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Singular hyperbolicity for transitive attractors with singular points of 3-dimensional C^2-flows

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Abstract. In the context of C^r-flows on 3-manifolds ($r \geqslant 1$), the notion of singular hyperbolicity, inspired on the Lorenz Attractor, is the right generalization of hyperbolicity (in the sense of Smale) for C^1-robustly transitive sets with singularities. We estabish conditions (on the associated linear Poincaré flow and on the nature of the singular set) under which a transitive attractor with singularities of a C^2-flow on a 3-manifold is singular hyperbolic.

Keywords: singular attractor, singular hyperbolic, flow.

Mathematical subject classification: 37C10, 37D45, 37D30.

1 Introduction

Differential equations given by a continuous vector field on a riemannian manifold have two different kinds of solutions from a dynamical viewpoint: regular orbits and equilibrium (fixed) points. When these two kinds of orbits are isolated from each other, the notion of hyperbolicity (in the sense of Smale in [8]) provides a complete portrait of the dynamics, like in the case of diffeomorphisms. However, when fixed points are accumulated by regular orbits of the same transitive piece it is impossible to find a continuous hyperbolic splitting. This is due to the absence of the flow direction on the tangent space over equilibria. Notwithstanding, in a series of papers, Morales, Pacífico and Pujals have been developing a weaker notion called *singular hyperbolicity*, which turns out to be adequate in this context. This notion is inspired in the Lorenz Attractor (see [5], [4], [10] and [9]), which is the paradigm of this scenario.

Regarding the conjecture about generic dynamics for 3-dimensional flows: hyperbolicity, singular hyperbolicity and homoclinic tangencies being C^1-dense

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among 3-dimensional C^1-flows, stated in [7], it seems important to obtain conditions in which a transitive set with singularities is singular hyperbolic.

Main Result in this paper is to prove that a transitive *attractor* set with singularities is singular hyperbolic provided it satisfies two basic conditions. The first one is about the nature of the singular set, and is an extension of the notion of Lorenz-like singular point for several equilibirum points in the same transitive piece. The second condition is about the existence of a *dominated splitting* for the associated linear Poincaré flow (defined below). Second condition is the natural setting for a flow which is not C^1-approximated by systems exhibiting homoclinic tangencies between periodic orbits; in particular, in [2] is proved that in this situation, regular points in the closure of the set of hyperbolic orbits do have a dominated splitting for the linear Poincaré flow.

This characterization of singular hyperbolicity plays an important role towards a proof of the conjecture about generic dynamics for 3-dimensional flows.

Let us state some precise definitions. Let M be a 3-dimensional closed differentiable manifold endowed with an auxiliary Riemannian metric. A C^r-flow $\Phi : \mathbb{R} \times M \to M$ is an action of the group $\mathbb{R}$ on M by C^r-diffeomorphisms $\Phi_t := \Phi(t, \cdot)$, with $r \geqslant 1$. A singular point of Φ is a fixed point of the action. Vector fields on M are closely related to flows; in fact, given a C^r-flow on M, one obtain a C^r-vector field $X(p) := \frac{d}{dt}\Phi(t, p)|_{t=0}$ associated to Φ (and vice-versa by integration). A compact subset $\Lambda \subset M$ is an isolated invariant set of Φ if there is a neighborhood U containing Λ such that $\bigcap_{t\in\mathbb{R}} \Phi(U) = \Lambda$. The set Λ is transitive if it contains a dense orbit; and it is C^1-robustly transitive if $\bigcap_{t\in\mathbb{R}} \Psi(U)$ is transitive for any Ψ in a C^1-neighborhood of Φ.

A continuous $D\Phi_t$-invariant splitting $T_\Lambda M = E \oplus F$ over Λ is *dominated* if there are constants $C > 0$ and $\rho < 0$ such that,

$$\forall x \in \Lambda : \|D\Phi_t|_{E(x)}\| \, \|D\Phi_{-t}|_F(\Phi_t(x))\| < C \exp(t\rho), \quad \text{for any} \quad t > 0.$$

Such dominated splitting is *partially hyperbolic* if E is one dimensional and contracting, that is, there are $\tilde{C}$ and $\tilde{\rho} < 0$ such that $\forall x \in \Lambda$: $\|D\Phi_t|_{E(x)}\| < \tilde{C}\exp(t\tilde{\rho})$, for any $t > 0$.

