

# A holomorphic extension theorem from a fractal hypersurface in $\mathbb{C}^2$

Ricardo Abreu Blaya, Juan Bory Reyes  
and Tania Moreno García

**Abstract.** We consider the problem of identifying boundary values of holomorphic functions on bounded domains in  $\mathbb{C}^2$ . We use the quaternionic analysis techniques to extending the *CR* structure to a pure function theoretical nature. The advantage of our procedure lies in the fact that it also runs for domains with fractal boundary.

**Keywords:** *CR*-functions, holomorphic functions, quaternionic analysis.

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## 1 Introduction

Holomorphic functions of several complex variables admit many unexpected and intriguing phenomena and the attempt to extend the remarkable feature of this theory to more general analogous has a long history. For a general background on complex analysis with several variables the reader may consult the books [7, 22, 23] and the references given there.

Quaternionic analysis is a natural generalization of the holomorphic functions theory in the complex plane to four-dimensional Euclidean space that preserves many of its basic features. It is centered around the notion of a hyperholomorphic function, i.e., null solutions of the Cauchy Riemann operator rewritten as a quaternionic one. For a general account of this theory we refer the reader to [14, 15, 25].

The aim of this paper is to bring together these two theories in order to shed some new light on the problem of holomorphic extension from the boundary in absence of smoothness of the domain.

## 2 Preliminares

### 2.1 $\mathbb{C}^2$ and quaternions

Let  $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$  denote the *two*-dimensional complex Euclidean space with product topology. The coordinates of  $\mathbb{C}^2$  will be denoted by  $z = (z_1, z_2)$  with  $z_s = x_{2s} + ix_{2s+1}$ ,  $s = 0, 1$ . Thus,  $\mathbb{C}^2$  can be identified with  $\mathbb{R}^4$  in a natural manner.

We embed the usual complex linear space  $\mathbb{C}^2$  into the skew-field  $\mathbb{H}$  of Hamilton's quaternions by means of the mapping that associates the pair

$$(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$$

with the quaternion

$$z = z_1 + jz_2 = x_0 + ix_1 + jx_2 - kx_3 \in \mathbb{H}.$$

Here we denote by  $i, j, k$  the basic quaternions, and the imaginary unit of  $\mathbb{C}$  is identified with  $i$ .

Throughout the paper  $z_1$  and  $z_2$  stand for the complex components of the quaternion  $z$ .

If  $\{\zeta, z\} \subset \mathbb{C}^2$ , then  $\langle \zeta, z \rangle := \zeta_1 z_1 + \zeta_2 z_2$  and  $|z|^2 := |z_1|^2 + |z_2|^2$ .

As a matter of fact, the above embedding means that  $\mathbb{C}^2$  becomes endowed with an additional structure of a skew field. In particular, the commutation rule is then:  $aj = j\bar{a}$  for every  $a \in \mathbb{C} \subset \mathbb{H}$ , and the two quaternions  $z = z_1 + jz_2$  and  $\zeta = \zeta_1 + j\zeta_2$  are multiplied according to the rule:

$$z\zeta = (z_1\zeta_1 - \bar{z}_2\zeta_2) + j(\bar{z}_1\zeta_2 + z_2\zeta_1).$$

Corresponding to each quaternion  $z = z_1 + jz_2$  is the conjugate quaternion  $\bar{z} = \bar{z}_1 - \bar{z}_2j$ . An important property of the quaternionic conjugation is:

$$z\bar{z} = \bar{z}z = (z_1 + jz_2)(\bar{z}_1 - \bar{z}_2j) = |z|^2.$$

Note that the introduced terminology agrees with the one given in [20, 21]. For a recent account of the properties of the quaternions taken in the form  $z = z_1 + jz_2$  we refer the reader to [16], Appendix 2, pp. 216–217.

### 2.2 Holomorphic and hyperholomorphic functions

As for prerequisites, the reader is expected to be familiar with the elementary aspects of the quaternionic and complex analyses in  $\mathbb{C}^2$ , reason why in this section we will touch only a few of the standard facts.

A  $\mathbb{C}$ -valued differentiable function  $f = f(z_1, z_2)$  is said to be holomorphic in a domain  $\Omega \subset \mathbb{C}^2$  if

$$\frac{\partial f}{\partial \bar{z}_1} = 0, \quad \frac{\partial f}{\partial \bar{z}_2} = 0 \quad \text{in } \Omega.$$

Let  $\mathcal{O}(\Omega)$  denote the ring of  $\mathbb{C}$ -valued functions that are holomorphic in the domain  $\Omega$  of  $\mathbb{C}^2$ . Using the complex 1-form

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}_1} d\bar{z}_1 + \frac{\partial}{\partial \bar{z}_2} d\bar{z}_2,$$

which is referred to as Cauchy-Riemann operator, we have that  $f \in \mathcal{O}(\Omega)$  iff  $\bar{\partial} f = 0$  in  $\Omega$ .

