BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 38(3), 437-454

L^p estimates and asymptotic behavior of extremal function to Hardy-Sobolev type inequality on the H-type group*

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Abstract. This paper is devoted to discuss the regularity of the weak solution to a class of non-linear equations corresponding to Hardy-Sobolev type inequality on the H-type group. Combining the Serrin's idea and the Moser's iteration, L^p estimates of the weak solution are obtained, which generalize the results of Garofalo and Vassilev in [6, 14]. As an application, asymptotic behavior of the weak solution has been discussed. Finally, doubling property and unique continuation of the weak solution are given.

Keywords: L^p estimate, Moser's iteration, unique continuation, Hardy-Sobolev type inequality, H-type group.

Mathematical subject classification: 22E30, 35H20.

1 Introduction

On the Carnot group, Garofalo and Vassilev (see [6, 13]) had studied the non-linear Dirichlet problem

$$\begin{cases} \mathcal{L}u = -u^{\frac{Q+2}{Q-2}}, \\ u \in D_0^{1,2}(\Omega), \ u \geq 0, \end{cases} \tag{1.1}$$

corresponding to Sobolev inequality due to Folland and Stein [2, 3]

$$\left(\int_\Omega |u|^{p^*}\, dH\right)^{\frac{1}{p^*}} \leq C \left(\int_\Omega |Xu|^p\, dH\right)^{\frac{1}{p}}, \quad u \in C_0^\infty(\Omega), \tag{1.2}$$

Received 16 June 2006.
*This material is based upon work funded by Zhejiang Provincial Natural Science Foundation of China under Grant No. Y606144.

where X is the horizontal gradient, $\mathcal{L}$ is the sub-Laplacian associated to X, Q is the homogeneous dimension, $p^* = \frac{Qp}{Q-p}$ is the Sobolev conjugate, $D_0^{1,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \left(\int_\Omega |Xu|^2 \, dH\right)^{\frac{1}{2}}.$$

They proved the existence of extremal function of (1.2) by Concentration-Compactness principle of Lions (see [12]) and got the L^p estimates by combining the Serrin's idea (see [11]) and Moser's iteration technique (see [10]).

In 2005, the first author of this paper and Prof. Niu (see [8]) generalized Sobolev inequality (1.2) and obtained the Hardy-Sobolev type inequality on the H-type group.

Proposition 1.1 (Hardy-Sobolev type inequality). *Let* $1 < p < Q$, $0 \le s \le p$ *and* $p_*(s) = \frac{p(Q-s)}{Q-p}$. *For any* $u \in D_0^{1,p}(\Omega)$, *the following inequality holds:*

$$\left(\int_\Omega \frac{|x|^s}{\rho^s} \frac{|u|^{p_*(s)}}{\rho^s} \, dxdy\right)^{\frac{1}{p_*(s)}} \le C(s, p, Q) \left(\int_\Omega |Xu|^p \, dxdy\right)^{\frac{1}{p}}, \tag{1.3}$$

where $\Omega \subset G$ *(H-type group) is an open set,* $D_0^{1,p}(\Omega)$ *is the closure of* $C_0^\infty(\Omega)$ *with respect to the norm*

$$\|u\| = \left(\int_\Omega |Xu|^p \, dxdy\right)^{\frac{1}{p}}$$

and $C(s, p, Q)$ *is a positive constant independent of* u.

Remark 1.2. In the Euclidean space, Sobolev-Hardy inequalities similar to (1.3) were obtained by Badiale and Tarantello (see [1]) in 2002.

Let

$$\begin{aligned} I_{s,p} = \inf\Bigg\{ &\left(\int_G |Xu|^p \, dxdy\right)^{\frac{1}{p}} \Big| u \in D_0^{1,p}(G), \\ &\int_G \frac{|x|^s}{\rho^s} \frac{|u|^{p_*(s)}}{\rho^s} \, dxdy = 1, 0 \le s < p \Bigg\}. \end{aligned} \tag{1.4}$$

Using Lions' idea of Concentration-Compactness principle and choosing the suitable concentration function, the existence of extremal function of (1.4) was given.

Proposition 1.3. *In the extremal problem* (1.4) *the infimun is attained at a function* $u \in D_0^{1,p}(G)$ *which satisfies*

$$\int_G |Xu|^p \, dxdy = I(s,p)^p, \quad \int_G \frac{|x|^s}{\rho^s} \frac{|u|^{p_*(s)}}{\rho^s} \, dxdy = 1.$$

Henceforth, non-linear equation

$$\begin{cases} -\sum_{j=1}^m X_j(|Xv|^{p-2} X_j v) = \frac{|x|^s}{\rho^{2s}} v^{p_*(s)-1}, \\ v > 0, \ v \in D^{1,p}(G), 0 \le s < p \end{cases} \tag{1.5}$$

admits the nontrivial solution $v = I_{s,p}^{\frac{p}{p_*(s)-p}} u$, *where* u *is an extremal function for* (1.4).

The proof of the above proposition is given by Garofalo and Vassilev (see [6, 13]) for the case $s = 0$ and by us (see [8]) for the case $0 < s < p$.

In this paper, we will discuss its regularity by Serrin's idea and Moser's iteration technique, which generalize the results obtained by Vassilev (see [14]) in the n-dimensional Euclidean space.

The paper is organized as follows. In Section 2, we present some basic definitions and notations. Section 3 is devoted to L^p estimate for the extremal function of (1.4). As an application, we can study its asymptotic behavior. In Section 4, its doubling property is obtained and this leads to the strong unique continuation.