Definition. *An invariant set Λ of a C^1-flow is singular hyperbolic if: any singular point in Λ is hyperbolic and, either for Φ_t or for Φ_{-t}, there is a partially hyperbolic $D\Phi_t$-invariant splitting $T_\Lambda M = E^s \oplus E^{cu}$; and there exists $C > 0$ and $\rho < 0$ such that for any $t > 0$ we have that:*

$$|\det(D\Phi_{-t}|_{E^{cu}})| < C \exp(t\rho).$$

A compact invariant set Λ is an *attractor* if there is an open set U such that $\Lambda = \bigcap_{t\geqslant 0} \Phi(\mathrm{cl}(U))$; $\mathrm{cl}(U)$ denotes the closure of U. The set U is called the basin of attraction of Λ. In [6], they prove that C^1-robustly transitive sets with singularities are singular hyperbolic attractors, either for Φ_t or Φ_{-t}; and more, all fixed points are like the one in the Lorenz Attractor, that is, eigenvalues of all singular points satisfy the following relation:

$$\lambda_{ss} < \lambda_s < 0 < -\lambda_s < \lambda_u, \tag{1}$$

and singular points must lay on the 'boundary' of the attractor, that is:

$$W^{ss}(\sigma) \cap \Lambda = \{\sigma\}. \tag{2}$$

Recall that any hyperbolic singular point of this kind has a stable, strong-stable and unstable manifold associated to each real eigenvalue. Let us call each connected component of $W^u_{\mathrm{loc}}(\sigma) - \{\sigma\}$ an unstable separatrix of σ, and let us extend the notion of Lorenz-like singularities for an invariant set Λ with several equilibrium points. Denote by $S(\Lambda)$ the set of singular points in Λ.

Definition. *The set $S(\Lambda)$ is Lorenz-like if any $\sigma \in S(\Lambda)$ is a hyperbolic, non-resonant, satisfy* (1) *and* (2), *and both unstable separatrices accumulates on $S(\Lambda)$.*

In order to state the theorem we need to introduce the notion of linear Poincaré flow. Consider $\mathcal{N}$, the normal bundle of Φ. Such bundle is well defined on the set of regular points $\tilde{U} = U - S(\Lambda)$, by the orthogonal complement of $X(p) \in T_pU$. On the other hand, for each $t \in \mathbb{R}$, the tangent map of Φ_t restricted to $\mathcal{N}$ induces an automorphism $L_t : \mathcal{N} \to \mathcal{N}$ wich covers Φ_t. The family $\{L_t\}$, of automorphisms of $\mathcal{N}$, is called the Linear Poincaré Flow associated to Φ.

Definition. *The linear Poincaré flow has dominated splitting on U if there is a splitting $\mathcal{N} = \mathcal{E} \oplus \mathcal{F}$ such that $L_t(\mathcal{E}(x)) = \mathcal{E}(\Phi_t(x))$ and $L_{-t}(\mathcal{F}(x)) = \mathcal{F}(\Phi_t(x))$, for any $x \in \tilde{U}$ and any $t \in \mathbb{R}$; and there are $C > 0$ and $\rho < 0$ such that:*

$$\|L_t|_{\mathcal{E}(x)}\| \|L_{-t}|_{\mathcal{F}(\Phi_t(x))}\| \leqslant C \exp(t\rho), \ \forall t \geqslant 0.$$

We say that the linear Poincaré flow has a *contracting direction* if there is $C > 0$ and $\lambda < 0$ such that for any $x \in \Lambda$ and any $t > 0$: $\|L_t|_{\mathcal{E}(x)}\| < C \exp(t\lambda)$. This hypothesis guarantees the existence of the local stable foliation on points of Λ; see [1].

