2,3]^2$ and $\lambda \geq 1$,

$$\begin{aligned} &\big\|\big[1-\chi_0\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\big]e^{-\Phi_\lambda(\tau,\xi,y)}1_{\mathbb{R}_-^*}(y)\big\|^2_{L^2(\mathbb{R}_y)} \\ &\leq \|1-\chi_0\|^2_{L^\infty(\mathbb{R})}\int_{\tilde{\Omega}_{\lambda,\tau,\xi}} e^{-2\Phi_\lambda(\tau,\xi,y)}dy \\ &\leq \|1-\chi_0\|^2_{L^\infty(\mathbb{R})}\int_{\tilde{\Omega}_{\lambda,\tau,\xi}} e^{-2c_3\lambda^{2\alpha+\frac{2}{k}}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})^2}dy \qquad (3.3.13) \\ &\leq e^{-\frac{c_3}{4}\lambda^{2\alpha+\frac{2}{k}-2\mu}}\|1-\chi_0\|^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} e^{-c_3\lambda^{2\alpha+\frac{2}{k}}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})^2}dy \\ &\leq \pi^{\frac{1}{2}}c_3^{-\frac{1}{2}}\|1-\chi_0\|^2_{L^\infty(\mathbb{R})}\lambda^{-\alpha-\frac{1}{k}}e^{-\frac{c_3}{4}\lambda^{2(\alpha+\frac{1}{k}-\mu)}}, \end{aligned}$$

since if $y \in \tilde{\Omega}_{\lambda,\tau,\xi}$, we have

$$e^{-2c_3\lambda^{2\alpha+\frac{2}{k}}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})^2} \leq e^{-\frac{c_3}{4}\lambda^{2\alpha+\frac{2}{k}-2\mu}}e^{-c_3\lambda^{2\alpha+\frac{2}{k}}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})^2}$$

and since a change of variables gives that

$$\int_{\mathbb{R}} e^{-c_3\lambda^{2\alpha+\frac{2}{k}}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})^2}dy = \int_{\mathbb{R}} e^{-c_3\lambda^{2\alpha+\frac{2}{k}}y^2}dy = \pi^{\frac{1}{2}}c_3^{-\frac{1}{2}}\lambda^{-\alpha-\frac{1}{k}}.$$

In view of (3.1.5) and (3.3.3), the use of the triangular inequality for all $(\tau,\xi) \in [2,3]^2$ and $\lambda \geq \lambda_0$,

$$\begin{aligned} \|g_{\lambda,\tau,\xi}\|_{L^2(\mathbb{R}_y)} &= \big\|\chi_0\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)e^{-\Phi_\lambda(\tau,\xi,y)}1_{\mathbb{R}_-^*}(y)\big\|_{L^2(\mathbb{R}_y)} \\ &\geq \|e^{-\Phi_\lambda(\tau,\xi,y)}1_{\mathbb{R}_-^*}(y)\|_{L^2(\mathbb{R}_y)} \\ &- \big\|\big[1-\chi_0\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\big]e^{-\Phi_\lambda(\tau,\xi,y)}1_{\mathbb{R}_-^*}(y)\big\|_{L^2(\mathbb{R}_y)}, \end{aligned}$$

with the estimates (3.3.11) and (3.3.13), shows that there exists a positive constant c_6 such that for all $(\tau,\xi) \in [2,3]^2$ and $\lambda \geq \lambda_0$,

$$\|g_{\lambda,\tau,\xi}\|_{L^2(\mathbb{R}_y)} \geq c_6\lambda^{-\frac{1+\alpha}{2}},$$

because from (3.1.3),

$$\alpha + \frac{1}{k} - \mu > 0.$$

This ends the proof of Lemma 3.3.1. □

The previous lemma permits us to obtain from the estimate (3.3.4) that for all $(\tau, \xi) \in [2, 3]^2$ and $\lambda \geq \lambda_0$,

$$\|g_{\lambda,\tau,\xi}\|_{H^{-N_0}(\mathbb{R}_y)} \;\geq\; c_6^2 c_5^{-1} \lambda^{-\alpha-1+\frac{\mu}{2}-(\mu+2+2\alpha)N_0}.$$

Then using (3.3.2), we obtain the following proposition.

Proposition 3.3.1. *There exists a positive constant c_9 such that for all $\lambda \geq \lambda_0$,*

$$\|u_\lambda\|_{(-N_0)} \geq c_9 \lambda^{-\alpha-1+\frac{\mu}{2}-(\mu+3+3\alpha)N_0}. \tag{3.3.14}$$

We need now to cutoff in the variables t, x to obtain a quasimode localized in an arbitrary neighbourhood of 0 in $\mathbb{R}^3$.

3.4 Cutoff in variables t and x

We need first to make the result of the lemma 3.2.1 more precise when there is no differentiation in the variable y.

Lemma 3.4.1. *For all $\rho \in \mathbb{N}^2$, there exist some functions a_l, $l = 0, \ldots, |\rho|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants β_l, $l = 0, \ldots, |\rho|(k+1)$ verifying*

$$\beta_l \;\leq\; 2\,|\rho| \left(\alpha - \frac{1}{k}\right),$$

such that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\partial_{\tau,\xi}^{\rho}\left(e^{-\Phi_\lambda}\right) = e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} a_l \left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right) \left(\lambda^{\frac{2}{k}} y\right)^l \lambda^{\beta_l}. \tag{3.4.1}$$

Proof. We prove again this lemma by induction on $|\rho|$. If $|\rho| = 0$, the expression (3.4.1) holds with $a_0 = 1$ and $\beta_0 = 0$. Let us assume now that for $\rho \in \mathbb{N}^2$, there exist some functions a_l, $l = 0, \ldots, |\rho|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants β_l, $l = 0, \ldots, |\rho|(k+1)$ verifying

$$\beta_l \;\leq\; 2\,|\rho| \left(\alpha - \frac{1}{k}\right)$$

such that for all

$$y \in \mathbb{R}_-^*, \quad (\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*, \quad \lambda \geq 1,$$

the expression (3.4.1) holds. Since we can write (3.2.5) and (3.2.6) in the following way,

$$\partial_\tau \Phi_\lambda(\tau,\xi,y) = \lambda^{2\alpha-\frac{2}{k}}\big(2\tau(\lambda^{\frac{2}{k}}y) + 2\xi^{-\frac{2}{k}}\tau^{\frac{k+2}{k}}\big), \tag{3.4.2}$$

