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Covariance information based on the t prior

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Abstract. The exact distribution of the covariance between two populations is derived. It is assumed that one of the population means has a Student's t prior while the other is taken to come from one of normal, Student's t, Laplace, logistic or Bessel families (the five well-known symmetric distributions). The exact distribution is given in terms of the characteristic function. The calculations involve several special functions.

Keywords: characteristic function, covariance, Student's t distribution.

Mathematical subject classification: 33C90, 62E99.

1 Introduction

Suppose we have two observations x and y arising from two populations with means μ_1 and μ_2, respectively. We are interested in studying the dependence between x and y. The most obvious measure is the covariance $(x-\mu_1)(y-\mu_2)$. Usually, the population means will be unknown but one might have some prior knowledge about them. For the past 40 to 50 years, the Student's t distribution has been the most popular prior distribution because elicitation of prior information in various physical, engineering, and financial phenomena is closely associated with that distribution. The Student's t distribution offers a more viable alternative to the normal distribution with respect to real-world data particularly because its tails are more realistic. Thus, we shall assume that $\mu_1 \sim f(\mu_1 - x)$ and that $\mu_2 \sim g(\mu_2 - y)$ independently of μ_1, where $f(\cdot)$ and $g(\cdot)$ are Student's t pdfs with degrees of freedom ν and b, respectively. The aim of this note is to derive the exact distribution of $(x-\mu_1)(y-\mu_2)$ under these assumptions. This amounts to deriving the exact distribution of the product XY when X and Y are independent random variables with the pdfs $f(\cdot)$ and $g(\cdot)$, respectively.

For completeness, in addition to the case that f and g are Student's t pdfs (considered in Section 3), we shall consider the four other cases: f is a Student's

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t pdf and g is a normal pdf (considered in Section 2); f is a Student's t pdf and g is a Laplace pdf (considered in Section 4); f is a Student's t pdf and g is a logistic pdf (considered in Section 5); and, f is a Student's t pdf and g is a Bessel pdf (considered in Section 6). For each of these cases, the exact distribution of XY is obtained by deriving its characteristic function (cf), which can be expressed as

$$\begin{aligned}\phi(t) &= E\left[\exp(itXY)\right] \\ &= \frac{1}{\sqrt{\nu}B\left(\nu/2, 1/2\right)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(itxy\right) \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} g(y)dxdy. \end{aligned} \tag{1}$$

The calculations involve several special functions, including the ${}_1F_1$ hypergeometric function (also known as the confluent hypergeometric function) defined by

$${}_1F_1\left(a; b; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!},$$

the Kummer function defined by

$$\begin{aligned}\Psi\left(a, b; x\right) &= \frac{\Gamma(1-b)\,{}_1F_1(a; b; x)}{\Gamma(1+a-b)} \\ &\quad + \frac{\Gamma(b-1)x^{1-b}\,{}_1F_1(1+a-b; 2-b; x)}{\Gamma(a)},\end{aligned}$$

the ${}_1F_2$ hypergeometric function defined by

$${}_1F_2\left(a; b, c; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k (c)_k} \frac{x^k}{k!},$$

the ${}_2F_1$ hypergeometric function (also known as the Gauss hypergeometric function) defined by

$${}_2F_1\left(a, b; c; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

and the modified Bessel function of the third kind defined by

$$K_\nu(x) = \frac{\sqrt{\pi} x^\nu}{2^\nu \Gamma\left(\nu + 1/2\right)} \int_1^{\infty} \left(t^2 - 1\right)^{\nu - 1/2} \exp\left(-xt\right) dt,$$

where $(e)_k = e(e+1) \cdots (e+k-1)$ denotes the ascending factorial. We also need the following important lemmas.