2 Preliminary

Consider a Carnot group G of step 2, with Lie algebra $\mathbf{g} = V_1 \oplus V_2$. We assume that $\mathbf{g}$ is equipped with a scalar product $< \cdot, \cdot >$, with respect to which the V_j's are mutually orthogonal. We use the exponential mapping $\exp\colon \mathbf{g} \to G$ to define analytic maps $\xi_i\colon G \to V_i, i = 1, 2$, through the equation $g = \exp(\xi_1(g) + \xi_2(g))$. Here, $\xi(g) = \xi_1(g) + \xi_2(g)$ is such that $g = \exp(\xi(g))$. With $m = \dim(V_1)$ and $V_1 = \mathrm{span}\{X_1, \ldots, X_m\}$ ($X_1, \ldots, X_m$ are orthonormal), the coordinates of the projection ξ_1 in the basis $X_1, \ldots, X_m$ are denoted by $x_1 = x_1(g), \ldots, x_m = x_m(g)$; that is,

$$x_j(g) = \langle \xi(g), X_j \rangle, \qquad j = 1, \ldots, m,$$

and we set $x = x(g) = (x_1(g), \ldots, x_m(g)) \in R^m$. Similarly, we fix an orthonormal basis $Y_1, \ldots, Y_k$ of V_2 and define the exponential coordinates in the second layer V_2 of a point $g \in G$ by letting

$$y_i(g) = \langle \xi(g), Y_i \rangle, \qquad i = 1, \ldots, k$$

and $y = (y_1, \cdots, y_k) \in R^k$.

For each $v \in V_1$, consider the orthogonal decomposition

$$V_1 = K_v \oplus R_v,$$

where $K_v = \ker(ad_v \colon V_1 \to V_2) = \{v' \in V_1 \colon [v, v'] = 0\}$. We shall say that the Lie algebra $\mathbf{g}$ is of H-type if the mapping $ad_v : R_v \to V_2$ is a surjective isometry for every unit vector $v \in V_1$ and the corresponding simple connected group G is named H-type group, which was introduced first by Kaplan[9] and extensively investigated by many authors (see [6, 7] etc.). If the dimension of the center of Lie algebra of an H-type is trivial, then the H-type group is isomorphic to a Heisenberg group.

In [6, 7], Garofalo and Vassilev pointed out that

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{i=1}^{k} \langle [\xi, X_j], Y_i \rangle \frac{\partial}{\partial y_i} \\ &= \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{i=1}^{k} \langle [\xi_1, X_j], Y_i \rangle \frac{\partial}{\partial y_i}, \qquad j = 1, \ldots, m, \end{aligned}$$

where

$$\xi = \xi_1 + \xi_2 \in \mathbf{g} = V_1 \bigoplus V_2, x = (x_1, \ldots, x_m) \in R^m, y = (y_1, \ldots, y_k) \in R^k.$$

For any differentiable function u on G, let $Xu = (X_1 u, \ldots, X_m u)$ denote the horizontal gradient and $|Xu| = \left(\sum_{j=1}^m |X_j u|^2\right)^{\frac{1}{2}}$.

A family of dilations is defined by

$$\delta_\lambda(x, y) = (\lambda x, \lambda^2 y), \qquad \text{for any } \lambda > 0, (x, y) \in G.$$

The homogeneous dimension of G with respect to dilations is $Q = m + 2k$. The norm function on G has the form

$$\rho(g) = d(g, 0) = \left(|x(g)|^4 + 16|y(g)|^2\right)^{\frac{1}{4}},$$

where 0 is the unit element of G. It is clear that ρ is quasi-homogeneous of degree one with respect to dilations above. Denote by $B_R(\xi) \equiv B(\xi, R) = \{\eta \in G | d(\xi, \eta) = \rho(\eta^{-1}\xi) < R\}$ the ball centered at ξ with radius R.

3 Regularity and asymptotic of extremal function

Let $1 < p < Q,\ 0 \leq s < p$. Denote by $p_*(s)$ the Hardy-Sobolev conjugate

$$p_*(s) = \frac{p(Q-s)}{Q-p} \tag{3.1}$$

and by p' the Hölder conjugate $p' = \frac{p}{p-1}$. For any s as above we define the exponent $r = r(s)$ to be the Hölder conjugate of the exponent $r' = r'(s)$ defined by

$$r' = \frac{p^*}{p_*(s) - p} = \frac{Q}{p-s}, \tag{3.2}$$

thus $r = \frac{Q}{Q-p+s} \geq 1$, $0 \leq rs \leq p$ and

$$rp = p_*(rs). \tag{3.3}$$

In the sequel, we usually let $\Omega \subset G$ be an open set (not necessarily bounded), $1 < p < Q$ and $0 \leq s < p$.

Theorem 3.1. *Assume that $u \in D_0^{1,p}(\Omega)$ is a non-negative weak solution of the inequality*

$$-\sum_{j=1}^{m} X_j \left(|Xu|^{p-2} X_j u\right) \leq V \frac{|x|^s}{\rho^s} \frac{|u|^{p-2}}{\rho^s} u \quad in \quad \Omega \tag{3.4}$$

i.e., for any non-negative function $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega |Xu|^{p-2} \langle Xu, X\phi \rangle \, dxdy \leq \int_\Omega V \frac{|x|^s}{\rho^s} \frac{|u|^{p-2}}{\rho^s} u\phi \, dxdy. \tag{3.5}$$

1) *If $V \in L^{r'}(\Omega)$, then $u \in L^q\left(\frac{|x|^t}{\rho^{2t}}\, dxdy\right)$ for any $0 \leq t < s,\ q \geq p_*(s)$;*

2) *If $V \in L^{t_0}(\Omega) \cap L^{r'}(\Omega)$ for some $t_0 > r'$, then $u \in L^\infty(\Omega)$.*

In particular $u \in L^q(\Omega)$ for every $p^* \leq q < \infty$.