$$\partial_\xi \Phi_\lambda(\tau,\xi,y) = \frac{2}{k+1}\lambda^{2\alpha-\frac{2}{k}}\big(-\tau^{\frac{2k+2}{k}}\xi^{-\frac{k+2}{k}} + (-1)^{k+1}\xi(\lambda^{\frac{2}{k}}y)^{k+1}\big) \tag{3.4.3}$$

and

$$\begin{aligned} &\partial_\tau\Big(e^{-\Phi_\lambda}\sum_{l=0}^{|\rho|(k+1)} a_l(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})(\lambda^{\frac{2}{k}}y)^l\lambda^{\beta_l}\Big) \\ &\quad = e^{-\Phi_\lambda}\Big(\sum_{l=0}^{|\rho|(k+1)}\big[-(\partial_\tau\Phi_\lambda)a_l(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}) \\ &\qquad + \partial_\tau\big(a_l(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})\big)\big](\lambda^{\frac{2}{k}}y)^l\lambda^{\beta_l}\Big), \end{aligned} \tag{3.4.4}$$

$$\begin{aligned} &\partial_\xi\Big(e^{-\Phi_\lambda}\sum_{l=0}^{|\rho|(k+1)} a_l(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})(\lambda^{\frac{2}{k}}y)^l\lambda^{\beta_l}\Big) \\ &\quad = e^{-\Phi_\lambda}\Big(\sum_{l=0}^{|\rho|(k+1)}\big[-(\partial_\xi\Phi_\lambda)a_l(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}) \\ &\qquad + \partial_\xi\big(a_l(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})\big)\big](\lambda^{\frac{2}{k}}y)^l\lambda^{\beta_l}\Big), \end{aligned} \tag{3.4.5}$$

we obtain that

$$\begin{aligned} &\partial_\tau\Big(e^{-\Phi_\lambda}\sum_{l=0}^{|\rho|(k+1)} a_l(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})(\lambda^{\frac{2}{k}}y)^l\lambda^{\beta_l}\Big) \\ &= -e^{-\Phi_\lambda}\sum_{l=0}^{|\rho|(k+1)} 2(\tau^{\frac{1}{k}})^k a_l(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})(\lambda^{\frac{2}{k}}y)^{l+1}\lambda^{\beta_l+2(\alpha-\frac{1}{k})} \\ &\quad - e^{-\Phi_\lambda}\sum_{l=0}^{|\rho|(k+1)} 2(\xi^{-\frac{1}{k}})^2(\tau^{\frac{1}{k}})^{k+2}a_l(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})(\lambda^{\frac{2}{k}}y)^l\lambda^{\beta_l+2(\alpha-\frac{1}{k})} \\ &\quad + e^{-\Phi_\lambda}\sum_{l=0}^{|\rho|(k+1)}\Big(\frac{1}{k}\Big[\tau^{\frac{1}{k}}(\tau^{-\frac{1}{k}})^k(\partial_1 a_l)(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}) \\ &\quad - (\tau^{-\frac{1}{k}})^{k+1}(\partial_2 a_l)(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})\Big](\lambda^{\frac{2}{k}}y)^l\lambda^{\beta_l}\Big) \end{aligned} \tag{3.4.6}$$

and

$$\begin{aligned}
&\partial_\xi\Big(e^{-\Phi_\lambda}\sum_{l=0}^{|\rho|(k+1)} a_l(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})(\lambda^{\frac{2}{k}}y)^l\lambda^{\beta_l}\Big)\\
&\quad = e^{-\Phi_\lambda}\sum_{l=0}^{|\rho|(k+1)}\frac{2}{k+1}(\tau^{\frac{1}{k}})^{2k+2}(\xi^{-\frac{1}{k}})^{k+2}a_l(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})\\
&\qquad\qquad (\lambda^{\frac{2}{k}}y)^l\lambda^{\beta_l+2(\alpha-\frac{1}{k})}\\
&\qquad - e^{-\Phi_\lambda}\sum_{l=0}^{|\rho|(k+1)}\frac{2}{k+1}(-1)^{k+1}(\xi^{\frac{1}{k}})^k a_l(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})\\
&\qquad\qquad (\lambda^{\frac{2}{k}}y)^{l+k+1}\lambda^{\beta_l+2(\alpha-\frac{1}{k})}\\
&\qquad + e^{-\Phi_\lambda}\sum_{l=0}^{|\rho|(k+1)}\Big(\frac{1}{k}\Big[\xi^{\frac{1}{k}}(\xi^{-\frac{1}{k}})^k(\partial_3 a_l)(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})\\
&\qquad\qquad - (\xi^{-\frac{1}{k}})^{k+1}(\partial_4 a_l)(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})\Big](\lambda^{\frac{2}{k}}y)^l\lambda^{\beta_l}\Big).
\end{aligned} \tag{3.4.7}$$

Since from (3.1.3), $\alpha > 1/k$, we deduce from (3.4.6) and (3.4.7) that if $\tilde{\rho} \in \{\rho+(1,0), \rho+(0,1)\}$, there exist some functions $\tilde{a}_l$, $l = 0, \dots, |\tilde{\rho}|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants $\tilde{\beta}_l$, $l = 0, \dots, |\tilde{\rho}|(k+1)$ verifying $\tilde{\beta}_l \leq 2\,|\tilde{\rho}|\,(\alpha - 1/k)$ such that for all $y \in \mathbb{R}_-^*$, $(\tau, \xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\partial_{\tau,\xi}^{\tilde{\rho}}\big(e^{-\Phi_\lambda}\big) = e^{-\Phi_\lambda}\sum_{l=0}^{|\tilde{\rho}|(k+1)}\tilde{a}_l\left(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}\right)\left(\lambda^{\frac{2}{k}}y\right)^l\lambda^{\tilde{\beta}_l}.$$

This proves the induction property at the rank $|\rho| + 1$ and ends the proof of the lemma 3.4.1. □

We can now prove the following lemma.