In what follows,  $\mathcal{D}$  stands for the quaternionic Cauchy Riemann operator

$$\mathcal{D} = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} - k \frac{\partial}{\partial x_3} = 2\left\{ \frac{\partial}{\partial \bar{z}_1} + j \frac{\partial}{\partial \bar{z}_2} \right\},$$

associated to the structural vector  $(1, i, j, -k)$ .

A quaternion valued continuously differentiable function  $F$  is called hyperholomorphic in a domain  $\Omega \subset \mathbb{C}^2$ , denoted by  $F \in \mathcal{H}(\Omega)$ , if  $\mathcal{D}[F] = 0$  in  $\Omega$ .

With the notation  $F = f_1 + f_2 j$ , hyperholomorphy of  $F$  is equivalent to the following system of complex differential equations

$$\frac{\partial f_1}{\partial \bar{z}_1} = \frac{\partial \bar{f}_2}{\partial z_2}, \quad \frac{\partial f_1}{\partial \bar{z}_2} = -\frac{\partial \bar{f}_2}{\partial z_1}.$$

We remark that one may naturally identify the operator  $\mathcal{D}$  with the complex 1-form  $\bar{\partial}$ . In [8] such an identification is detailedly reviewed in a more general setting.

It is easy to check that any holomorphic mapping  $(f_1, f_2)$  in  $\Omega$  defines a hyperholomorphic function  $F = f_1 + f_2 j$ . Furthermore, if one of the functions (say  $f_1$ ) is holomorphic, then  $F = f_1 + f_2 j$  is hyperholomorphic in  $\Omega$  iff  $f_2$  is holomorphic there.

In particular a  $\mathbb{C}$ -valued function is holomorphic in  $\Omega$  iff it is hyperholomorphic in  $\Omega$ .

To cause no confusion, we will use small letters to denote  $\mathbb{C}$ -valued functions otherwise capital letters to denote those  $\mathbb{H}$ -valued.

Accordingly this, we use the same symbol to denote the usual spaces of either  $\mathbb{C}$ -valued functions or  $\mathbb{H}$ -valued functions. In this manner, the spaces of all continuous,  $k$ -time continuous differentiable and  $\alpha$ -Hölder continuous ( $0 < \alpha \leq 1$ ) functions are denoted by  $C(E)$ ,  $C^k(E)$  and  $\Lambda^\alpha(E)$ , respectively, where  $E$  can be any suitable subset of  $\mathbb{C}^2$ .

### 2.3 Cauchy and Bochner-Martinelli kernels

If  $\omega_4$  stands for the surface area of the unit sphere in  $\mathbb{C}^2$  then

$$K(z) := \frac{1}{\omega_4} \frac{\bar{z}}{|z|^4} = \frac{1}{\omega_4} \frac{\bar{z}_1 - \bar{z}_2 j}{|z|^4}, \quad z \neq 0$$

generalizes the Cauchy kernel in the plane, more precisely  $K(z)$  is the fundamental solution (in the distributional sense) of the operator  $\mathcal{D}$ . Clearly, it is hyperholomorphic in  $\mathbb{C}^2 \setminus \{0\}$ .

From now on,  $\Omega \subset \mathbb{C}^2$  stands for an oriented bounded domain whose boundary is a compact topological hypersurface  $\Gamma$ . We shall first assume that  $\Gamma$  is a properly smooth hypersurface guaranteeing existence of a unit normal vector  $\nu$  everywhere on  $\Gamma$ . Nonetheless, starting from Subsection 3.1 instead of the smooth restriction, certainly much more weaker geometric assumptions will be required on the boundary of the domains under consideration.

It is easy to establish by direct calculation that

$$K(\zeta - z)\nu(\zeta) = u(\zeta, z) + jm(\zeta, z), \quad \zeta \in \Gamma, \quad z \neq \zeta, \quad (1)$$

where

$$u(\zeta, z) := \frac{1}{\omega_4} \frac{\langle \bar{\zeta} - \bar{z}, \nu(\zeta) \rangle}{|\zeta - z|^4},$$

is the classical Bochner-Martinelli kernel and  $m(\zeta, z)$  is given by

$$m(\zeta, z) := -\frac{1}{\omega_4} \frac{\langle \zeta - z, j\nu(\zeta) \rangle}{|\zeta - z|^4}.$$

This construction follows [1, 4, 26] and elsewhere, but the idea goes back at least as far as [13].