Proof. The assumption $V \in L^{r'}(\Omega)$ and Hardy-Sobolev type inequality (1.3) show that (3.5) holds for any non-negative $\phi \in D_0^{1,p}(\Omega)$. In fact, take a sequence of test functions $\phi_n \in C_0^\infty(\Omega)$ which converge to ϕ in $D_0^{1,p}(\Omega)$. Observing $|Xu|^{p-1} \in L^{p'}$, we can put the limit function in the left-hand side of (3.5). On the other hand, for any $\phi \in C_0^\infty(\Omega)$, Hölder inequality and Hardy-Sobolev type inequality (1.3) lead to

$$\begin{aligned}
&\int_\Omega |V| \frac{|x|^s}{\rho^s} \frac{|u|^{p-1}}{\rho^s} \phi \, dxdy \\
&\leq \|V\|_{L^{r'}} \left(\int_\Omega \frac{|x|^{rs}}{\rho^{rs}} \frac{|u|^{r(p-1)}}{\rho^{rs}} |\phi|^r \, dxdy \right)^{\frac{1}{r}} \\
&\leq \|V\|_{L^{r'}} \left(\int_\Omega \frac{|x|^{rs}}{\rho^{rs}} \frac{|u|^{rp'(p-1)}}{\rho^{rs}} \, dxdy \right)^{\frac{1}{rp'}} \left(\int_\Omega \frac{|x|^{rs}}{\rho^{rs}} \frac{|\phi|^{rp}}{\rho^{rs}} \, dxdy \right)^{\frac{1}{rp}} \qquad (3.6) \\
&= \|V\|_{L^{r'}} \left(\int_\Omega \frac{|x|^{rs}}{\rho^{rs}} \frac{|u|^{p_*(rs)}}{\rho^{rs}} \, dxdy \right)^{\frac{p-1}{p_*(rs)}} \left(\int_\Omega \frac{|x|^{rs}}{\rho^{rs}} \frac{|\phi|^{p_*(rs)}}{\rho^{rs}} \, dxdy \right)^{\frac{1}{p_*(rs)}} \\
&\leq I_{rs,p}^p \|V\|_{L^{r'}} \|Xu\|_{L^p}^{p-1} \|X\phi\|_{L^p},
\end{aligned}$$

which allows to pass to the limit in the right-hand side of (3.5).

1) Define $G(t)(t \in \mathbb{R})$ on the real line as follows

$$G(t) = \begin{cases} \operatorname{sign}(t)|t|^{\frac{q}{p}} & \text{if } \; 0 \leq |t| \leq l, \\ l^{\frac{q}{p}-1} t & \text{if } \; l < |t|, \end{cases} \tag{3.7}$$

Obviously, G is a piece-wise smooth, globally Lipschitz function. Set

$$F(u) = \int_0^u |G'(t)|^p \, dt. \tag{3.8}$$

It is easy to obtain that

$$|u|^{p-1}|F(u)| \leq C(q)|G(u)|^p \leq C(q)|u|^q, \tag{3.9}$$

where $C(q) \leq Cq^p$, C is dependent on the constant p, but independent of q and l. From the chain rule, $G(u), F(u) \in D^{1,p}(\Omega)$. We claim that for any $q \geq p_*(s),\ 0 \leq t < s$, if $u \in L^q\left(\frac{|x|^s}{\rho^{2s}} \, dxdy\right)$, then $u \in L^{\kappa_t q}\left(\frac{|x|^t}{\rho^{2t}} \, dxdy\right)$, where $\kappa_t = \frac{p_*(t)}{p}$, and there exists a constant C depending on $p,\ q,\ \|V\|_{L^{r'}}$ such that

$$\|u\|_{L^{\kappa_t q}(\frac{|x|^t}{\rho^{2t}} \, dxdy)} \leq C\|u\|_{L^q(\frac{|x|^s}{\rho^{2s}} \, dxdy)}. \tag{3.10}$$

Taking $\phi = F(u)$ and substituting into (3.5), the left-hand side can be rewritten as

$$\int_\Omega |Xu|^{p-2} < Xu, XF(u) > \, dxdy = \int_\Omega |XG(u)|^p \, dxdy. \tag{3.11}$$

Let $M > 0$ be a constant that will be fixed in the sequel and estimate the integral in the right-hand side of (3.5) as follows.

$$\begin{aligned}
&\int_\Omega V \frac{|x|^s}{\rho^s} \frac{|u|^{p-1}}{\rho^s} F(u)\, dxdy \\
=\ & \int_{(|V| \le M)} V \frac{|x|^s}{\rho^s} \frac{|u|^{p-1}}{\rho^s} F(u)\, dxdy \\
&+ \int_{(|V| > M)} V \frac{|x|^s}{\rho^s} \frac{|u|^{p-1}}{\rho^s} F(u)\, dxdy \\
\le\ & C(q) \int_{(|V| \le M)} |V| \frac{|x|^s}{\rho^s} \frac{|G(u)|^p}{\rho^s}\, dxdy \\
&+ C(q) \|V\|_{L^{r'}(|V|>M)} \left(\int_\Omega \frac{|x|^{rs}}{\rho^{rs}} \frac{|G(u)|^{rp}}{\rho^{rs}}\, dxdy \right)^{\frac{1}{r}} \\
\le\ & C(q) M \|u\|^q_{L^q\left(\frac{|x|^s}{\rho^{2s}}\, dxdy\right)} + C(q) I^p_{rs,p} \|V\|_{L^{r'}(|V|>M)} \|XG(u)\|^p_{L^p}.
\end{aligned} \tag{3.12}$$

Because of $V \in L^{r'}$, so we can take M sufficiently large such that

$$C(q) I^p_{rs,p} \|V\|_{L^{r'}(|V|>M)} \le \frac{1}{2}.$$

Combining (3.11), (3.12) and Hardy-Sobolev type inequality (1.3), we have

$$\|G(u)\|^p_{L^{p*(t)}\left(\frac{|x|^t}{\rho^{2t}}\, dxdy\right)} \le 2C(q) M I^p_{s,p} \|u\|^q_{L^q\left(\frac{|x|^s}{\rho^{2s}}\, dxdy\right)}. \tag{3.13}$$

Let $l \to \infty$ in the definition of G. By Fatou's theorem we get

$$\|u\|_{L^{\kappa_t q}\left(\frac{|x|^t}{\rho^{2t}}\, dxdy\right)} \le (2C(q) M I^p_{s,p})^{\frac{1}{q}} \|u\|_{L^q\left(\frac{|x|^s}{\rho^{2s}}\, dxdy\right)},$$

The proof of part 1) is finished.