Lemma 3.4.2. *For all $\rho \in \mathbb{N}^2$, there exists a positive constant M_ρ such that for all $(\tau, \xi) \in [1, 4]^2$, $\lambda \geq \lambda_0$ and $y \in \operatorname{supp}\chi_0\big(\lambda^\mu(\cdot + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)$,*

$$\left|\partial_{\tau,\xi}^{\rho}\big(e^{-\Phi_\lambda(\tau,\xi,y)}\big)\right| \leq M_\rho\lambda^{2|\rho|(\alpha-\frac{1}{k})}. \tag{3.4.8}$$

Proof. We recall that the above notation $\operatorname{supp}\chi_0\big(\lambda^\mu(\cdot + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)$ stands for the support of the function

$$y \mapsto \chi_0\big(\lambda^\mu(y + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big).$$

It follows from the previous lemma that there exist some functions a_l, $l = 0, \dots, |\rho|(k+1)$, which are polynomial in $\mathbb{R}^4$ and some constants β_l, $l = 0, \dots, |\rho|(k+1)$ verifying

$$\beta_l \leq 2\,|\rho|\left(\alpha - \frac{1}{k}\right), \tag{3.4.9}$$

such that for all $y \in \mathbb{R}^*_-$, $(\tau, \xi) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+$ and $\lambda \geq 1$,

$$\partial^{\rho}_{\tau,\xi}\big(e^{-\Phi_\lambda}\big) = e^{-\Phi_\lambda} \sum_{l=0}^{|\rho|(k+1)} a_l\left(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}}\right)\left(\lambda^{\frac{2}{k}} y\right)^l \lambda^{\beta_l}. \tag{3.4.10}$$

Using the non-negativity of the phase function Φ_λ (see (3.1.10) and (3.1.11)), we deduce from (3.1.5) and (3.4.9) that for $l = 0, \dots, |\rho|(k+1)$, there exists a positive constant $c_{10,l}$ such that for all $(\tau, \xi) \in [1,4]^2$, $\lambda \geq \lambda_0$ and

$$y \in \operatorname{supp} \chi_0\big(\lambda^{\mu}(\cdot + (\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big),$$

$$\left|e^{-\Phi_\lambda} a_l(\tau^{\frac{1}{k}}, \tau^{-\frac{1}{k}}, \xi^{\frac{1}{k}}, \xi^{-\frac{1}{k}})\lambda^{\beta_l}\right| \leq c_{10,l}\lambda^{2|\rho|(\alpha - \frac{1}{k})}, \tag{3.4.11}$$

since from (3.1.3), $\alpha > 1/k$. Finally, in view of (3.1.5), (3.4.10) and (3.4.11), we deduce that there exists a positive constant M_ρ such that the estimate (3.4.8) holds. This ends the proof of the lemma 3.4.2. □

Let us now consider the function v_λ defined by

$$v_\lambda(t,x,y) = \chi_1(\lambda^{\gamma} t, \lambda^{\gamma} x) u_\lambda(t,x,y), \tag{3.4.12}$$

where γ is a parameter verifying

$$0 < \gamma < \frac{1}{k} \quad \text{and} \quad \gamma + \alpha < \frac{2}{k}. \tag{3.4.13}$$

This choice is possible in view of (3.1.3). The function χ_1 is taken in the space $C_0^{\infty}(\mathbb{R}^2, \mathbb{R})$ such that

$$\operatorname{supp} \chi_1 \subset B(0,1) \text{ and } \chi_1 = 1 \text{ on } B\left(0, \frac{1}{2}\right), \tag{3.4.14}$$

where the notation $B(0,r)$ stands for the closed Euclidean ball centered in 0 with a radius r. We start by getting a lower bound for the quantity $\|v_\lambda\|_{(-N_0)}$. To do this, we prove the following lemma.

Lemma 3.4.3. *For all $M \in \mathbb{N}$, there exists a positive constant K_M such that for all $\lambda \geq \lambda_0$,*

$$\big\|\big(1 - \chi_1(\lambda^{\gamma} t, \lambda^{\gamma} x)\big) u_\lambda(t,x,y)\big\|_{(-N_0)} \leq K_M \lambda^{-M}. \tag{3.4.15}$$

Proof. Since from (3.1.4), (3.1.11), (3.1.13) and a change of variables

$$u_\lambda(t,x,y)=\frac{\lambda^{\frac{1}{2}+\alpha}}{(2\pi)^2} \times \int_{\mathbb{R}^2} e^{i(x\xi\lambda^{1+\alpha}+t\tau\lambda^\alpha)}\psi_1(\tau,\xi)\chi_0\big(\lambda^\mu(y+(\tau\xi^{-1}\lambda^{-1})^{\frac{2}{k}})\big)e^{-\Phi_\lambda(\tau,\xi,y)}d\tau d\xi, \tag{3.4.16}$$

we deduce from (3.1.1), (3.1.5) and (3.4.14) that for all $\lambda \geq \lambda_0$,

$$\begin{aligned}&\big\|\big(1-\chi_1(\lambda^\gamma t,\lambda^\gamma x)\big)u_\lambda(t,x,y)\big\|_{(-N_0)}\\ &\leq \big\|\big(1-\chi_1(\lambda^\gamma t,\lambda^\gamma x)\big)u_\lambda(t,x,y)\big\|_{L^2(\mathbb{R}^3)} \leq \|1-\chi_1\|_{L^\infty(\mathbb{R}^2)}\\ &\quad\times\Big(\int_{-c_1\lambda^{-\frac{2}{k}}}^{-c_2\lambda^{-\frac{2}{k}}}\Big[\int_{\{(t,x)\in\mathbb{R}^2:\ t^2+x^2\geq 4^{-1}\lambda^{-2\gamma}\}}|u_\lambda(t,x,y)|^2 dtdx\Big]dy\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.4.17}$$

Now, some integrations by parts on (3.4.16) show that for all $q \in \mathbb{N}$, $(t,x) \neq (0,0)$, $y \in \mathbb{R}$ and $\lambda \geq \lambda_0$,

$$\begin{aligned}u_\lambda(t,x,y) &= \frac{\lambda^{\alpha+\frac{1}{2}}(i\lambda^{1+\alpha}x+\lambda^\alpha t)^{-q}}{(2\pi)^2}\int_{\mathbb{R}^2}(\partial_\xi - i\partial_\tau)^q\big(e^{i(\lambda^{1+\alpha}x\xi+\lambda^\alpha t\tau)}\big)\\ &\quad\times\ \psi_1(\tau,\xi)\chi_0\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)e^{-\Phi_\lambda(\tau,\xi,y)}d\tau d\xi\\ &= \frac{\lambda^{\alpha+\frac{1}{2}}(i\lambda^{1+\alpha}x+\lambda^\alpha t)^{-q}}{(2\pi)^2}\int_{\mathbb{R}^2}e^{i(\lambda^{1+\alpha}x\xi+\lambda^\alpha t\tau)}(i\partial_\tau-\partial_\xi)^q\\ &\quad\Big[\chi_0\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\psi_1(\tau,\xi)e^{-\Phi_\lambda(\tau,\xi,y)}\Big]d\tau d\xi.\end{aligned} \tag{3.4.18}$$

We need the following lemma.