For  $F \in C(\Gamma)$  the Cauchy transform of  $F$ , given by

$$\mathcal{K}F(z) := \int_{\Gamma} K(\zeta - z)\nu(\zeta)F(\zeta)d\zeta, \quad z \notin \Gamma,$$

defines a hyperholomorphic function  $\Gamma$ . Its boundary limit values

$$\begin{aligned} \mathcal{K}^+F(t) &:= \lim_{\Omega \ni z \rightarrow t} \mathcal{K}F(z), \\ \mathcal{K}^-F(t) &:= \lim_{\mathbb{C}^2 \setminus \bar{\Omega} \ni z \rightarrow t} \mathcal{K}F(z), \end{aligned}$$

and the Hilbert transform of  $F$ , i.e.,

$$SF(t) := 2 \lim_{\epsilon \rightarrow 0+} \int_{\{\zeta \in \Gamma: |\zeta - t| > \epsilon\}} K(\zeta - t)\nu(\zeta)F(\zeta)d\zeta, \quad t \in \Gamma, \quad (2)$$

are related by the so-called Plemelj-Sokhotski formulae, namely:

$$\mathcal{K}^\pm F(t) = \frac{1}{2}(SF(t) \pm F(t)), \quad t \in \Gamma.$$

Of course, one must be worried about the existence of the limits everywhere on  $\Gamma$ .

There is obvious evidence that if the smooth property of  $\Gamma$  is not any more assumed, then even for  $F = 1$  the Hilbert transform  $S1$  could fail to be a continuous function on  $\Gamma$ . To avoid this difficulty we could replace (2) into the following

$$SF(t) := 2 \lim_{\epsilon \rightarrow 0+} \int_{\{\zeta \in \Gamma: |\zeta - t| > \epsilon\}} K(\zeta - t) \nu(\zeta) (F(\zeta) - F(t)) d\zeta + F(t), \quad t \in \Gamma.$$

However, restricting to  $C^1$ -smooth boundary, it is easy to see that both Hilbert transforms coincide. For that reason here and subsequently, we use the same letter  $S$  for both Hilbert transforms.

The principal significance of the relation (1) is that it allows one to express the classical Bochner-Martinelli transform in terms of the first complex component of the Cauchy transform above defined. In fact, for a  $\mathbb{C}$ -valued function  $f$  we have

$$\mathcal{K}f(z) = \mathcal{U}f(z) + j\mathcal{M}f(z), \quad z \notin \Gamma, \quad (3)$$

where

$$\mathcal{U}f(z) := \int_{\Gamma} u(\zeta - z) f(\zeta) d\zeta$$

is the Bochner-Martinelli transform of  $f$  and

$$\mathcal{M}f(z) := \int_{\Gamma} m(\zeta - z) f(\zeta) d\zeta.$$

A nice presentation of the study of the Bochner-Martinelli transform can be found in [17].

According to the complex structure of the formula (3), we have

$$\mathcal{K}^\pm f(t) = \mathcal{U}^\pm f(t) + j\mathcal{M}^\pm f(t), \quad t \in \Gamma. \quad (4)$$

Let us remark that when  $\mathcal{M}f(z) = 0, z \notin \Gamma$ , then the Cauchy transform becomes  $\mathbb{C}$ -valued and coincides with the Bochner-Martinelli one. Under this assumption the machinery developed for the former could be directed toward setting conjectures about the latter.

### 3 Extension from the boundary

Let  $\Gamma$  be a  $C^1$ -smooth hypersurface in  $\mathbb{C}^2$  and suppose that  $f \in C^1(\Gamma)$ . Then  $\bar{\partial}f$  can be decompose on  $\Gamma$  into two terms, one of which,  $\bar{\partial}_N f$  is directed in the direction of the complex normal vector to  $\Gamma$  at this point, and the other, the so-called *tangential Cauchy Riemann operator*,  $\bar{\partial}_T f := \bar{\partial}f - \bar{\partial}_N f$ , is directed complex-orthogonally to this normal.