2) The assumption $t_0 > r'$ implies that $t_0' = \frac{t_0}{t_0-1} < r$ and then $0 \le t_0's < rs$. Therefore,

$$\left(\int_\Omega \frac{|x|^{st_0'}}{\rho^{2st_0'}}|u|^{qt_0'}\,dxdy\right)^{\frac{1}{qt_0'}} < +\infty$$

due to part 1). In the following, we will prove that for any $q \ge q_0 = t_0'p_*(s)$, $\|u\|_{L^q(\frac{|x|^{st_0'}}{\rho^{2st_0'}}\,dxdy)}$ are uniformly bounded and there exists constant C such that

$$\left(\int_\Omega \frac{|x|^{st_0'}}{\rho^{2st_0'}}|u|^{q}\,dxdy\right)^{\frac{1}{q}} \le C\left(\int_\Omega \frac{|x|^{st_0'}}{\rho^{2st_0'}}|u|^{q_0}\,dxdy\right)^{\frac{1}{q_0}}.$$

For any $q \ge p_*(s)$, taking $\phi = F(u)$ as in the part 1) and substituting in (3.5), by (3.11), we have

$$\begin{aligned}\|XG\|_{L^p}^p &\le \int_\Omega |V|\frac{|x|^s}{\rho^s}\frac{|u|^{p-1}}{\rho^s}F(u)\,dxdy\\ &\le C(q)\|V\|_{L^{t_0}}\|u\|^q_{L^{qt_0'}\left(\frac{|x|^{st_0'}}{\rho^{2st_0'}}\,dxdy\right)}.\end{aligned}$$

Noting that $C(q) \le C(p)q^p$, when $l \to \infty$, Hardy-Sobolev type inequality (1.3) and Fatou's theorem lead to

$$\begin{aligned}&\left(\int_\Omega \frac{|x|^{st_0'}}{\rho^{st_0'}}\frac{|u|^{\frac{q}{p}p_*(st_0')}}{\rho^{st_0'}}\,dxdy\right)^{\frac{p}{p_*(st_0')}}\\ &\le I^p_{st_0',p}C(p)q^p\|V\|_{L^{t_0}}\left(\int_\Omega \frac{|x|^{st_0'}}{\rho^{2st_0'}}|u|^{qt_0'}\,dxdy\right)^{\frac{1}{t_0'}}.\end{aligned} \tag{3.14}$$

Let $\delta = \frac{p_*(st_0')}{pt_0'} > \frac{p_*(st_0')}{rp} = \frac{p_*(st_0')}{p_*(rs)} > 1$. (3.14) can be rewritten as

$$\begin{aligned}&\|u\|_{L^{\delta qt_0'}(\frac{|x|^{st_0'}}{\rho^{2st_0'}}\,dxdy)}\\ &\le I^{\frac{p}{q}}_{st_0',p}C(p)^{q^{-1}}q^{\frac{p}{q}}\|V\|^{q^{-1}}_{L^{t_0}}\|u\|_{L^{qt_0'}\left(\frac{|x|^{st_0'}}{\rho^{2st_0'}}\,dxdy\right)}\\ &= C(p,s,t_0)^{\frac{1}{qt_0'}}(qt_0')^{\frac{pt_0'}{qt_0'}}\|V\|^{\frac{t_0'}{qt_0'}}_{L^{t_0}}\|u\|_{L^{qt_0'}\left(\frac{|x|^{st_0'}}{\rho^{2st_0'}}\,dxdy\right)}.\end{aligned} \tag{3.15}$$

Letting $q_0 = p_*(s)t_0',\ q_k = \delta^k q_0,\ C = C(p,s,t_0)$, we have

$$\|u\|_{L^{q_k}\left(\frac{|x|^{st_0'}}{\rho^{2st_0'}}\,dxdy\right)} \le \prod_{j=0}^{k-1}\left[Cq_j^{pt_0'}\right]^{\frac{1}{q_j}} \|V\|_{L^{t_0}}^{t_0'\sum_{j=0}^{k-1}\frac{1}{q_j}} \|u\|_{L^{q_0}\left(\frac{|x|^{st_0'}}{\rho^{2st_0'}}\,dxdy\right)}. \tag{3.16}$$

Because $\delta > 1$, thus

$$\sum_{j=0}^{\infty}\frac{1}{q_j} = \frac{1}{q_0}\sum_{j=0}^{\infty}\frac{1}{\delta^j} < \infty, \qquad \sum_{j=0}^{\infty}\frac{\log q_j}{q_j} < \infty. \tag{3.17}$$

When $k \to \infty$, (3.16) leads to

$$\|u\|_{L^\infty} \le C\|u\|_{L^{q_0}\left(\frac{|x|^{st_0'}}{\rho^{2st_0'}}\,dxdy\right)}. \qquad \square$$

Corollary 3.2. *If $R \in L^\infty$ and $u \in D_0^{1,p}(\Omega)$ is a non-negative weak solution of inequality*

$$-\sum_{j=1}^{m} X_j\left(|Xv|^{p-2}X_j u\right) \le R\frac{|x|^s}{\rho^s}\frac{|u|^{p_*(s)-2}}{\rho^s}u \quad in \quad \Omega,$$

then $u \in L^\infty(\Omega)$.