Now we can state the Main Theorem of this work:

Main Theorem. *Let Λ be a transitive attractor of a C^2-flow such that: the linear Poincaré flow on the set of regular points has a dominated splitting with contracting direction, all singularities are Lorenz-like and all periodic orbits are hyperbolic, then Λ is singular hyperbolic.*

An outline of the argument is the following: The key property we obtain is the volume expansion condition along a center unstable bundle over points of the attractor Λ. For that, we first care about invariant subsets without singularities contained in Λ. Corollary 1 assert that any compact invariant subset of a transitive attractor Λ without singular points of a C^2-flow is hyperbolic. Then we define inside linearizing neighborhoods of singularities two transversal sections (not connected) which controls all recurrence of regular orbits to equilibria (Lemma 3) and where the expansion property is obtained by the eigenvalues. Finnaly we care about pieces of orbits traveling outside the singular region proving that they should accumulate on some hyperbolic set, and then we traslate the expansion properties by means of the Lyapunov exponents of some limit of invariant measures.

2 Global properties

Consider a C^r-flow Φ on M, with $r \geqslant 1$, and let $\Lambda \subset M$ be a transitive attractor with singular points such that its linear Poincaré flow has a dominated splitting.

Lemma 1. *If the linear Poincaré flow of Φ has a dominated invariant splitting with contracting direction, there is a partial hyperbolic $D\Phi_t$-invariant splitting $T_\Lambda M = E^s \oplus E^{cu}$.*

Proof. The bundle $\mathcal{E}(\cdot)$, definied over $\Lambda - S(\Lambda)$, can be extended continuously to $S(\Lambda)$ setting $\mathcal{E}(\sigma) = E^{ss}(\sigma)$. The fact that L_t is exponentially contracting on $\mathcal{E}$ implies the existence of a one dimensional stable bundle for $D\Phi_t$, denoted by E^s; following [3]. On the other hand, we can define a two dimensional bundle, first on regular points by $E^{cu}(p) := \mathcal{F}(p) \oplus [X(p)]$, where $[X(p)] \subset T_p M$ is the linear space spanned by $X(p)$; and we can extend it continuously to $S(\Lambda)$ defining $E^{cu}(\sigma) = E^s(\sigma) \oplus E^u(\sigma)$. Observe that E^{cu} is $D\Phi_t$-invariant. A straightforward calculation yields the domination property of $E^s \oplus E^{cu}$ and, together with the exponential contraction on E^{ss} one obtains partialy hyperbolicity for $E^s \oplus E^{cu}$. □

As a consequence of this lemma we obtain local stable manifolds $W^s_{\text{loc}}(x)$ on every point of Λ. Moreover, these local stable manifolds exist on every point of some open subset, containing Λ, of the basin attraction; see Lemma 3.2 and Corollary 3.4 in [2]; and Lemma 1 of [1]. Once we have constructed the bundle E^{cu}, it is left to prove that it expands volume to obtain singular hyperbolicity.

2.1 Global properties

Denote by $\omega(x)$ the ω-limit of a point $x \in M$, and if $V \subset M$ denote $\omega(V) = \cup_{x\in V}\omega(x)$.

Lemma 2. *Let $V \subset U$ be any open subset of the basin of attraction of Λ, then $\omega(V) = \Lambda$.*

Proof. The set $\Lambda_0 = \cup_{x\in V}\omega(x)$ is an invariant set for Φ contained in Λ. If $\Lambda_0 \neq \Lambda$, then Λ is not transitive whether is a periodic orbit or a non trivial hyperbolic set. See Lemma 6 of [1]. □

Let us recall the following theorem:

Theorem B of [2]. *Let Γ be a compact invariant set of a C^2-flow such that: the linear Poincaré flow on the set of regular points has a dominated splitting, all periodic orbits are hyperbolic of saddle type and such that $S(\Gamma) = \emptyset$; then either it is hyperbolic or it contains a normally hyperbolic invariant torus, where it is conjugated to an irrational-slope linear flow on it.*

Corollary 1. *If $\Lambda_0 \subset \Lambda$ is a compact invariant set that $\Lambda_0 \cap S(\Lambda) = \emptyset$, then Λ_0 is hyperbolic of saddle type.*