Lemma 3.4.4. *For all $\rho \in \mathbb{N}^2$ and $l \in \mathbb{N}$, there exists a positive constant $c_{11,\rho,l}$ such that for all $y \in \mathbb{R}$, $(\tau,\xi) \in [1,4]^2$ and $\lambda \geq 1$,*

$$\Big|\partial^\rho_{\tau,\xi}\Big[\chi_0^{(l)}\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\Big]\Big| \leq c_{11,\rho,l}\lambda^{|\rho|(\mu-\frac{2}{k})}. \tag{3.4.19}$$

Proof of the lemma 3.4.4. We start by proving that for all $\rho \in \mathbb{N}^2$ and $l \in \mathbb{N}$, there exist some polynomial functions $P_{\rho,l,j}$ in $\mathbb{R}^4$, $j = 0,\dots,|\rho|$, such that for all $y \in \mathbb{R}$, $(\tau,\xi) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\lambda \geq 1$,

$$\begin{aligned}&\partial^\rho_{\tau,\xi}\Big[\chi_0^{(l)}\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\Big]\\ &=\sum_{j=0}^{|\rho|}P_{\rho,l,j}(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})\chi_0^{(l+j)}\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\lambda^{j(\mu-\frac{2}{k})}.\end{aligned} \tag{3.4.20}$$

Let us consider $l \in \mathbb{N}$. We prove (3.4.20) by induction on $|\rho|$. If $|\rho| = 0$, the identity (3.4.20) holds with $P_{\rho,l,0} = 1$. Let us assume now that the identity (3.4.20) holds for $\rho \in \mathbb{N}^2$. The two direct computations using (3.4.20),

$$
\begin{aligned}
&\partial_{\tau,\xi}^{\tilde{\rho}}\Big[\chi_0^{(l)}\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\Big] \\
&= \sum_{j=0}^{|\tilde{\rho}|-1}\Big[\frac{2}{k}(\xi^{-\frac{1}{k}})^2(\tau^{\frac{1}{k}})^{2-k}P_{\rho,l,j}(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}) \\
&\qquad \times \chi_0^{(l+j+1)}\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\lambda^{(j+1)(\mu-\frac{2}{k})}\Big] \\
&\quad + \sum_{j=0}^{|\tilde{\rho}|-1}\Big[\frac{1}{k}(\tau^{\frac{1}{k}})^{1-k}(\partial_1 P_{\rho,l,j})(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}) \\
&\qquad \times \chi_0^{(l+j)}\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\lambda^{j(\mu-\frac{2}{k})}\Big] \\
&\quad - \sum_{j=0}^{|\tilde{\rho}|-1}\Big[\frac{1}{k}(\tau^{-\frac{1}{k}})^{k+1}(\partial_2 P_{\rho,l,j})(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}) \\
&\qquad \times \chi_0^{(l+j)}\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\lambda^{j(\mu-\frac{2}{k})}\Big],
\end{aligned}
$$

if $\tilde{\rho} = \rho + (1,0)$ and

$$
\begin{aligned}
&\partial_{\tau,\xi}^{\tilde{\rho}}\Big[\chi_0^{(l)}\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\Big] \\
&= -\sum_{j=0}^{|\tilde{\rho}|-1}\Big[\frac{2}{k}(\xi^{-\frac{1}{k}})^{k+2}(\tau^{\frac{1}{k}})^2P_{\rho,l,j}(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}) \\
&\qquad \times \chi_0^{(l+j+1)}\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\lambda^{(j+1)(\mu-\frac{2}{k})}\Big] \\
&\quad + \sum_{j=0}^{|\tilde{\rho}|-1}\Big[\frac{1}{k}(\xi^{\frac{1}{k}})^{1-k}(\partial_3 P_{\rho,l,j})(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}) \\
&\qquad \times \chi_0^{(l+j)}\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\lambda^{j(\mu-\frac{2}{k})}\Big] \\
&\quad - \sum_{j=0}^{|\tilde{\rho}|-1}\Big[\frac{1}{k}(\xi^{-\frac{1}{k}})^{k+1}(\partial_4 P_{\rho,l,j})(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}}) \\
&\qquad \times \chi_0^{(l+j)}\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\lambda^{j(\mu-\frac{2}{k})}\Big],
\end{aligned}
$$

if $\tilde{\rho} = \rho + (0,1)$, prove that the induction property holds at the rank $|\rho|+1$. This proves (3.4.20). Since from (3.1.2), $\chi_0 \in C_0^{\infty}(\mathbb{R},\mathbb{R})$ and from (3.1.3), $\mu > 2/k$, we deduce from (3.4.20) that there exists a positive constant $c_{11,\rho,l}$ such that for all $y \in \mathbb{R}$, $(\tau,\xi) \in [1,4]^2$ and $\lambda \geq 1$,

$$\Big|\partial^{\rho}_{\tau,\xi}\Big[\chi_0^{(l)}\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\Big]\Big| \leq c_{11,\rho,l}\lambda^{|\rho|(\mu-\frac{2}{k})},$$

which ends the proof of the lemma 3.4.4. □

Then, we obtain using the lemma 3.4.2, (3.1.1), (3.4.19) and the Leibniz formula on the expression

$$(i\partial_\tau - \partial_\xi)^q\Big[\chi_0\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\psi_1(\tau,\xi)e^{-\Phi_\lambda(\tau,\xi,y)}\Big],$$

where $q \in \mathbb{N}$, that there exist some positive constants $c_{12,j}$, $j=0,\dots,q$ and c_{13} such that for all $y \in \mathbb{R}$, $(\tau,\xi) \in \mathbb{R}^2$ and $\lambda \geq \lambda_0$,

$$\begin{aligned}
&\Big|(i\partial_\tau - \partial_\xi)^q\Big[\chi_0\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\psi_1(\tau,\xi)e^{-\Phi_\lambda(\tau,\xi,y)}\Big]\Big| \\
&\quad\leq \sum_{j=0}^{q} c_{12,j}\lambda^{(\mu-\frac{2}{k})j}\lambda^{2(q-j)(\alpha-\frac{1}{k})} \\
&\quad\leq c_{13}\lambda^{q\max(\mu-\frac{2}{k},2\alpha-\frac{2}{k})},
\end{aligned} \tag{3.4.21}$$

since from (3.1.3),

$$\mu > \frac{2}{k} \quad \text{and} \quad \alpha > \frac{1}{k}.$$

Since from (3.1.1), supp $\psi_1 \subset [1,4]^2$, it follows from (3.4.18) and (3.4.21) that for all $q \in \mathbb{N}$, there exists a positive constant $c_{14,q}$ such that for all $(t,x) \neq (0,0)$, $y \in \mathbb{R}$ and $\lambda \geq \lambda_0$,

$$|u_\lambda(t,x,y)| \leq c_{14,q}\lambda^{q\max(\mu-\frac{2}{k},2\alpha-\frac{2}{k})-q\alpha+\alpha+\frac{1}{2}}|x^2+t^2|^{-\frac{q}{2}}. \tag{3.4.22}$$