Proof. Let $V = R|u|^{p_*(s)-p}$ and then $V \in L^{\frac{p^*}{p_*(s)-p}}(\Omega)$. By the part 1) of Theorem 3.1, $u \in L^q(\Omega)(p^* \le q < \infty)$ and $V \in L^{\frac{q}{p_*(s)-p}}(\Omega)$. Combining the part 2) of Theorem 3.1, $u \in L^\infty(\Omega)$ holds. $\square$

Theorem 3.3. *Assume that $V \in L^{r'},\ V_0 \in L^1 \cap L^{r'}$. If $u \in D_0^{1,p}(\Omega)$ is non-negative locally bounded weak solution of the inequality*

$$-\sum_{j=1}^{m} X_j\left(|Xu|^{p-2}X_j u\right) \le V\frac{|x|^s}{\rho^s}\frac{|u|^{p-2}}{\rho^s}u + V_0, \tag{3.18}$$

then $u \in L^q\left(\frac{|x|^s}{\rho^{2s}}\,dxdy\right)$ for every $\frac{p_*(s)}{p'} < q \le p_*(s)$. In particular, $u \in L^q$ for every $\frac{p^*}{p'} < q \le p^*$.

Proof. Obviously, $u \in L^{p_*(s)}\left(\frac{|x|^s}{\rho^{2s}}\,dxdy\right)$. Henceforth, it is enough to discuss the case $\frac{p_*(s)}{p'} < q < p_*(s)$. Let $0 < \theta < \frac{1}{p}$ and define the function $f = u^{1-\theta}$. If we can prove $\|Xf\|_{L^p} < \infty$, then we can get $f \in L^{p_*(s)}\left(\frac{|x|^s}{\rho^{2s}}\,dxdy\right)$ and complete the proof.

Define function $\eta_m(t)$ as follows

$$\eta_m(t) = \begin{cases} t^{1-\theta p}, & \text{if } \; t \geq \frac{1}{m} \\ 2m^{\theta p}(t - \frac{1}{2m}), & \text{if } \; \frac{1}{2m} < t < \frac{1}{m} \\ 0, & \text{if } \; t \leq \frac{1}{2m}. \end{cases} \tag{3.19}$$

It is easy to prove that $\eta_m(t)$ is a piece-wise smooth, globally Lipschitz function and there exists a constant $C(p,\theta)$ such that

$$0 \leq \eta_m(t) \leq 2^{2-\theta p}|t|^{1-\theta p} \equiv C_{p,\theta}|t|^{1-\theta p}. \tag{3.20}$$

Take $\phi_m = \eta_m(u) \geq 0$ and then $\phi_m \in D^{1,p}(\Omega)$.

Let

$$j = \left(\frac{p^*}{1-\theta p}\right)' = \frac{p^*}{p^* - (1-\theta p)}.$$

Obviously,

$$r' = \frac{p^*}{p_*(s) - p} > j > 1. \tag{3.21}$$

The fact $V_0 \in L^1 \cap L^{r'}$ implies that $V_0 \in L^j$. Taking $\phi = \phi_m$ as a test function, substituting it into (3.18) and noting $pr = p_*(rs)$, we have the following three estimates:

$$\begin{aligned} &\int_\Omega |Xu|^{p-2}Xu \cdot X\phi_m \, dxdy \\ &= \int_{\frac{1}{2m}<u<\frac{1}{m}} |Xu|^{p-2}Xu \cdot X\phi_m \, dxdy \\ &\quad + \int_{\frac{1}{m}\leq u} |Xu|^{p-2}Xu \cdot X\phi_m \, dxdy \\ &= 2m^{\theta p}\int_{\frac{1}{2m}<u<\frac{1}{m}} |Xu|^p \, dxdy + \frac{1-\theta p}{(1-\theta)^p}\int_{\frac{1}{m}\leq u} |Xf|^p \, dxdy, \end{aligned} \tag{3.22}$$

$$
\begin{aligned}
&\int_\Omega V \frac{|x|^s}{\rho^s} \frac{|u|^{p-1}}{\rho^s} \phi_m \, dxdy \\
&\quad = \int_{u \geq \frac{1}{m}} V \frac{|x|^s}{\rho^s} \frac{|u|^{p-1}}{\rho^s} \phi_m \, dxdy + \int_{\frac{1}{2m} < u < \frac{1}{m}} V \frac{|x|^s}{\rho^s} \frac{|u|^{p-1}}{\rho^s} \phi_m \, dxdy \\
&\quad \leq C_{p,\theta} \int_{u \geq \frac{1}{m}} |V| \frac{|x|^s}{\rho^s} \frac{|f|^p}{\rho^s} \, dxdy + 2m^{\theta p} \int_{\frac{1}{2m} < u < \frac{1}{m}} |V| \frac{|x|^s}{\rho^s} \frac{|u|^p}{\rho^s} \, dxdy \\
&\quad \leq C_{p,\theta} \|V\|_{L^{r'}} \left(\int_{u \geq \frac{1}{m}} \frac{|x|^{rs}}{\rho^{rs}} \frac{|f|^{rp}}{\rho^{rs}} \, dxdy \right)^{\frac{1}{r}} \\
&\qquad + 2m^{\theta p} \|V\|_{L^{r'}} \left(\int_{\frac{1}{2m} < u < \frac{1}{m}} \frac{|x|^{rs}}{\rho^{rs}} \frac{|u|^{rp}}{\rho^{rs}} \, dxdy \right)^{\frac{1}{r}} \\
&\leq C_{p,\theta} \|V\|_{L^{r'}} I^p_{rs,p} \int_{u \geq \frac{1}{m}} |Xf|^p \, dxdy \\
&\qquad + 2m^{\theta p} \|V\|_{L^{r'}} I^p_{rs,p} \int_{\frac{1}{2m} < u < \frac{1}{m}} |Xu|^p \, dxdy,
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&\int_\Omega V_0 \phi_m \, dxdy \leq C_{p,\theta} \int_{\frac{1}{2m} < u < \frac{1}{m}} |V_0| u^{1-\theta p} \, dxdy \\
&\qquad + C_{p,\theta} \int_{u \geq \frac{1}{m}} |V_0| u^{1-\theta p} \, dxdy \leq C_{p,\theta} m^{\theta p - 1} \int_{\frac{1}{2m} < u < \frac{1}{m}} |V_0| \, dxdy \\
&\qquad + C_{p,\theta} \|V_0\|_{L^j\left(u \geq \frac{1}{m}\right)} \|u\|^{1-\theta p}_{L^{p^*}\left(u \geq \frac{1}{m}\right)}.
\end{aligned}
\tag{3.24}
$$