Proof. Notice that all periodic orbits in Λ are of saddle type. In our setting, Theorem B of [2] implies that if Λ_0 is not hyperbolic then it contains a normally hyperbolic (contracting) invariant torus $\mathcal{T}$. Of course, $\mathcal{T}$ does not coincides with Λ because the second contains singular points and the first one does not. Therefore, $\mathcal{T}$ is a proper attractor contained in Λ. Since Λ is transitive, this is a contradiction to Lemma 2. Hence Λ_0 is hyperbolic of saddle type. □

3 Analysis around singular points

Consider $\sigma \in S(\Lambda)$. Since it is non-resonant, there is a linearizing chart $V \subset \mathbb{R}^3$ containing the origin. Denote by λ_u, λ_{ss} and λ_s the eigenvalues of σ, corresponding to the x, y and z-axis respectively. Recall that they satisfy (1). Inside V we can draw a picture of the stable, strong-stable and unstable manifolds according to the standard basis of $\mathbb{R}^3$. In fact, inside of V the flow can be expressed by a linear differential equation given by the diagonal matrix with λ_u, λ_{ss} and λ_s on the diagonal. In these coordinates, local invariant manifolds $W^{ss,u}_{\text{loc}}(\sigma)$, correspond to the y and x-axis, respectively; and $W^s_{\text{loc}}(\sigma)$ corresponds to the plane $[x = 0]$.

Given four positive real numbers: r_1, r_2, ε and η, we can construct a cylinder inside V in the following way: $C = (-\varepsilon, \varepsilon) \times [y^2 + z^2 = r_1^2]$; and two disks Δ^+ and Δ^-, also contained in V by: $\Delta^\pm := \{(\pm\eta, u, v) | u^2 + v^2 = r_2^2\}$. Denote $\Delta = \Delta^+ \cup \Delta^-$. These sections will be called *output* sections. Denote by $\pi_x, \pi_y : V \to \mathbb{R}$ the canonical projections on the x and y-axis, respectively.

Lemma 3. *For each $\sigma \in S(\Lambda)$ there is $\delta > 0$ and a choice of positive numbers: $r_1, r_2, \varepsilon, \eta$ such that:*

1. *If $q \in B_\delta(\sigma)$ there are two positive numbers T_- and T_+ that $\Phi_{-T_-}(q) \in C$ and $\Phi_{T_+}(q) \in \Delta$.*

2. *There is some $\gamma > 0$ such that if one denotes by $\tilde{\Sigma} = \{p \in C | \, |\pi_y(p)| < \gamma\}$, then $\Lambda \cap C \subset \tilde{\Sigma}$.*

Proof. The first item is trivial, since the dynamics inside V is linear. The proof of second item stands on the hypothesis that $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$ and is precisely the content of Lemma 2 in [1]. □

Observe that we can change $\tilde{\Sigma}$ by two planar transversal sections $\Sigma^\pm$ contained in the plane $[z = \pm z_0]$, respectively, flowing it a small (bounded) amount of time. Let $\Sigma = \Sigma^+ \cup \Sigma^-$; these sections will be called *input* sections. Define $Q := W^s_{\text{loc}}(\sigma) \cap \Sigma$. Since any $\sigma \in S(\Lambda)$ is Lorenz-like then $\lambda_u(\sigma) + \lambda_s(\sigma) > 0$.

Corollary 2. *For any point $p \in \Sigma - Q$, there is $t > 0$ such that $\Phi_t(p) \in \Delta$ and* $\det(D\Phi_t(p)|E^{cu}) = \exp(t(\lambda_u(\sigma) + \lambda_s(\sigma)))$.

The union of all pieces of orbits between the input and the output sections, together with the local stable, strong-stable and unstable manifolds of σ, they build an open neighborhood of σ. Denote this neighborhood by V_σ.

4 The volume expanding condition

Consider $V = \cup_{\sigma \in S(\Lambda)} V_\sigma$; where V_σ is obtained after Lemma 3, applied to each $\sigma \in S(\Lambda)$. Denote by Σ the union of all *input* sections, and Δ the union of all *output* sections associated to all singular points. Next lemma analyzes the past orbit of any point of the attractor.