We deduce by getting back to (3.4.17), using (3.4.22) and a change of variables that for all $q \in \mathbb{N}\setminus\{0,1\}$ and $\lambda \geq \lambda_0$,

$$\begin{aligned}
&\big\|\big(1-\chi_1(\lambda^{\gamma}t,\lambda^{\gamma}x)\big)u_\lambda\big\|_{(-N_0)} \\
&\quad\leq c_{14,q}(c_1-c_2)^{\frac{1}{2}}\lambda^{q\max(\mu-\alpha-\frac{2}{k},\alpha-\frac{2}{k})+\alpha+\frac{1}{2}-\frac{1}{k}}\|1-\chi_1\|_{L^\infty(\mathbb{R}^2)} \\
&\qquad\times\Big(\int_{\{(t,x)\in\mathbb{R}^2:\ t^2+x^2\geq 4^{-1}\lambda^{-2\gamma}\}}|t^2+x^2|^{-q}dtdx\Big)^{\frac{1}{2}} \\
&\quad\leq c_{14,q}(c_1-c_2)^{\frac{1}{2}}\lambda^{q\max(\mu-\alpha-\frac{2}{k}+\gamma,\alpha+\gamma-\frac{2}{k})+\alpha+\frac{1}{2}-\frac{1}{k}-\gamma}\|1-\chi_1\|_{L^\infty(\mathbb{R}^2)} \\
&\qquad\times\Big(\int_{\{(t,x)\in\mathbb{R}^2:\ t^2+x^2\geq 4^{-1}\}}|t^2+x^2|^{-q}dtdx\Big)^{\frac{1}{2}}.
\end{aligned} \tag{3.4.23}$$

We recall in view of (3.1.5) that $c_1 > c_2$. Let us notice that

$$\max\Big(\mu-\alpha-\frac{2}{k}+\gamma, \alpha+\gamma-\frac{2}{k}\Big) < 0, \tag{3.4.24}$$

because from (3.4.13),

$$\alpha+\gamma < \frac{2}{k}$$

and from (3.1.3) and (3.4.13),

$$\mu-\alpha-\frac{2}{k}+\gamma < \gamma-\frac{1}{k} < 0.$$

Finally, we obtain using (3.4.23) and (3.4.24) the estimate (3.4.15). This ends the proof of the lemma 3.4.3. □

We can now use (3.4.12) and the triangular inequality for all $\lambda \geq \lambda_0$,

$$\|u_\lambda\|_{(-N_0)} - \big\|\big(1-\chi_1(\lambda^\gamma t, \lambda^\gamma x)\big)u_\lambda\big\|_{(-N_0)} \leq \|v_\lambda\|_{(-N_0)},$$

with the estimates (3.3.14) and (3.4.15) to prove the following result.

Proposition 3.4.1. *There exists a positive constant c_{15} such that for all $\lambda \geq \lambda_0$,*

$$\|v_\lambda\|_{(-N_0)} \geq c_{15}\lambda^{-\alpha-1+\frac{\mu}{2}-(\mu+3+3\alpha)N_0}. \tag{3.4.25}$$

We now need to get an upper bound for the quantity $\|L_2^* v_\lambda\|_{(N_0)}$, $N_0 \in \mathbb{N}$, with respect to the parameter λ. It follows from (3.1.6) and (3.4.12) that for all $\lambda \geq \lambda_0$,

$$\begin{aligned} L_2^* v_\lambda \;=\; & \chi_1(\lambda^\gamma t, \lambda^\gamma x)L_2^* u_\lambda - \lambda^{2\gamma}\theta_k(y)(\partial_x^2\chi_1)(\lambda^\gamma t, \lambda^\gamma x)u_\lambda \\ & - 2\lambda^\gamma\theta_k(y)(\partial_x\chi_1)(\lambda^\gamma t, \lambda^\gamma x)\partial_x u_\lambda + \lambda^{2\gamma}(\partial_t^2\chi_1)(\lambda^\gamma t, \lambda^\gamma x)u_\lambda \\ & + 2\lambda^\gamma(\partial_t\chi_1)(\lambda^\gamma t, \lambda^\gamma x)\partial_t u_\lambda. \end{aligned} \tag{3.4.26}$$

We note respectively A_λ, B_λ, C_λ, D_λ and E_λ the terms appearing in the right-hand-side of the last expression. Let us first notice that these five terms are C^∞ on $\mathbb{R}^3$. Indeed, we have already proved after (3.1.15) that $u_\lambda \in C^\infty\big(\mathbb{R}_y, S(\mathbb{R}^2_{t,x})\big)$ and, it follows from (3.1.1), (3.1.5) and (3.4.16) that for all $(t,x) \in \mathbb{R}^2$, $\lambda \geq \lambda_0$,

$$\operatorname{supp} u_\lambda(t,x,\cdot) \subset \Big[-c_1\lambda^{-\frac{2}{k}}, -c_2\lambda^{-\frac{2}{k}}\Big] \tag{3.4.27}$$

and from (3.1.6), $\theta_k(y) = (-y)^k$ if $y \in \mathbb{R}_-$. Moreover, we have already proved in (3.1.16) that $L_2^* u_\lambda$ is C^∞ on $\mathbb{R}^3$. Then, we want to get an upper bound for the