Combining (3.18), (3.22), (3.23) and (3.24), we have

$$
\begin{aligned}
&2m^{\theta p} \left(1 - I^p_{rs,p} \|V\|_{L^{r'}}\right) \int_{\frac{1}{2m} < u < \frac{1}{m}} |Xu|^p \, dxdy \\
&\qquad + \left(\frac{1 - \theta p}{(1-\theta)^p} - C_{p,\theta} I^p_{rs,p} \|V\|_{L^{r'}} \right) \int_{u \geq \frac{1}{m}} |Xf|^p \, dxdy \\
&\quad \leq C_{p,\theta} m^{\theta p - 1} \int_{\frac{1}{2m} < u < \frac{1}{m}} |V_0| \, dxdy + C_{p,\theta} \|V_0\|_{L^j\left(u \geq \frac{1}{m}\right)} \|u\|^{1-\theta p}_{L^{p^*}\left(u \geq \frac{1}{m}\right)}.
\end{aligned}
\tag{3.25}
$$

Let

$$C_1 = 2m^{\theta p}\left(1 - I^p_{rs,p}\|V\|_{L^{r'}}\right),\ C_2 = \frac{1-\theta p}{(1-\theta)^p} - C_{p,\theta} I^p_{rs,p}\|V\|_{L^{r'}}.$$

If $C_1, C_2 > 0$, then for every θ, $\|Xf\|_{L^p} < \infty$.

With this in mind, fix a sufficiently large constant $R_0 > 0$ and a sufficiently small constant $\delta > 0$, which we shall choose in a moment. Let $B_{2\delta,R_0} = G\backslash B_{R_0}(0) \cup B_{2\delta}(0)$ and take the function $\alpha \in C_0^\infty(B_{2\delta,R_0})$ satisfying $0 \le \alpha \le 1$ and $\alpha \equiv 1$ on $B_{\delta,2R_0}$. Let $V' = \alpha V$ and then for every θ, we choose δ and R_0 such that

$$2m^{\theta p}\left(1 - I^p_{rs,p}\|V'\|_{L^{r'}}\right) > 0, \quad \frac{1-\theta p}{(1-\theta)^p} - C_{p,\theta} I^p_{rs,p}\|V'\|_{L^{r'}} > 0.$$

Take

$$g = (1-\alpha)V\frac{|x|^s}{\rho^s}\frac{|u|^{p-2}}{\rho^s}u + V_0 \equiv g_0 + V_0$$

and then $g \in L^1$. On the other hand, the fact $\operatorname{supp} g_0 \subset B_{2\delta,R_0}$ and the fact that u is locally bounded lead to

$$\int_{2\delta\le\rho\le R_0} |g_0|^{r'}\,dxdy \le C\max\left\{\frac{1}{(2\delta)^s}, \frac{1}{R_0^s}\right\}\int_{2\delta\le\rho\le R_0} |V|^{r'}\,dxdy < \infty.$$

Because of $V_0 \in L^{r'}$, thus $g \in L^{r'}$. Henceforth, $g \in L^j$.

Rewriting the inequality (3.18) in the form

$$-\sum_{j=1}^m X_j\left(|Xu|^{p-2}X_j u\right) \le V'\frac{|x|^s}{\rho^s}\frac{|u|^{p-2}}{\rho^s}u + g, \tag{3.26}$$

replacing V, V_0 with V', g and applying (3.25) to (3.26), we obtain that $\|Xf\|_{L^p} < \infty$ holds for every θ. □

Remark 3.4. If $u \in D_0^{1,p}(\Omega)$ is a non-negative solution of nonlinear equation (1.5), then Theorem 3.2 and Theorem 3.3 show that $u \in L^q$, where $\frac{p^*}{p} < q \le +\infty$.

Theorem 3.5. *Let $u \in D_0^{1,p}(\Omega)$ be a non-negative weak solution to the equation*

$$-\sum_{j=1}^m X_j\left(|Xu|^{p-2}X_j u\right) = \frac{|x|^s}{\rho^s}\frac{|u|^{p_*(s)-2}}{\rho^s}u \quad \text{in } \Omega, \tag{3.27}$$

i.e., for every $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega |Xu|^{p-2}\langle Xu, X\phi\rangle\, dxdy = \int_\Omega \frac{|x|^s}{\rho^s}\frac{|u|^{p_*(s)-2}}{\rho^s} u\phi\, dxdy. \tag{3.28}$$

We assume that u *has been extended with zero outside* Ω. *Suppose that* $q \geq p$ *is an exponent such that* $u \in L^q(\Omega)$. *There exists* $C = C(p) > 0$ *such that for every* $\xi \in G \backslash B_3(0)$

$$\operatorname*{ess\,sup}_{B(\xi,1)} u \leq C\left(\frac{1}{|B(\xi,2)|}\int_{B(\xi,2)} |u|^q\, dxdy\right)^{\frac{1}{q}}. \tag{3.29}$$

In particular, we can take $q = p^*$ *in the above inequality.*

Proof. Given a non-negative function $\alpha \in C_0^\infty(G)$, for $\gamma \geq 1$ we consider the function $\phi = \alpha^p u^\gamma \in D_0^{1,p}(\Omega)$. Using ϕ as a test function in (3.28) we find

$$\begin{aligned} \int_\Omega \alpha^p \frac{|x|^s}{\rho^s}\frac{|u|^{\gamma+p_*(s)-1}}{\rho^s}\, dxdy &= \int_\Omega \frac{|x|^s}{\rho^s}\frac{|u|^{p_*(s)-2}}{\rho^s} u\alpha^p u^\gamma\, dxdy \\ &= \gamma \int_\Omega \alpha^p u^{\gamma-1}|Xu|^p\, dxdy \\ &\quad + p\int_\Omega \alpha^{p-1}u^\gamma |Xu|^{p-2}\langle Xu, X\alpha\rangle\, dxdy, \end{aligned}$$