Lemma 1. (Equation (2.3.6.9)**, Prudnikov et al., 1986, volume 1).** *For* $\alpha > 0$ *and* $p > 0$,

$$\int_0^\infty \frac{x^{\alpha-1}\exp(-px)}{(x+z)^\rho}dx = \Gamma(\alpha)z^{\alpha-p}\Psi\left(\alpha, \alpha+1-\rho;\, pz\right).$$

Lemma 2. (Equation (2.5.6.4)**, Prudnikov et al., 1986, volume 1).** *For* $b > 0$, $\rho > 0$ *and* $z > 0$,

$$\int_0^\infty \frac{\cos(bx)}{\left(x^2+z^2\right)^\rho}dx = \left(\frac{2z}{b}\right)^{1/2-\rho}\frac{\sqrt{\pi}}{\Gamma(\rho)}K_{1/2-\rho}(bz).$$

Lemma 3. (Equation (2.3.8.4)**, Prudnikov et al., 1986, volume 1).** *For* $y > 0$ *and* $z > 0$,

$$\int_0^\infty \frac{\exp\left(i\lambda x\right)}{\left(x^2+z^2\right)\left(y \pm ix\right)}dx = \frac{\pi}{2}\exp\left(\pm\lambda z\right)\left(y \pm z\right)^{-1}.$$

Lemma 4. (Equation (2.16.3.13)**, Prudnikov et al., 1986, volume 2).** *For* $c > 0$, $z > 0$ *and* $\alpha >\mid \nu \mid$,

$$\begin{aligned}
&\int_0^\infty \frac{x^{\alpha-1}}{\left(x^2+z^2\right)^\rho}K_\nu(cx)dx \\
&= 2^{\nu-2}c^{-\nu}z^{\alpha-2\rho-\nu}\Gamma(\nu)B\left(\rho+\frac{\nu-\alpha}{2}, \frac{\alpha-\nu}{2}\right) \\
&\qquad \times {}_2F_1\left(\frac{\alpha-\nu}{2}; 1-\nu, 1-\rho+\frac{\alpha-\nu}{2}; -\frac{c^2z^2}{4}\right) \\
&\quad + 2^{-(\nu+2)}c^{\nu}z^{\alpha-2\rho+\nu}\Gamma(-\nu)B\left(\rho-\frac{\nu+\alpha}{2}, \frac{\alpha+\nu}{2}\right) \\
&\qquad \times {}_2F_1\left(\frac{\alpha+\nu}{2}; 1+\nu, 1-\rho+\frac{\alpha+\nu}{2}; -\frac{c^2z^2}{4}\right) \\
&\quad + 2^{\alpha-2\rho-2}c^{2\rho-\alpha}\Gamma\left(\frac{\alpha+\nu}{2}-\rho\right)\Gamma\left(\frac{\alpha-\nu}{2}-\rho\right) \\
&\qquad \times {}_2F_1\left(\rho; 1+\rho-\frac{\alpha+\nu}{2}, 1+\rho-\frac{\alpha-\nu}{2}; -\frac{c^2z^2}{4}\right).
\end{aligned}$$

Lemma 5. (Equation (2.16.33.1)**, Prudnikov et al., 1986, volume 2).** *For* $b + c > 0$ *and* $\alpha > \mid \mu \mid + \mid \nu \mid$,

$$\begin{aligned}\int_0^\infty & x^{\alpha-1} K_\mu(bx) K_\nu(cx) dx \\ = & \frac{2^{\alpha-3} b^\mu}{c^{\alpha+\mu} \Gamma(\alpha)} \Gamma\left(\frac{\alpha+\mu+\nu}{2}\right) \Gamma\left(\frac{\alpha+\mu-\nu}{2}\right) \Gamma\left(\frac{\alpha-\mu+\nu}{2}\right) \\ & \times \Gamma\left(\frac{\alpha-\mu-\nu}{2}\right) {}_1F_2\left(\frac{\alpha+\mu+\nu}{2}, \frac{\alpha-\mu+\nu}{2}; \alpha; 1 - \frac{b^2}{c^2}\right).\end{aligned}$$