$H^{N_0}(\mathbb{R}^3)$ norm of the term A_λ. To do this, it is enough to get an upper bound for the quantity

$$\|\partial_{t,x}^{M_1}\big(\chi_1(\lambda^\gamma t,\lambda^\gamma x)\big)\partial_{t,x,y}^{M_2}(L_2^* u_\lambda)\|_{L^2(\mathbb{R}^3)},$$

where $M_1 \in \mathbb{N}^2$ and $M_2 \in \mathbb{N}^3$ verify $|M_1| + |M_2| = N_0$. Since

$$\|\partial_{t,x}^{M_1}\big(\chi_1(\lambda^\gamma t,\lambda^\gamma x)\big)\partial_{t,x,y}^{M_2}(L_2^* u_\lambda)\|_{L^2(\mathbb{R}^3)} \le \lambda^{\gamma|M_1|}\|\partial_{t,x}^{M_1}\chi_1\|_{L^\infty(\mathbb{R}^2)}\|L_2^* u_\lambda\|_{(|M_2|)},$$

it follows that there exists a positive constant c_{16} such that for all $\lambda \ge \lambda_0$,

$$\|A_\lambda\|_{(N_0)} \le c_{16}\lambda^{\gamma N_0}\Big(\sup_{|j|\le N_0}\|\partial_{t,x}^j\chi_1\|_{L^\infty(\mathbb{R}^2)}\Big)\|L_2^* u_\lambda\|_{(N_0)}. \tag{3.4.28}$$

Thus, we deduce from (3.1.3), (3.2.18) and (3.4.28) that

$$\|A_\lambda\|_{(N_0)} = O\big(e^{-\frac{c_3}{16}\lambda^{2(\alpha+\frac{1}{k}-\mu)}}\big) \text{ when } \lambda \to +\infty. \tag{3.4.29}$$

Let us now consider $(j_1, j_2, j_3) \in \mathbb{N}^3$. It follows from (3.4.16) that for all $\lambda \ge \lambda_0$,

$$\begin{aligned}\partial_t^{j_1}\partial_x^{j_2}\partial_y^{j_3}u_\lambda(t,x,y) &= \frac{i^{j_1+j_2}}{(2\pi)^2}\lambda^{\alpha+\frac{1}{2}+j_1\alpha+j_2(1+\alpha)}\int_{\mathbb{R}^2} e^{i(\lambda^{1+\alpha}x\xi+\lambda^\alpha t\tau)}\\ &\times \tau^{j_1}\xi^{j_2}\psi_1(\tau,\xi)\partial_y^{j_3}\Big[\chi_0\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)e^{-\Phi_\lambda(\tau,\xi,y)}\Big]d\tau d\xi.\end{aligned} \tag{3.4.30}$$

We can make again some integrations by parts in (3.4.30) as in (3.4.18). Thus, we obtain that for all $q \in \mathbb{N}$, $(t,x) \ne (0,0)$, $y \in \mathbb{R}$ and $\lambda \ge \lambda_0$,

$$\begin{aligned}&\partial_t^{j_1}\partial_x^{j_2}\partial_y^{j_3}u_\lambda(t,x,y)\\ &= \frac{i^{j_1+j_2}}{(2\pi)^2}\lambda^{\alpha+\frac{1}{2}+j_1\alpha+j_2(1+\alpha)}(i\lambda^{1+\alpha}x+\lambda^\alpha t)^{-q}\int_{\mathbb{R}^2} e^{i(\lambda^{1+\alpha}x\xi+\lambda^\alpha t\tau)}(i\partial_\tau-\partial_\xi)^q\\ &\quad\Big(\tau^{j_1}\xi^{j_2}\psi_1(\tau,\xi)\partial_y^{j_3}\Big[\chi_0\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)e^{-\Phi_\lambda(\tau,\xi,y)}\Big]\Big)d\tau d\xi.\end{aligned} \tag{3.4.31}$$

Let us prove the following lemma.

Lemma 3.4.5. *For all $\rho \in \mathbb{N}^2$, $j_3 \in \mathbb{N}$, there exists a positive constant c_{17,ρ,j_3} such that for all $(\tau,\xi) \in [1,4]^2$, $\lambda \ge \lambda_0$ and $y \in \operatorname{supp}\chi_0\big(\lambda^\mu(\cdot+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)$,*

$$\begin{aligned}&\Big|\partial_{\tau,\xi}^\rho\partial_y^{j_3}\Big[\chi_0\big(\lambda^\mu(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)e^{-\Phi_\lambda(\tau,\xi,y)}\Big]\Big|\\ &\quad\le c_{17,\rho,j_3}\lambda^{(2+2\alpha+\mu)j_3+|\rho|\max(2\alpha-\frac{2}{k},\mu-\frac{2}{k})}.\end{aligned} \tag{3.4.32}$$

Proof. The Leibniz formula first proves that

$$\partial_y^{j_3}\Big[\chi_0\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)e^{-\Phi_\lambda}\Big] = \sum_{l=0}^{j_3} C_{j_3}^{l}\lambda^{l\mu}\chi_0^{(l)}\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\partial_y^{j_3-l}\big(e^{-\Phi_\lambda}\big). \tag{3.4.33}$$

We deduce from the lemma 3.2.1 and (3.4.33) that there exist some functions $a_{r,l}$, $r=0,\dots,j_3(k+1)$, $l=0,\dots,j_3$, which are polynomial in $\mathbb{R}^4$ and some constants $\beta_{r,l}$, $r=0,\dots,j_3(k+1)$, $l=0,\dots,j_3$, verifying

$$\beta_{r,l} \ \leq\ 2j_3(\alpha+1), \tag{3.4.34}$$

such that for all $y\in\mathbb{R}_-^*$, $(\tau,\xi)\in\mathbb{R}_+^*\times\mathbb{R}_+^*$ and $\lambda\geq 1$,

$$\partial_y^{j_3}\Big[\chi_0\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)e^{-\Phi_\lambda}\Big] = \sum_{\substack{0\leq l\leq j_3\\ 0\leq r\leq j_3(k+1)}} \chi_0^{(l)}\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)y^r a_{r,l}(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})\lambda^{l\mu+\beta_{r,l}}e^{-\Phi_\lambda}. \tag{3.4.35}$$