Lemma 4. *For any $p \in \Lambda$ one of the following happens: $p \in \bigcup_{\sigma \in S(\Lambda)} W^u(\sigma)$; $\mathcal{O}^-(p) \cap \Sigma$ is an infinite set; or $\alpha(p)$ is a hyperbolic set.*

Proof. Let $p \in \Lambda$ such that $p \notin \bigcup_{\sigma \in S(\Lambda)} W^u(\sigma)$ and such that $\mathcal{O}^-(p) \cap \Sigma = \emptyset$. If the cardinality of this set is finite, we can consider $\Phi_{-T}(p)$ instead of p. Recall Σ is a 2-dimensional open subset. If the orbit of p remains on the boundary of Σ by Lemma 4.3 de [2] the α-limit of p is a hyperbolic periodic orbit. Otherwise, for any $T > 0$ there is an open set V containig p that $\Phi_T(V) \cap \bigcup_{\sigma \in S(\Lambda)} B_{\delta_\sigma}(\sigma) = \emptyset$. Hence, $\alpha(p) \cap S(\Lambda) = \emptyset$ and is closed. Corollary 1 assert that $\alpha(p)$ is hyperbolic. □

In §3.5 of [6] it is proved that in order to obtain the volume expansion condition on the bundle E^{cu} it is enough to bound the rate of expansion for the past of each orbit in Λ. This is the content of the next Lemma.

Lemma 5. *For any $x \in \Lambda$ we have that*

$$\liminf_{t \to \infty} \det(D_x \Phi_{-t}(x)|_{E^{cu}(x)}) = 0. \tag{3}$$

Proof. Recall the open set V containing $S(\Lambda)$. Let $U' = \mathrm{cl}(U) - V$. Consider any point $x \in \Lambda$. Lemma 4 gives us three possibilities.

[A] If $x \in W^u(\sigma) - \{\sigma\}$, for some $\sigma \in S(\Lambda)$, there is $T > 0$ such that $x_0 := \Phi_{-T}(x) \in \Delta$ and $\Phi_{-t}(x_0) \in V$, for all $t \geqslant 0$. Since the dynamic inside V is linear, then

$$\det\left(D_{x_0} \Phi_{-t}|_{E^{cu}(x)}\right) = \exp\left(-t(\lambda_s(\sigma) + \lambda_u(\sigma))\right).$$

$S(\Lambda)$ is Lorenz-like, then $\lambda_s(\sigma) + \lambda_u(\sigma) > 0$. Hence, we are done in this case. If $x \in S(\Lambda)$ the same calculation applies.

[B] If the α-limit of x is a hyperbolic set then:

$$\lim_{t \to +\infty} \|D_x \Phi_{-t}|_{E^{cu}(x)}\| = 0.$$

Hence, $\det\left(D_x \Phi_{-t}|_{E^{cu}(x)}\right) \leqslant \|X\| \|D_x \Phi_{-t}|_{E^{cu}(x)}\|$; and we are done in this case.

[C] Assume that $\#\mathcal{O}^-(x) \cap \Sigma = +\infty$. Without lost of generality we can assume that $x \in \Sigma$. Hence, there is a partition of the set of positive real numbers given by a sequence of not empty intervals of the form $K_n := [a_n, a_{n+1})$ for which $\Phi_{-t}(x) \in \Sigma$ iff $t \in \{a_n | n \in \mathbb{N}\}$.

Lemma 3 imply that for any integer n there is $b_n \in \mathbb{R}$ such that $a_n < b_n < a_{n+1}$ and $\Phi_{-b_n}(x) \in \Delta$. This induces a refinament of the previous partition:

$$\mathbb{R}^+ = \bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} K_n^0 \cup K_n^1;$$

where $K_n^0 = [a_n, b_n]$ and $K_n^1 = (b_n, a_{n+1})$. Notice that $t \in \cup_n K_n^0$ iff $\Phi_{-t}(x) \in U'$, and $t \in \cup_n K_n^1$ iff $\Phi_{-t}(x) \in V$.