Further properties of the above special functions can be found in Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

2 Normal g

If g is the normal pdf given by

$$g(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}$$

then (1) can be reduced as

$$\begin{aligned}\phi(t) = & \frac{1}{\sqrt{\nu} B(\nu/2, 1/2)} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^\infty \int_{-\infty}^\infty \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} \\ & \times \exp\left\{itxy - \frac{(y-\mu)^2}{2\sigma^2}\right\} dy dx \\ = & \frac{1}{\sqrt{\nu} B(\nu/2, 1/2)} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^\infty \int_{-\infty}^\infty \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} \\ & \times \exp\left\{\frac{2\sigma^2 itxy - y^2 + 2\mu y - \mu^2}{2\sigma^2}\right\} dy dx \\ = & \frac{1}{\sqrt{\nu} B(\nu/2, 1/2)} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^\infty \int_{-\infty}^\infty \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} \\ & \times \exp\left\{-\frac{\left(y - \mu - \sigma^2 itx\right)^2}{2\sigma^2} + \mu itx - \frac{\sigma^2 t^2 x^2}{2}\right\} dy dx \\ = & \frac{1}{\sqrt{\nu} B(\nu/2, 1/2)} \int_{-\infty}^\infty \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} \exp\left(\mu itx - \frac{\sigma^2 t^2 x^2}{2}\right) dx \\ = & \frac{2}{\sqrt{\nu} B(\nu/2, 1/2)} \int_0^\infty \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} \cos(\mu tx) \exp\left(-\frac{\sigma^2 t^2 x^2}{2}\right) dx.\end{aligned} \tag{2}$$

The integral in (2) cannot be calculated directly in its general form. However, using the series expansion for cosine

$$\cos(x) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \tag{3}$$

one can rewrite (2) as

$$\begin{aligned}
\phi(t) \;&=\; \frac{2}{\sqrt{\nu} B\left(\nu/2, 1/2\right)} \int_0^{\infty} \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k (\mu t x)^{2k}}{(2k)!} \right\} \\
&\quad \times \exp\left(-\frac{\sigma^2 t^2 x^2}{2}\right) dx \\
&=\; \frac{2\nu^{\nu/2}}{B\left(\nu/2, 1/2\right)} \sum_{k=0}^{\infty} \frac{(-1)^k (\mu t)^{2k}}{(2k)!} \int_0^{\infty} x^{2k} \left(\nu + x^2\right)^{-(1+\nu)/2} \\
&\quad \times \exp\left(-\frac{\sigma^2 t^2 x^2}{2}\right) dx \\
&=\; \frac{\nu^{\nu/2}}{B\left(\nu/2, 1/2\right)} \sum_{k=0}^{\infty} \frac{(-1)^k (\mu t)^{2k}}{(2k)!} \int_0^{\infty} w^{k-1/2} \left(\nu + w\right)^{-(1+\nu)/2} \\
&\quad \times \exp\left(-\frac{\sigma^2 t^2 w}{2}\right) dw,
\end{aligned} \tag{4}$$

where the last step follows by the substitution $w = x^2$. The integral in (4) can be calculated by direct application of Lemma 1 to yield

$$\begin{aligned}
\phi(t) \;&=\; \frac{1}{B\left(\nu/2, 1/2\right)} \sum_{k=0}^{\infty} \frac{(-1)^k \nu^k (\mu t)^{2k}}{(2k)!} \Gamma\left(k + \frac{1}{2}\right) \\
&\quad \Psi\left(k + \frac{1}{2}, k - \frac{\nu}{2} + 1; \frac{\sigma^2 t^2 \nu}{2}\right),
\end{aligned} \tag{5}$$

an infinite sum involving the Kummer function. If $\mu = 0$ then (5) reduces to the simple form

$$\phi(t) \;=\; \frac{\Gamma\left((\nu+1)/2\right)}{\Gamma\left(\nu/2\right)} \Psi\left(\frac{1}{2}, \frac{2-\nu}{2}; \frac{\sigma^2 t^2 \nu}{2}\right). \tag{6}$$