Since using the Leibniz formula on (3.4.35), we can write

$$\begin{aligned}\partial_{\tau,\xi}^{\rho}\partial_y^{j_3}&\Big[\chi_0\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)e^{-\Phi_\lambda}\Big]\\ &= \sum_{l,r,\rho_1,\rho_2,\rho_3}\Big[c_{18,l,r,\rho_1,\rho_2,\rho_3}\partial_{\tau,\xi}^{\rho_1}\Big(\chi_0^{(l)}\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)\Big)\\ &\qquad\times\ y^r\partial_{\tau,\xi}^{\rho_2}\big(a_{r,l}(\tau^{\frac{1}{k}},\tau^{-\frac{1}{k}},\xi^{\frac{1}{k}},\xi^{-\frac{1}{k}})\big)\partial_{\tau,\xi}^{\rho_3}\big(e^{-\Phi_\lambda}\big)\lambda^{l\mu+\beta_{r,l}}\Big],\end{aligned}$$

where the above sum is taken on $0\leq l\leq j_3$, $0\leq r\leq j_3(k+1)$, $(\rho_1,\rho_2,\rho_3)\in(\mathbb{N}^2)^3$, $\rho_1+\rho_2+\rho_3=\rho$ and where $c_{18,l,r,\rho_1,\rho_2,\rho_3}$ are some constants, we deduce from (3.1.5), (3.4.8), (3.4.19) and (3.4.34), that there exist some positive constant $c_{19,l,r,\rho_1,\rho_2,\rho_3}$ and c_{20} such that for all $(\tau,\xi)\in[1,4]^2$, $\lambda\geq\lambda_0$ and $y\in\operatorname{supp}\chi_0\big(\lambda^{\mu}(\cdot+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)$,

$$\begin{aligned}\Big|\partial_{\tau,\xi}^{\rho}\partial_y^{j_3}&\Big[\chi_0\big(\lambda^{\mu}(y+(\lambda^{-1}\xi^{-1}\tau)^{\frac{2}{k}})\big)e^{-\Phi_\lambda}\Big]\Big|\\ &\leq \sum_{l,r,\rho_1,\rho_2,\rho_3} c_{19,l,r,\rho_1,\rho_2,\rho_3}\lambda^{|\rho_1|(\mu-\frac{2}{k})}\lambda^{2|\rho_3|(\alpha-\frac{1}{k})}\lambda^{l\mu+2j_3(\alpha+1)}\\ &\leq c_{20}\lambda^{j_3(2+2\alpha+\mu)+|\rho|\max(2\alpha-\frac{2}{k},\mu-\frac{2}{k})},\end{aligned}$$

where the sum of the previous expression is taken on $0\leq l\leq j_3$, $0\leq r\leq j_3(k+1)$, $(\rho_1,\rho_2,\rho_3)\in(\mathbb{N}^2)^3$, $\rho_1+\rho_2+\rho_3=\rho$. This proves (3.4.32) and ends the proof of the lemma 3.4.5. □

Thus, since from (3.1.1) and (3.1.3),

$$\text{supp } \psi_1 \subset [1,4]^2 \text{ and } \max\Big(2\alpha-\frac{2}{k},\mu-\frac{2}{k}\Big)>0,$$

we can deduce from (3.4.31) and the previous lemma that for all $q \in \mathbb{N}$, there exists a positive constant $c_{21,q}$ such that for all $(t,x)\neq(0,0)$, $y\in\mathbb{R}$ and $\lambda\geq\lambda_0$,

$$\begin{aligned}&|\partial_t^{j_1}\partial_x^{j_2}\partial_y^{j_3}u_\lambda(t,x,y)|\\ &\leq c_{21,q}\lambda^{\alpha+\frac{1}{2}+j_1\alpha+j_2(1+\alpha)+j_3(\mu+2+2\alpha)+q\max(2\alpha-\frac{2}{k},\mu-\frac{2}{k})-q\alpha}|t^2+x^2|^{-\frac{q}{2}}.\end{aligned} \tag{3.4.36}$$

It follows from (3.4.27) and from (3.1.6), $\theta_k(y)=(-y)^k$ for all $y\in\mathbb{R}_-$ that there exists a positive constant c_{22} such that we have the following estimate of the $H^{N_0}(\mathbb{R}^3)$ norm of the terms B_λ, C_λ, D_λ and E_λ defined in (3.4.26),

$$\begin{aligned}&\max\big(\|B_\lambda\|_{(N_0)},\|C_\lambda\|_{(N_0)},\|D_\lambda\|_{(N_0)},\|E_\lambda\|_{(N_0)}\big)\\ &\leq\ c_{22}\sum_{\beta_1,\beta_2}\Big\|\partial_{t,x}^{\beta_1}\big(\chi_1(\lambda^\gamma t,\lambda^\gamma x)\big)\partial_{t,x,y}^{\beta_2}u_\lambda\Big\|_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{3.4.37}$$

where the sum is taken on $\beta_1=(l_1,l_2)$, $\beta_2=(l_3,l_4,l_5)$ with l_j, $j=1,\ldots,5$ some integers verifying $0\leq l_j\leq N_0+2$ and $l_1+l_2\geq 1$. Using these notations, let us consider some integers l_j, $j=1,\ldots,5$ verifying

$$0\leq l_j\leq N_0+2 \quad\text{and}\quad l_1+l_2\geq 1. \tag{3.4.38}$$

We set $\beta_1=(l_1,l_2)$ and $\beta_2=(l_3,l_4,l_5)$. Since from (3.4.14) and (3.4.38),

$$\text{supp } \partial_{t,x}^{\beta_1}\chi_1\subset\big\{(t,x)\in\mathbb{R}^2:t^2+x^2\geq 1/4\big\},$$

we deduce from (3.4.27) and (3.4.36) that for all $q\geq 2$, there exists a positive constant $c_{23,q}$ such that for all $\lambda\geq\lambda_0$,

$$\begin{aligned}&\Big\|\partial_{t,x}^{\beta_1}\big(\chi_1(\lambda^\gamma t,\lambda^\gamma x)\big)\partial_{t,x,y}^{\beta_2}u_\lambda\Big\|_{L^2(\mathbb{R}^3)}\\ &\leq\ c_{23,q}\lambda^{\alpha+\frac{1}{2}+\gamma(l_1+l_2)+\alpha l_3+(1+\alpha)l_4+(\mu+2+2\alpha)l_5+q\max(\alpha-\frac{2}{k},\mu-\alpha-\frac{2}{k})}\|\partial_{t,x}^{\beta_1}\chi_1\|_{L^\infty(\mathbb{R}^2)}\\ &\quad\times\left(\int_{-c_1\lambda^{-\frac{2}{k}}}^{-c_2\lambda^{-\frac{2}{k}}}\Big(\int_{\{(t,x)\in\mathbb{R}^2:\ t^2+x^2\geq 4^{-1}\lambda^{-2\gamma}\}}|t^2+x^2|^{-q}dtdx\Big)dy\right)^{\frac{1}{2}}.\end{aligned} \tag{3.4.39}$$