Given a fixed $t \in \mathbb{R}^+$, denote by $C(t) = \det(D_x\Phi_{-t}|_{E^{cu}(x)})$. Applying the chain rule, we obtain that for each $N \in \mathbb{N}$ we have:

$$C(a_N) = \prod_{n=0}^{N} A(n) \cdot B(n);$$

where:

$$A(n) = \det(D_{\Phi_{-b_n}(x)}\Phi_{-(a_{n+1}-b_n)}|_{E^{cu}(\Phi_{-b_n}(x))})$$

and

$$B(n) = \det(D_{\Phi_{-a_n}(x)}\Phi_{-(b_n-a_n)}|_{E^{cu}(\Phi_{-a_n}(x))})$$

If the sequence $|K_n^1|$ is bounded, then $\alpha(x)$ do not accumulates on $S(\Lambda)$, and hence it is hyperbolic. In fact, it is possible to find a smaller section $\tilde{\Sigma}$ where $\mathcal{O}^-(x) \cap \tilde{\Sigma} = \emptyset$. Therefore there is a subsequence n_j such that $|K_{n_j}^1| \to +\infty$ as $j \to +\infty$. Moreover, Corollary 2 implies that there is $\rho = \min\{\lambda_u(\sigma) + \lambda_s(\sigma) | \sigma \in S(\Lambda)\} > 0$ such that:

$$B(n) < \exp(-\rho|K_n^1|) < 1;\ \forall n \in \mathbb{N}$$

In particular $B(n_j) \to 0$ as $j \to +\infty$.

On the other hand if there is $T > 0$ such that $|K_n^0| \leqslant T$, for all n, there is $\tilde{C} > 0$ such that $|A(n)| \leqslant \tilde{C}$. Therefore,

$$C(a_N) \leqslant \tilde{C} \prod_{n=0}^{N} B(n) \leqslant \tilde{C} \prod_{j=0}^{N^*} B(n_j)$$

for N^* being the greatest n_j that $n_j \leqslant N$. This implies that $C(a_N) \to 0$ as $N \to +\infty$, and we are done.

Now it is left the case when $|K_n^0|$ is unbounded. For that, denote by $p_n = \Phi_{-a_n}(x) \in \Sigma$ and $s_n = b_n - a_n = |K_n^0|$, to simplify the notation. Observe that $\Phi_{-s}(p_n) \in U'$ for all $s \in [0, s_n]$, and $\Phi_{-s_n}(p_n) \in \Delta$.

We claim that for any subsequence $s_{n_j} \to \infty$ we have that:

$$\|D_{p_{n_j}} \Phi_{-s_{n_j}} |_{E^{cu}(p_{n_j})}\| \to 0 \text{ as } j \to +\infty.$$

If the claim is true, then $C(a_N) \to 0$ as $N \to +\infty$ since $A(n) \leqslant \|X\| \cdot \|D_{p_{n_j}} \Phi_{-s_{n_j}} |_{E^{cu}(p_{n_j})}\|$; and we are done.

To obtain the claim assume there is a subsequence $\{s_n\}$ (we drop the subindex to simplify notation) and a number $\gamma > 0$ that:

$$\|D_{p_n} \Phi_{-s_n} |_{E^{cu}(p_n)}\| \geqslant \gamma > 0. \tag{4}$$

Consider a sequence of probability measures μ_n in $\mathrm{cl}(U)$ defined in the following way: If A is any Borel set A denote χ_A its characteristic function and let:

$$\mu_n(A) := \frac{1}{s_n} \int_0^{s_n} \chi_A(\Phi_{-t}(p_n))dt$$

Observe that if $A \subset V$ then $\mu_n(A) = 0$ and $\mu_n(U') = 1$, for any $n \in \mathbb{N}$.