If $\mu = 0$ and $\nu = 1$ then (6) can be reduced further to

$$\phi(t) \;=\; \exp\left(\frac{\sigma^2 t^2}{2}\right) \operatorname{erfc}\left(\frac{\sigma \mid t \mid}{\sqrt{2}}\right).$$

3 Student's t g

Suppose g is the Student's t pdf with degrees of freedom b given by

$$g(y) = \frac{1}{\sqrt{b} B\left(b/2, 1/2\right)} \left(1 + \frac{y^2}{b}\right)^{-(1+b)/2}. \quad (7)$$

Note that (1) can be expressed as

$$\begin{aligned} \phi(t) &= \frac{1}{\sqrt{\nu} B\left(\nu/2, 1/2\right)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(itxy\right) \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} g(y) dx dy \\ &= \frac{4\nu^{\nu/2}}{B\left(\nu/2, 1/2\right)} \int_{0}^{\infty} \int_{0}^{\infty} \cos\left(txy\right) \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} g(y) dx dy. \end{aligned} \quad (8)$$

The inner integral in (8) can be calculated by direct application of Lemma 2. This yields

$$\phi(t) = \frac{2^{2-\nu/2} \nu^{\nu/4} t^{\nu/2}}{\Gamma\left(\nu/2\right)} \int_{0}^{\infty} y^{\nu/2} K_{-\nu/2}\left(t\sqrt{\nu} y\right) g(y) dy. \quad (9)$$

For the form of $g(\cdot)$ given by (7), direct application of Lemma 4 shows that (9) can be calculated as

$$\begin{aligned} \phi(t) &= \frac{2^{2-\nu/2} \nu^{\nu/4} b^{b/2} t^{\nu/2}}{\Gamma\left(\nu/2\right) B\left(b/2, 1/2\right)} \int_{0}^{\infty} y^{\nu/2} K_{-\nu/2}\left(t\sqrt{\nu} y\right) \left(y^2 + b\right)^{-(1+b)/2} dy \\ &= \frac{2^{-\nu/2} \nu^{\nu/4} b^{b/2} t^{\nu/2}}{\Gamma\left(\nu/2\right) B\left(b/2, 1/2\right)} \Bigg[2^{-\nu/2} \left(t\sqrt{\nu}\right)^{\nu/2} \left(\sqrt{b}\right)^{\nu-b} \\ &\quad \times \Gamma\left(-\frac{\nu}{2}\right) B\left(\frac{b-\nu}{2}, \frac{\nu+1}{2}\right) \\ &\quad \times {}_2F_1\left(\frac{\nu+1}{2}; 1 + \frac{\nu}{2}, 1 + \frac{\nu-b}{2}; -\frac{t^2 \nu b}{4}\right) \\ &\quad + 2^{\nu/2} \left(t\sqrt{\nu}\right)^{-\nu/2} \left(\sqrt{b}\right)^{-b} \Gamma\left(\frac{\nu}{2}\right) B\left(\frac{b}{2}, \frac{1}{2}\right) \\ &\quad \times {}_2F_1\left(\frac{1}{2}; 1 - \frac{\nu}{2}, 1 - \frac{b}{2}; -\frac{t^2 \nu b}{4}\right) \\ &\quad + 2^{\nu/2-b-2} \left(t\sqrt{\nu}\right)^{b-\nu/2} \Gamma\left(-\frac{b}{2}\right) \Gamma\left(\frac{a-b}{2}\right) \\ &\quad \times {}_2F_1\left(\frac{1+b}{2}; 1 + \frac{b}{2}, 1 + \frac{b-\nu}{2}; -\frac{t^2 \nu b}{4}\right) \Bigg]. \end{aligned} \quad (10)$$