A  $\mathbb{C}$ -valued function  $f \in C^1(\Gamma)$  satisfies the *tangential Cauchy-Riemann condition* on  $\Gamma$ , or more briefly,  $f$  is a *CR-function* on  $\Gamma$ , if  $\bar{\partial}_T f = 0$  on  $\Gamma$ .

These notations overlap with the book of Shabat [23].

The following theorem, to which this section is devoted, asserts that every function that is holomorphic, in two alternative senses, on the boundary of a domain can be extended holomorphically into the whole domain.

The meaning of holomorphy on the boundary is considered in terms of either the tangential Cauchy-Riemann condition or a conservation law on the Hilbert transform. The advantage in using the latter lies in the fact that it can be considered under a relaxed assumption on the boundary. Moreover, the picture thus obtained offers an essentially enriched approach to this topic.

**Theorem 1.** *Suppose that in  $\mathbb{C}^2$  we are given a domain  $\Omega$  with  $C^1$ -smooth boundary  $\Gamma$  and let  $f \in C^1(\Gamma)$ . Then, the following conditions are equivalent:*

- (i)  $f$  is a CR-function on  $\Gamma$
- (ii)  $f$  has holomorphic extension to  $\Omega$
- (iii)  $Sf = f$  on  $\Gamma$ .

**Proof.** A direct proof of (i)  $\Leftrightarrow$  (ii) (Bochner-Severi Theorem) can be found in many sources, see, e.g., [23] §11, Theorem 2 or [18] §8, Theorem 8.20.

Next, let us prove that (ii)  $\Leftrightarrow$  (iii).

It follows from (ii) that there exists a  $\mathbb{C}$ -valued function  $\tilde{f}$  on  $\overline{\Omega}$  which is holomorphic on  $\Omega$ , continuous in  $\overline{\Omega}$ , with  $\tilde{f}|_{\Gamma} = f$ .

Note that, guaranteed by the assumption, we have

$$\mathcal{K}f(z) = \begin{cases} \tilde{f}(z), & z \in \Omega, \\ 0, & z \in \mathbb{C}^2 \setminus \overline{\Omega}. \end{cases}$$

Therefore,  $\mathcal{K}^- f(t) = 0$  and, in consequence,  $Sf = f$ .

Conversely, suppose that (iii) holds. We claim that the function  $\mathcal{K}f$  is a holomorphic extension of  $f$  to  $\Omega$ . Indeed, from (iii) we conclude that  $\mathcal{K}f$  is

a hyperholomorphic extension of  $f$ , then we are left with the task of proving that  $\mathcal{K}f$  is  $\mathbb{C}$ -valued. In order to get this, we can use the Plemelj-Sokhotski formulae to conclude that the boundary limit values  $\mathcal{M}^\pm f(t) = 0$ . Since  $\mathcal{M}f$  is real harmonic off  $\Gamma$ , we have by the classical Dirichlet problem that  $\mathcal{M}f \equiv 0$  in  $\mathbb{C}^2$ . Then,  $\mathcal{K}f \equiv \mathcal{U}f$  and the proof is complete.  $\square$

### 3.1 Some comments and more delicate results

Although we have been working under the assumption that  $f \in C^1(\Gamma)$ , it should be noted, however, that Theorem 3 still holds if it is just assumed that the function  $f$  is merely continuous on  $\Gamma$  and condition (i) is understood in the weak sense of Definition 8.17 in [18], but we will not develop this point here.

After that only the continuity of  $f$  is assumed, the proof of (ii)  $\Leftrightarrow$  (iii) more strongly depends on the assumption that  $Sf = f$ , but now uniformly, since the Cauchy transform of a continuous function has not in general continuous boundary limit values even for  $C^1$ -smooth boundary.

A more sophisticated arguments, see [9], enables one to prove a generalized version of Theorem 3, if  $\Gamma$  can be thought of as a regular and rectifiable hypersurface. In this more general context, the a priori smoothness restrictions on  $\Gamma$  may be entirely avoided, however, if the pointwise normal vector occurring in the definition of the Cauchy transform is replaced by the exterior normal of  $\Omega$  in the Federer's sense which is defined in [12].

**Definition 1.** We say that  $\Gamma$  is a rectifiable hypersurface if it is the Lipschitz image of some bounded subset of  $\mathbb{R}^3$ . Meanwhile, we call  $\Gamma$  a regular hypersurface, if there exists a constant  $c > 0$  such that

$$c^{-1}r^3 \leq \int_{\{\zeta \in \Gamma: |\zeta - t| \leq r\}} d\zeta \leq cr^3,$$

for all  $t \in \Gamma$  and  $0 < r \leq \text{diam } \Gamma$ .