Since with our choice of the integers l_j in (3.4.38), we have

$$\gamma(l_1+l_2)+\alpha l_3+(1+\alpha)l_4+(\mu+2+2\alpha)l_5\leq(N_0+2)(3+2\gamma+4\alpha+\mu). \tag{3.4.40}$$

We deduce from (3.4.38), (3.4.39), (3.4.40) and a change of variables that

$$\begin{aligned}&\big\|\partial_{t,x}^{\beta_1}\big(\chi_1(\lambda^\gamma t,\lambda^\gamma x)\big)\partial_{t,x,y}^{\beta_2}u_\lambda\big\|_{L^2(\mathbb{R}^3)} \\ &\le\ c_{23,q}(c_1-c_2)^{\frac{1}{2}}\lambda^{\alpha+\frac{1}{2}-\frac{1}{k}-\gamma+(N_0+2)(3+2\gamma+4\alpha+\mu)+q\max(\alpha-\frac{2}{k},\mu-\alpha-\frac{2}{k})+q\gamma} \\ &\quad\Big(\sup_{|\beta_1|\le 2N_0+4}\|\partial_{t,x}^{\beta_1}\chi_1\|_{L^\infty(\mathbb{R}^2)}\Big)\Big(\int_{\{(T,X)\in\mathbb{R}^2:\ T^2+X^2\ge 4^{-1}\}}|T^2+X^2|^{-q}dTdX\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.4.41}$$

We recall that in view of (3.1.5), we have $c_1>c_2$. Then, it follows from (3.4.37) and (3.4.41) that for all $q\ge 2$, there exists a positive constant $c_{24,q}$ such that

$$\begin{aligned}&\max\big(\|B_\lambda\|_{(N_0)},\|C_\lambda\|_{(N_0)},\|D_\lambda\|_{(N_0)},\|E_\lambda\|_{(N_0)}\big) \\ &\le c_{24,q}\lambda^{\alpha+\frac{1}{2}-\frac{1}{k}-\gamma+(N_0+2)(3+2\gamma+4\alpha+\mu)+q\max(\alpha+\gamma-\frac{2}{k},\mu+\gamma-\alpha-\frac{2}{k})}.\end{aligned} \tag{3.4.42}$$

Since from (3.1.3) and (3.4.13),

$$\max\Big(\alpha+\gamma-\frac{2}{k},\mu+\gamma-\alpha-\frac{2}{k}\Big)<0, \tag{3.4.43}$$

because

$$\mu-\alpha-\frac{1}{k}<0\quad\text{and}\quad\gamma-\frac{1}{k}<0,$$

we obtain from (3.4.26), (3.4.29), (3.4.42) and (3.4.43) that for all $M\in\mathbb{N}$, there exists a positive constant C_M such that for all $\lambda\ge\lambda_0$,

$$\|L_2^*v_\lambda\|_{(N_0)}\le C_M\lambda^{-M}. \tag{3.4.44}$$

To sum up, we have built a family $\big(v_\lambda(t,x,y)\big)_{\lambda\ge\lambda_0}$ in (3.4.12), which is C^∞ on $\mathbb{R}^3$ and has according to (3.4.14) and (3.4.27), its support in the compact set

$$B(0,\lambda^{-\gamma})\times\Big[-c_1\lambda^{-\frac{2}{k}},-c_2\lambda^{-\frac{2}{k}}\Big]. \tag{3.4.45}$$

The estimates obtained in (3.4.25) and (3.4.44), for all $\lambda\ge\lambda_0$,

$$\|v_\lambda\|_{(-N_0)}\ge c_{15}\lambda^{-\alpha-1+\frac{\mu}{2}-(\mu+3+3\alpha)N_0}, \tag{3.4.46}$$

$$\forall M\in\mathbb{N},\exists C_M>0,\ \|L_2^*v_\lambda\|_{(N_0)}\le C_M\lambda^{-M}, \tag{3.4.47}$$

allow us to prove that **no** a priori estimates of the following type can hold

$$\begin{gathered}\exists C_0>0,\exists N_0\in\mathbb{N},\exists V_0\text{ an open neighbourhood of 0 in }\mathbb{R}^3\text{ such that}\\ \forall u\in C_0^\infty(V_0),\ C_0\|L_2^*u\|_{(k-3)}\ge\|u\|_{(-N_0)}.\end{gathered}$$

This proves that the operator L_2 is nonsolvable in any neighbourhood of 0 in $\mathbb{R}^3$ in the sense where there do **not** exist an integer $N_0 \in \mathbb{N}$ and an open neighbourhood V_0 of 0 in $\mathbb{R}^3$ such that for all $f \in H^{N_0}(V_0)$, there exists $u \in H^{-k+3}(\mathbb{R}^3)$ such that

$$L_2 u = f \quad \text{on} \quad V_0.$$

References

[1] F. Colombini, L. Pernazza and F. Treves. *Solvability and nonsolvability of second-order evolution equations*. Hyperbolic problems and related topics, 111-120, Grad. Ser. Anal., Int. Press, Somerville, MA (2003).

[2] Yu. V. Egorov. *On an example of a linear hyperbolic equation without solutions*. C.R. Acad. Sci. Paris Sér. 1 Math. **317**(12) (1993), 1149–1153.

[3] Yu. V. Egorov. *Sur un exemple d'équation linéaire hyperbolique n'ayant pas de solution*, Journées "Equations aux Dérivées Partielles" (Saint-Jean-de-Monts, 1992), Exp. No. XIV, 6 pp., École Polytech. Palaiseau (1992).

[4] L. Hörmander. *The analysis of Linear Partial Differential Operators*, Springer-Verlag (1983-1985).

[5] N. Lerner. *On a class of doubly characteristic operators with a principal-type behaviour*, preprint (2006).

[6] G.A. Mendoza and G.A. Uhlmann. *A sufficient condition for local solvability for a class of operators with double characteristics*. Am. Jour. Math. **106** (1984), 187–217.

[7] G.A. Mendoza and G.A. Uhlmann. *A necessary condition for local solvability for a class of operators with double characteristics*. J. Funct. Anal. **52**(2) (1983), 252–256.

[8] F. Treves. *On a question of Louis Nirenberg*. Math. Res. Lett. **10**(5-6) (2003), 729–735.

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