4 Laplace g

If g is the Laplace pdf given by

$$g(y) \;=\; \frac{\lambda}{2}\exp\left(-\lambda \mid y-\theta \mid\right)$$

then (1) can be reduced as

$$\begin{aligned}
\phi(t) \;&=\; \frac{\lambda\nu^{\nu/2}}{2B\left(\nu/2, 1/2\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{itxy-\lambda \mid x-\theta \mid\right\}\left(y^2+\nu\right)^{-(1+\nu)/2}dxdy \\
&=\; \frac{\lambda\nu^{\nu/2}}{2B\left(\nu/2, 1/2\right)}\int_{-\infty}^{\infty}\left(y^2+\nu\right)^{-(1+\nu)/2}\left[\int_{-\infty}^{\theta}\exp\left\{-\lambda(\theta-x)+itxy\right\}dx\right. \\
&\qquad \left. +\int_{\theta}^{\infty}\exp\left\{-\lambda(x-\theta)+itxy\right\}dx\right]dy \\
&=\; \frac{\lambda\nu^{\nu/2}}{2B\left(\nu/2, 1/2\right)}\int_{-\infty}^{\infty}\left(y^2+\nu\right)^{-(1+\nu)/2}\left[\frac{\exp\left(it\theta y\right)}{\lambda+ity}+\frac{\exp\left(it\theta y\right)}{\lambda-ity}\right]dy.
\end{aligned} \tag{11}$$

The integral in (11) cannot be calculated in its general form. However, in the particular case $\nu = 1$, one can reduce (11) as

$$\begin{aligned}
\phi(t) \;&=\; \frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\left(y^2+1\right)^{-1}\left[\frac{\exp\left(it\theta y\right)}{\lambda+ity}+\frac{\exp\left(it\theta y\right)}{\lambda-ity}\right]dy \\
&=\; \frac{\lambda}{2\pi t}\left[\int_{-\infty}^{\infty}\frac{\exp\left(it\theta y\right)}{\left(y^2+1\right)\left(\lambda/t+iy\right)}dy+\int_{-\infty}^{\infty}\frac{\exp\left(it\theta y\right)}{\left(y^2+1\right)\left(\lambda/t-iy\right)}dy\right] \\
&=\; \frac{\lambda}{2}\left[\frac{\exp\left(t\theta\right)}{\lambda+t}+\frac{\exp\left(-t\theta\right)}{\lambda-t}\right],
\end{aligned} \tag{12}$$

where the last step follows by direct application of Lemma 3.

5 Logistic g

Suppose g is the logistic pdf given by

$$g(y) \;=\; \frac{\lambda\exp\left\{-\lambda(y-\theta)\right\}}{\left[1+\exp\left\{-\lambda(y-\theta)\right\}\right]^2}.$$

Note that this can be reexpressed as the mixture of Laplace pdfs:

$$g(y) \;=\; \sum_{k=0}^{\infty}\frac{2}{k+1}\binom{-2}{k}\frac{\lambda(k+1)}{2}\exp\left\{-\lambda(k+1) \mid y-\theta \mid\right\}.$$

Thus, using the result for the Laplace distribution given by (12), one obtains

$$\phi(t) = \lambda \sum_{k=0}^{\infty} \binom{-2}{k} \left[\frac{\exp(t\theta)}{(k+1)\lambda + t} + \frac{\exp(-t\theta)}{(k+1)\lambda - t} \right] \tag{13}$$

for the particular case $\nu = 1$.