For a deeper discussion of these concepts we refer the reader to [10, 12, 19].

**Theorem 2.** Suppose that in  $\mathbb{C}^2$ , we are given a domain  $\Omega$  with regular and rectifiable boundary  $\Gamma$ , and let  $f \in C(\Gamma)$ . Then,  $f$  has holomorphic extension to  $\Omega$  if and only if  $Sf = f$  uniformly on  $\Gamma$ .

**Proof.** The proof follows by the same method as in Theorem 1, but it strongly depends on the uniform existence of  $Sf$  and the conclusion of Theorem 6 in [9].  $\square$

The following result may be proved in much the same way as Theorem 2. It makes no appeal to the uniform existence assumption on  $Sf$  with “continuity” of  $f$  replaced by “Hölder continuity”.

**Theorem 3.** *Let  $\Omega \subset \mathbb{C}^2$  with regular boundary  $\Gamma$  and let  $f \in \Lambda^\alpha(\Gamma)$ ,  $0 < \alpha \leq 1$ . Then,  $f$  has holomorphic extension to  $\Omega$  if and only if  $Sf = f$  at all point of  $\Gamma$ .*

**Proof.** By means of Theorem 4.1 in [5], §4, we can compute the jump of the Cauchy transform  $\mathcal{K}f$  passing across the boundary  $\Gamma$  having in proving the formula

$$f = \mathcal{K}^+ f + \mathcal{K}^- f.$$

For the assumption, we then obtain that  $\mathcal{K}^+ f$  is continuous on  $\overline{\Omega}$  and  $\mathcal{K}^+ f = f$  on  $\Gamma$ .  $\square$

## 4 Extension from a fractal boundary

### 4.1 Whitney extension theorem

In this section we will be concerned with the problem of extending holomorphically a  $\mathbb{C}$ -valued Hölder continuous function given on the fractal boundary  $\Gamma$  of a domain  $\Omega$ . The fractal requirement on  $\Gamma$  is understanding in the Mandelbrot sense, i.e., if its Hausdorff dimension is strictly greater than 3 (see [11]).

The properties of the Whitney decomposition of the complement of a compact set in Euclidean spaces, see [24], enable us to write the Whitney extension theorem as follows.

**Theorem 4.** *Let  $F \in \Lambda^\alpha(\Gamma)$ ,  $0 < \alpha \leq 1$ . Then there exists a function  $\mathcal{E}_0 F \in \Lambda^\alpha(\mathbb{C}^2)$  with compact support satisfying*

- (i)  $\mathcal{E}_0 F|_\Gamma = F$ ,
- (ii)  $\mathcal{E}_0 F \in C^\infty(\mathbb{C}^2 \setminus \Gamma)$ ,
- (iii)  $|\mathcal{D}\mathcal{E}_0 F(z)| \leq C [\text{dist}(z, \Gamma)]^{\alpha-1}$ ,  $z \in \mathbb{C}^2 \setminus \Gamma$ .

### 4.2 The main theorem

In Theorem 2, the boundary  $\Gamma$  is required to be a regular and rectifiable hypersurface both assuring the finiteness of the three-dimensional Hausdorff measure



of it. If  $\Gamma$  is one of kind to which this measurable nature can be missed, serious difficulties appear, because the Cauchy transform loses its meaning and becomes necessary to use a new method which does not use boundary integration and can thus be used on fractal domains.

For the convinience of the reader we repeat the relevant material from [3, 6], thus making our exposition self-contained:

Let  $\Omega \subset \mathbb{C}^2$  with boundary  $\Gamma$ . They were able to show that for  $F \in \Lambda^\alpha(\Gamma)$ ,  $0 < \alpha \leq 1$ , when the exponent  $\alpha$  and the Minkowski dimension  $M(\Gamma)$  of the hypersurface  $\Gamma$ , see [19], satisfy the relation

$$\alpha > \frac{M(\Gamma)}{4}, \quad (5)$$

then the functions given by the formulas

$$F^+(z) = E_0 F(z) + \mathcal{T}_\Omega \mathcal{D} E_0 F(z)$$

$$F^-(z) = \mathcal{T}_\Omega \mathcal{D} E_0 F(z)$$

are hyperholomorphic in  $\Omega$  and  $\mathbb{C}^2 \setminus \overline{\Omega}$ , respectively, continuous in the corresponding closed domains and satisfy on  $\Gamma$  the jump relation:

$$F(t) = F^+(t) - F^-(t).$$

Here and subsequently  $\mathcal{T}_\Omega G$  denote the Theodoresco transform of a function  $G$  defined by

$$\mathcal{T}_\Omega G(z) := - \int_{\Omega} K(\eta - z) G(\eta) d\eta.$$

For details regarding the basic properties of Theoderesco transform, we refer the reader to [14, 15].