Lemma 6. *Any convergent subsequence of μ_n converges in the weak* topology to an invariant measure μ.*

Proof. Let μ be any limit of a subsequence of μ_n. Consider a Borel set A, some fixed $T > 0$ and any $n \in \mathbb{N}$. Observe that:

$$\int_0^{s_n} \chi_{\Phi_{-T}(A)}(\Phi_{-t}(p_n))dt = \int_0^{s_n} \chi_A(\Phi_{T-t}(p_n)) = \int_{-T}^{s_n - T} \chi_A(\Phi_{-t}(p_n))dt.$$

Hence,

$$\begin{aligned} |\mu_n(\Phi_{-T}(A)) - \mu_n(A)| &= \frac{1}{s_n} \left| \int_{-T}^{s_n - T} \chi_A(\Phi_{-t}(p_n)) - \int_0^{s_n} \chi_A(\Phi_{-t}(p_n)) \right| \\ &= \frac{1}{s_n} \left| \int_{-T}^{0} \chi_A(\Phi_{-t}(p_n)) - \int_{s_n - T}^{s_n} \chi_A(\Phi_{-t}(p_n)) \right| \leqslant \frac{2T}{s_n}. \end{aligned}$$

Therefore, $|\mu_n(\Phi_{-T}(A)) - \mu_n(A)| \to 0$ as $n \to \infty$, and this implies that the limit measure μ is invariant. □

Let $\{\mu_j\}$ be any convergent subsequence and denote by μ_∞ its limit. The support of the limit measure, $\mathrm{supp}(\mu_\infty)$, is a compact invariant set contained in U'. Therefore Corollary 1 implies it is hyperbolic of saddle type, since it do not contains singular points. As a consequence, this set has two Lyapunov exponents $\rho_s < 0$ and $\rho_u > 0$.

On the other hand, if we denote by $q_n = \Phi_{-s_n}(p_n)$, we can estimate ρ_u in the following way:

$$\rho_u = \lim_{n\to\infty} \frac{1}{s_n} \log \|D_{q_n}\Phi_{s_n}|_{E^{cu}(q_n)}\|.$$

However, $\|D_{q_n}\Phi_{s_n}|_{E^{cu}(q_n)}\| = \|D_{p_n}\Phi_{-s_n}|_{E^{cu}(p_n)}\|^{-1} \leqslant \gamma^{-1}$, by (4). Hence,

$$\rho_u \leqslant \lim_{n\to\infty} \frac{\log(\gamma^{-1})}{s_n} = 0$$

This is a contradiction to the fact of $\mathrm{supp}(\mu_\infty)$ is a hyperbolic invariant set of saddle type. So, we obtain the claim and we are done. □

Once we have completed the proof of the previous Lemma, we can conclude that $D\Phi_t$ expands volume on the bundle E^{cu}; and hence, the splitting over Λ, $T_\Lambda M = E^s \oplus E^{cu}$ is singular hyperbolic.

References

[1] Aubin Arroyo and Enrique Pujals. *Dynamical properties of singular hyperbolic attractors.* To appear.

[2] Aubin Arroyo and Federico Rodriguez Hertz. *Homoclinic bifurcations and uniform hyperbolicity for three-dimensional flows.* Ann. Inst. H. Poincaré Anal. Non Linéaire, **20**(5) (2003), 805–841.

[3] C.I. Doering. *Persistently transitive vector fields on three-dimensional manifolds.* In *Dynamical systems and bifurcation theory (Rio de Janeiro, 1985)*, volume 160 of Pitman Res. Notes Math. Ser., pages 59–89. Longman Sci. Tech., Harlow, 1987.

[4] John Guckenheimer and R.F. Williams. *Structural stability of Lorenz attractors.* Inst. Hautes Études Sci. Publ. Math., **50** (1979), 59–72.

[5] E.N. Lorenz. *Deterministic nonperiodic flow.* J. Atmosph. Sci., **20** (1963), 130–141.

[6] C.A. Morales, M.J. Pacifico and E.R. Pujals. *Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers.* Ann. of Math. (2), **160**(2) (2004), 375–432.

[7] J. Palis. *A global perspective for non-conservative dynamics.* Ann. Inst. H. Poincaré Anal. Non Linéaire, **22**(4) (2005), 485–507.

[8] S. Smale. *Differentiable dynamical systems.* Bull. Amer. Math. Soc., **73** (1967), 747–817.

[9] Warwick Tucker. *The Lorenz attractor exists.* C.R. Acad. Sci. Paris Sér. I Math., **328**(12) (1999), 1197–1202.

[10] R.F. Williams. *The structure of Lorenz attractors.* Inst. Hautes Études Sci. Publ. Math., **50** (1979), 73–99.

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