At this point we choose $\gamma = q - p + 1$ to obtain

$$\begin{aligned} \int_\Omega \alpha^p \frac{|x|^s}{\rho^s}\frac{|u|^{q+p_*(s)-p}}{\rho^s}\, dxdy \ &= \ (q-p+1)\int_\Omega \alpha^p u^{q-p}|Xu|^p \\ &+ \ p\int_\Omega \alpha^{p-1}u^{q-p+1}|Xu|^{p-2}\langle Xu, X\alpha\rangle\, dxdy \\ &\geq \ (q-p+1)\int_\Omega \alpha^p u^{q-p}|Xu|^p\, dxdy \\ &- \ p\int_\Omega \alpha^{p-1}u^{q-p+1}|Xu|^{p-1}|X\alpha|\, dxdy. \end{aligned} \tag{3.30}$$

Now Young's inequality gives

$$
\begin{aligned}
&\int_\Omega \alpha^{p-1} u^{q-p+1} |Xu|^{p-1} |X\alpha| \, dxdy \\
&\quad = \int_\Omega \alpha^{p-1} u^{\frac{q-p}{p'}} |Xu|^{p-1} u^{\frac{q}{p}} |X\alpha| \, dxdy \\
&\quad \le \left(\int_\Omega \alpha^p u^{q-p} |Xu|^p \, dxdy \right)^{\frac{1}{p'}} \left(\int_\Omega u^q |X\alpha|^p \, dxdy \right)^{\frac{1}{p}} \\
&\quad \le \frac{\epsilon}{p'} \int_\Omega \alpha^p u^{q-p} |Xu|^p \, dxdy + \frac{\epsilon^{-\frac{p}{p'}}}{p} \int_\Omega u^q |X\alpha|^p \, dxdy,
\end{aligned} \tag{3.31}
$$

where $\epsilon > 0$ is an arbitrary constant fixed in a moment. Substituting (3.31) into (3.30) and choosing $\epsilon = \frac{1}{p}$, we have

$$
\begin{aligned}
&\int_\Omega \alpha^p \frac{|x|^s}{\rho^s} \frac{|u|^{q+p_*(s)-p}}{\rho^s} \, dxdy \\
&\quad \ge \left(q - p + \frac{1}{p} \right) \int_\Omega \alpha^p u^{q-p} |Xu|^p \, dxdy \\
&\qquad - p^{p-1} \int_\Omega u^q |X\alpha|^p \, dxdy \\
&\quad \ge \frac{1}{p} \int_\Omega \alpha^p u^{q-p} |Xu|^p \, dxdy - p^{p-1} \int_\Omega u^q |X\alpha|^p \, dxdy.
\end{aligned} \tag{3.32}
$$

Let $\psi = u^{\frac{q}{p}}$ and then $|X\psi|^p = (\frac{q}{p})^p u^{q-p} |Xu|^p$. We obtain from (3.32)

$$
\begin{aligned}
&\int_\Omega \alpha^p |X\psi|^p \, dxdy = \left(\frac{q}{p} \right)^p \int_\Omega \alpha^p u^{q-p} |Xu|^p \, dxdy \\
&\le p \left(\frac{q}{p} \right)^p \int_\Omega \alpha^p \frac{|x|^s}{\rho^s} \frac{|u|^{\gamma+p_*(s)-1}}{\rho^s} \, dxdy + q^p \int_\Omega u^q |X\alpha|^p \, dxdy.
\end{aligned} \tag{3.33}
$$

For $\xi \in B_3(0)$ and $1 \le r < R \le 2$, let $\alpha \in C_0^\infty(B(\xi, R))$ such that $\alpha \equiv 1$ in the $B(\xi, r)$ and $|X\alpha| \le \frac{C}{R-r}$. Assuming $\delta = \frac{p^*}{p} > 1$, Hardy-Sobolev type inequality (1.3) and Theorem 3.1 lead to

$$
\begin{aligned}
\left(\int_{B(\xi,r)} u^{\delta q}\,dxdy\right)^{\frac{1}{\delta}} &= \left(\int_{B(\xi,r)} \psi^{p^*}\,dxdy\right)^{\frac{1}{\delta}} \\
&\leq I_{0,p}^p \int_{B(\xi,R)} |X(\alpha\psi)|^p\,dxdy \\
&\leq 2^p I_{0,p}^p \left(\int_{B(\xi,R)} \alpha^p |X\psi|^p\,dxdy + \int_{B(\xi,R)} |\psi|^p |X\alpha|^p\,dxdy\right) \\
&\leq 2^p I_{0,p}^p \left(p\left(\frac{q}{p}\right)^p \int_{B(\xi,R)} \alpha^p \frac{|x|^s}{\rho^s}\frac{|u|^{q+p_*(s)-p}}{\rho^s}\,dxdy\right. \\
&\qquad\left. + (q^p+1)\int_{B(\xi,R)} u^q |X\alpha|^p\,dxdy\right) \\
&\leq 2^p I_{0,p}^p \left(p\left(\frac{q}{p}\right)^p \|u\|_{L^\infty(\Omega)}^{p_*(s)-p} \int_{B(\xi,R)} \alpha^p u^q\,dxdy\right. \\
&\qquad\left. + (q^p+1)\int_{B(\xi,R)} u^q |X\alpha|^p\,dxdy\right),
\end{aligned}
\tag{3.34}
$$

namely, there exists $K = K(p, \|u\|_{L^\infty(\Omega)})$ such that

$$
\left(\int_{B(\xi,r)} u^{\delta q}\,dxdy\right)^{\frac{1}{\delta q}} \leq \frac{K^{\frac{1}{q}} q^{\frac{p}{q}}}{|R-r|^{\frac{p}{q}}} \left(\int_{B(\xi,R)} u^q\,dxdy\right)^{\frac{1}{q}}.
$$