It is essential to point out that, under condition (5),  $\mathcal{D}(E_0 F|_\Omega)$  is integrable in  $\Omega$  with any degree no greater than  $\frac{4-m}{1-\alpha}$ , i.e., under condition (5) it is integrable with certain exponent greater than 4.

At the same time,  $\mathcal{T}_\Omega \mathcal{D} E_0 F$  is a  $\mu$ -Hölder continuous function in the whole  $\mathbb{C}^2$  with

$$\mu < \frac{4\alpha - M(\Gamma)}{4 - M(\Gamma)},$$

which is due to the fact that  $\mathcal{T}_\Omega \mathcal{D} E_0 F$  satisfies the Hölder condition with exponent  $1 - \frac{4}{p}$ .

When condition (5) is violated then some obstructions can be constructed as we have proved in [2], Section 5.

In the sequel we assume  $\alpha$  and  $M(\Gamma)$  to be connected by (5).

Based in the previous arguments, thus we are led to the following theorem proved in [2].

**Theorem 5.** *Let  $F \in \Lambda^\alpha(\Gamma)$ . If  $F$  has an extension  $\tilde{F} \in \mathcal{H}(\Omega) \cap \Lambda^\alpha(\overline{\Omega})$ , then*

$$\mathcal{T}_\Omega[\mathcal{DE}_0 F]|_\Gamma = 0. \quad (6)$$

*Conversely, if (6) is satisfied, then  $F$  has an extension  $\tilde{F} \in \mathcal{H}(\Omega) \cap \Lambda^\mu(\overline{\Omega})$  for some  $\mu < \alpha$ .*

Having disposed of this result, we are in position to show our main holomorphic extension theorem for  $\mathbb{C}$ -valued functions on fractal domains.

**Theorem 6.** *Let  $f \in \Lambda^\alpha(\Gamma)$ . If  $f$  has an extension  $\tilde{f} \in \mathcal{O}(\Omega) \cap \Lambda^\alpha(\overline{\Omega})$ , then*

$$\mathcal{T}_\Omega[\mathcal{DE}_0 f]|_\Gamma = 0. \quad (7)$$

*Conversely, if (7) is satisfied, then  $f$  has an extension  $\tilde{f} \in \mathcal{O}(\Omega) \cap \Lambda^\mu(\overline{\Omega})$  for some  $\mu < \alpha$ .*

**Proof.** Let us suppose (7) holds. Then, by Theorem 4.2,  $f$  has hyperholomorphic extension  $\tilde{F}$  to  $\Omega$ . The task is now to prove that it is  $\mathbb{C}$ -valued.

In fact, writing  $\tilde{F} = \tilde{f}_1 + \tilde{f}_2 j$  we obtain for  $\tilde{f}_2$  the classical Dirichlet problem

$$\Delta \tilde{f}_2 = 0, \quad \tilde{f}_2|_\Gamma = 0,$$

which implies  $\tilde{f}_2 = 0$  as required.

On the other direction the proof is immediate. □

**Corollary 1.** *Let  $\Omega \subset \mathbb{C}^2$  with rectifiable boundary  $\Gamma$ . If a function  $f \in \Lambda^1(\Gamma)$  satisfies the condition (7), then  $f$  can be extended into the domain  $\Omega$  to a function that is holomorphic in  $\Omega$  and continuous in  $\overline{\Omega}$ .*

**Proof.** What is essential here is that the Minkowski dimension of a rectifiable hypersurface in  $\mathbb{C}^2$  is equal to three. Thus, the condition (5) is evidently fulfilled for the Hölder exponent  $\alpha = 1$ . □

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**Ricardo Abreu Blaya**

Facultad de Informática y Matemática  
Universidad de Holguín  
Holguín 80100  
CUBA

E-mail: rabreu@facinf.uho.edu.cu

**Juan Bory Reyes**

Departamento de Matemática  
Universidad de Oriente  
Santiago de Cuba 90500  
CUBA

E-mail: jbory@rect.uo.edu.cu

**Tania Moreno García**

Facultad de Informática y Matemática  
Universidad de Holguín  
Holguín 80100  
CUBA

E-mail: tmorenog@facinf.uho.edu.cu