6 Bessel g

Suppose g is the Bessel function pdf given by

$$g(y) = \frac{|\ y\ |^m}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} K_m \left(\left| \frac{y}{b} \right| \right).$$

For this form, (9) can be expressed as

$$\begin{aligned} \phi(t) &= \frac{2^{2-\nu/2} \nu^{\nu/4} t^{\nu/2}}{\sqrt{\pi} 2^m b^{m+1} \Gamma(\nu/2) \Gamma(m+1/2)} \\ &\quad \times \int_0^{\infty} y^{\nu/2+m} K_{-\nu/2}\left(t\sqrt{\nu} y \right) K_m(y/b)\, dy. \end{aligned}$$

An application of Lemma 5 shows that the above can be reduced to

$$\begin{aligned} \phi(t) &= \frac{\Gamma((\nu+1)/2)\Gamma(m+(\nu+1)/2)}{b^{m+1}\Gamma(\nu/2)\Gamma(\nu/2+m+1)} \\ &\quad \times {}_1F_2 \left(m + \frac{1}{2}, m + \frac{\nu+1}{2}; \frac{\nu}{2} + m + 1; 1 - t^2 \nu b^2 \right). \end{aligned} \tag{14}$$

Using special properties of the Gauss hypergeometric function, one can reduce (14) to elementary forms when m and ν take integer or half-integer values. For example, if $\nu = 1$ and $m = 2, 3, 4, 5, 6$ then (14) can be reduced to the simple forms

$$\phi(t) = \frac{16}{15\pi} \left\{ \frac{25x - 15}{8x^2 (x-1)^2} + \frac{15 \operatorname{arctanh}\left(\sqrt{x}\right)}{8x^{5/2}} \right\},$$

$$\phi(t) = \frac{32}{35\pi} \left\{ \frac{-231x^2 + 280x - 105}{48x^3 (x-1)^3} - \frac{35 \operatorname{arctanh}\left(\sqrt{x}\right)}{16x^{7/2}} \right\},$$

$$\phi(t) = \frac{256}{315\pi} \left\{ \frac{837x^3 - 1533x^2 + 1155x - 315}{128x^4 (x-1)^4} + \frac{315 \operatorname{arctanh}\left(\sqrt{x}\right)}{128x^{9/2}} \right\},$$

$$\phi(t) = \frac{512}{693\pi}\left\{\frac{-10615x^4 + 26070x^3 - 29568x^2 + 16170x - 3465}{1280\,(x-1)^5\,x^5} - \frac{693\,\mathrm{arctanh}\left(\sqrt{x}\right)}{256x^{11/2}}\right\},$$

$$\phi(t) = \frac{2048}{3003\pi}\left\{\frac{154635x^5 - 476905x^4 + 723294x^3 - 594594x^2 + 255255x - 45045}{15360x^6\,(x-1)^6}\right.$$

$$\left. + \frac{3003\,\mathrm{arctanh}\left(\sqrt{x}\right)}{1024x^{13/2}}\right\},$$

where $x = 1 - t^2 b^2$. Also, if $\nu = 1$ and $m = 3/2, 5/2, 7/2, 9/2, 11/2$ then (14) reduces to

$$\phi(t) = \frac{1}{x^2} + \frac{(12\,x - 8)\sqrt{1-x}}{8x^2\,(-1+x)^2},$$

$$\phi(t) = -\frac{1}{x^3} + \frac{\left(-30x^2 + 40\,x - 16\right)\sqrt{1-x}}{16x^3\,(-1+x)^3},$$

$$\phi(t) = \frac{1}{x^4} + \frac{\left(280x^3 - 560x^2 + 448x - 128\right)\sqrt{1-x}}{128x^4\,(x-1)^4},$$

$$\phi(t) = -\frac{1}{x^5} + \frac{\left(-630\,x^4 + 1680\,x^3 - 2016\,x^2 + 1152\,x - 256\right)\sqrt{1-x}}{256x^5\,(x-1)^5},$$

$$\phi(t) = \frac{1}{x^6} + \frac{\left(2772x^5 - 9240x^4 + 14784x^3 - 12672x^2 + 5632x - 1024\right)\sqrt{1-x}}{1024x^6\,(x-1)^6},$$

where $x = 1 - t^2 b^2$.

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