Assuming the finiteness of the integral in the right-hand side of the latter inequality, Moser's iteration procedure finally gives (3.29). □

Corollary 3.6. *Let $\Omega \subset G$ be an unbounded open set. If $u \in D_0^{1,p}(\Omega)$ is a weak solution to the equation* (3.27)*, then*

$$
\lim_{\xi\in G,\, d(\xi,e)\to\infty} u(\xi) = 0.
$$

4 Unique continuation

Theorem 4.1. *Let $u \in D_0^{1,p}(\Omega)$ be a non-negative weak solution of the equation*

$$
-\sum_{j=1}^m X_j\left(|Xu|^{p-2}X_j u\right) = V\frac{|x|^s}{\rho^s}\frac{|u|^{p-2}}{\rho^s}u
$$

with $V \in L^{r'}(\Omega)$, i.e., for any $\phi \in D^{1,p}(\Omega)$

$$\int_\Omega |Xu|^{p-2}\langle Xu, X\phi\rangle\, dxdy = \int_\Omega V\frac{|x|^s}{\rho^s}\frac{|u|^{p-2}}{\rho^s}u\phi\, dxdy. \tag{4.1}$$

There exist $q = q(G, \|V\|_{L^{r'}(\Omega)}) > 0$ and $C = C(G, \|V\|_{L^{r'}(\Omega)}) > 0$ such that for every $\bar{B}(\xi, 2r) \subset \Omega$ one has

$$\int_{B(\xi,2r)} u^q\, dxdy \le C\int_{B(\xi,r)} u^q\, dxdy.$$

Proof. Let $\phi = \alpha^p(u+\epsilon)^{-p+1}$, $\epsilon > 0$ with $\alpha \in C_0^\infty(\Omega)$ satisfying $0 \le \alpha \le 1$, $\alpha \equiv 1$ in $B(\xi, r)$, $\alpha \equiv 0$ outside $B(\xi, 2r)$, $|X\alpha| \le C/r$. Substituting ϕ into (4.1), we have

$$\begin{aligned}&(p-1)\int_\Omega \alpha^p(u+\epsilon)^{-p}|Xu|^p\, dxdy\\ &\quad\le\ p\int_\Omega \alpha^{p-1}(u+\epsilon)^{-p+1}|Xu|^{p-1}|X\alpha|\, dxdy + \int_\Omega |V|\frac{|x|^s}{\rho^s}\frac{\alpha^p}{\rho^s}\, dxdy.\end{aligned}$$

Let $v = \log(u+\epsilon)$ and then the above formula can be rewritten as follows

$$(p-1)\int_\Omega \alpha^p|Xv|^p\, dxdy \le p\int_\Omega \alpha^{p-1}|Xv|^{p-1}|X\alpha|\, dxdy + \int_\Omega |V|\frac{|x|^s}{\rho^s}\frac{\alpha^p}{\rho^s}\, dxdy.$$

Applying Hölder inequality, Young inequality and Hardy-Sobolev type inequality (1.3), for every $\sigma > 0$ we have

$$\begin{aligned}&\int_\Omega \alpha^p|Xv|^p\, dxdy\\ &\quad\le\ p'\int_\Omega \alpha^{p-1}|Xv|^{p-1}|X\alpha|\, dxdy + \frac{1}{p-1}\int_\Omega |V|\frac{|x|^s}{\rho^s}\frac{\alpha^p}{\rho^s}\, dxdy\\ &\quad\le\ p'\left(\int_\omega \alpha^p|Xv|^p\, dxdy\right)^{\frac{1}{p'}}\left(\int_\Omega |X\alpha|^p\, dxdy\right)^{\frac{1}{p}}\\ &\qquad+\ \frac{1}{p-1}\|V\|_{L^{r'}(B(\xi,2r))}\left(\int_\Omega \frac{|x|^{rs}}{\rho^{rs}}\frac{\alpha^{rp}}{\rho^{rs}}\, dxdy\right)^{\frac{1}{r}}\\ &\quad\le\ \sigma\int_\Omega \alpha^p|Xv|^p\, dxdy + \frac{\sigma^{1-p}}{p-1}\int_\Omega |X\alpha|^p\, dxdy\\ &\qquad+\ \frac{I_{rs,p}^p}{p-1}\|V\|_{L^{r'}(B(\xi,2r))}\int_\Omega |X\alpha|^p\, dxdy.\end{aligned}$$

Choosing $\sigma = \frac{p-1}{p}$ we get

$$\begin{aligned}\int_\Omega \alpha^p |Xv|^p \, dxdy \ &\leq \ \left(\frac{p}{p-1}\right)^p \frac{C^p}{r^p} |B(\xi, 2r)| \\ &\quad + \frac{p}{p-1} \frac{C^p}{r^p} |B(\xi, 2r)| \|V\|_{L^{r'}(B(\xi,2r))} \\ &= \ C(p) \frac{|B(\xi, 2r)|}{r^p} \left(1 + \|V\|_{L^{r'}(B(\xi,2r))}\right),\end{aligned}$$

where $C(p) = \left(\frac{p}{p-1}\right)^p C^p$. Repeating the procedure of Theorem 10.6 in [6], we finish the proof. □

By Theorem 4.1, we immediately deduce a unique continuation as follows

Corollary 4.2. *Let $u \in D^{1,p}(\Omega)$ be a non-negative weak solution to* (3.27) *in a connected, open set $\Omega \subset G$. If u vanishes to infinite order at one point $g \in \Omega$, then $u \equiv 0$ in Ω.*

The proof is similar to the proof of Corollary 10.7 in [6].

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