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Stable bundles on 3-fold hypersurfaces

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Abstract. Using monads, we construct a large class of stable bundles of rank 2 and 3 on 3-fold hypersurfaces, and study the set of all possible Chern classes of stable vector bundles.

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1 Introduction

Given a projective variety X, a *linear monad* on X is a complex of holomorphic bundles of the form:

$$0 \to \mathcal{O}_X(-1)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0\,, \tag{1}$$

which is exact on the first and last terms. In other words, α is injective and β is surjective as bundle maps, and $\beta\alpha = 0$. The holomorphic bundle $E = \ker\beta/\operatorname{Im}\alpha$ is called the cohomology of the monad; bundles that can be obtained as the cohomology of a linear monad are known as linear bundles. We will also be interested in the kernel bundle $K = \ker\beta$, which is the dual to what is known in the literature as a Steiner bundle. Such objects have been extensively studied by a number of authors; more on linear monads and their cohomology bundles can be found in [1, 11, 12, 13] and the references therein. Steiner bundles were considered in [1, 3, 5, 12, 14].

The goal of this note is to use linear monads to construct stable bundles of rank 2 and 3 on hypersurfaces within $\mathbb{P}^4$. More precisely, we prove:

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Main Theorem. *Let X be a 3-dimensional non-singular complex projective variety with* $\mathrm{Pic}(X) = \mathbb{Z}$, *and let $H = c_1(\mathcal{O}_X(1))$. Consider the following linear monad:*

$$0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0 \quad (c \geq 1). \tag{2}$$

1. *The kernel $K = \ker \beta$ is a stable rank $c+2$ bundle with $c_1(K) = -c \cdot H$ and $c_2(K) = \frac{1}{2}(c^2 + c) \cdot H^2$;*
2. *the cohomology $E = \ker \beta / Im\, \alpha$ is a stable rank 2 bundle with $c_1(E) = 0$ and $c_2(E) = c \cdot H^2$.*

The cohomology of a linear monad of the form (2) is known as an instanton bundle [11, 12, 13].

Our motivation is twofold. First, we will see that it is actually impossible to have an inequality of the form:

$$\Delta(E) = \frac{1}{r^2}\left(2rc_2(E) - (r-1)c_1(E)^2\right) \cdot H \geq \kappa c_2(TX) \cdot H \tag{3}$$

where $E \to X$ is a stable bundle of rank r, H is the class of an ample line bundle on X and κ some universal positive constant, if the underlying variety X is allowed to be too general. Such a stronger version of the Bogomolov inequality was proposed, based on physical grounds, in [6, Conjecture 2.1] with $\kappa = 1/12$; this conjecture was withdrawn in a revised version of [6]. Second, we present a generalization to 3-fold hypersurfaces of a result due to Hartshorne on the characterization of all possible cohomology classes that arise as Chern classes of stable bundles on $\mathbb{P}^3$.

It is also worth mentioning that cohomologies of linear monads of the form

$$0 \to \mathcal{O}_X(-1)^{\oplus c+l} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 3+2c+l} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0 \quad (c, l \geq 1)$$

are stable rank 3 bundles E with $c_1(E) = l \cdot H$ and $c_2(E) = (c + l(l+1)/2) \cdot H^2$, see [13, Theorem 10]. In addition, cohomologies of linear monads of the form

$$0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus r+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0 \quad (c \geq 1,\ r = 3, 4, 5)$$

are semistable rank r bundles E with $c_1(E) = 0$ and $c_2(E) = c \cdot H^2$, see [13, Theorem 7]. Further results regarding the (semi)stability of linear monads on higher dimensional projective varieties were also estabished in [13].

Finally, the second part of our Main Theorem may be regarded as the simplest case of a more general conjecture: every rank $2n$ instanton bundle on a complex projective variety X of dimension $2n+1$ with cyclic Picard group is stable. Even for $X = \mathbb{P}^{2n+1}$, the conjecture has remained open for more than 20 years; it is known to be true only for $X = \mathbb{P}^5$ [1] and for the case $c = 1$ (a.k.a nullcorrelation bundles) on $X = \mathbb{P}^{2n+1}$ [7].

This note is organized as follows. After recalling some standard facts about hypersurfaces within complex projective spaces in Section 2, we explicity establish the existence of monads of the form (2) in Section 3. Possible generalizations of the Bogomolov inequality are discussed in Section 4. The proof of the Main Theorem is left to Section 5.

2 Hypersurfaces and monads on hypersurfaces

A hypersurface $X_{(d,n)} \hookrightarrow \mathbb{P}^n$ of degree d is the zero locus of a section $\sigma \in H^0(\mathcal{O}_{\mathbb{P}^n}(d))$; for generic σ, its zero locus is non-singular. It follows from the Lefschetz hyperplane theorem that every hypersurface is simply-connected and has cyclic Picard group [3]. It is also easy to see that hypersurfaces are arithmetically Cohen-Macaulay, that is $H^p(\mathcal{O}_X(k)) = 0$ for $1 \le p \le n-2$ and all $k \in \mathbb{Z}$. Finally, the restriction of the Kähler $\tilde{H}$ class of $\mathbb{P}^n$ induces a Kähler class H on $X_{(d,n)}$, which is the ample generator of $\mathrm{Pic}(X_{(d,n)})$. One can show that:

$$c_1(TX_{(d,n)}) = (n+1-d)\cdot H \qquad \text{and}$$

$$c_2(TX_{(d,n)}) = \left(d^2 - (n+1)d + \frac{1}{2}n(n+1)\right)\cdot H^2 \,. \tag{4}$$

Given a fixed ample invertible sheaf $\mathcal{L}$ with $c_1(\mathcal{L}) = H$ on a projective variety V of dimension m, recall that the slope $\mu(E)$ with respect to $\mathcal{L}$ of a torsion-free sheaf E on V is defined as follows:

$$\mu(E) := \frac{c_1(E)\cdot H^{m-1}}{\mathrm{rk}(E)}\,.$$

We say that E is stable with respect to $\mathcal{L}$ if for every coherent subsheaf $0 \neq F \hookrightarrow E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ we have $\mu(F) < \mu(E)$.

In the case at hand, stability will always be measured in relation to the tautological line bundle $\mathcal{O}_X(1)$ on the hypersurface $X_{(d,n)}$, whose first Chern class, denoted by H, is the ample generator of $\mathrm{Pic}(X_{(d,n)})$.

Given a linear monad on $X = X_{(d,n)}$ as in (1), note that

$$\mathrm{ch}(E) = b - a\cdot \mathrm{ch}(\mathcal{O}_X(-1)) - c\cdot \mathrm{ch}(\mathcal{O}_X(1))\,.$$

In particular,

$$\begin{aligned} \mathrm{rk}(E) &= b-a-c\,, \\ c_1(E) &= (a-c)\cdot H \quad \text{and} \\ c_2(E) &= \frac{1}{2}(a^2-2ac+c^2+a+c)\cdot H^2\,, \end{aligned} \tag{5}$$

where in this case $H = c_1(\mathcal{O}_X(1))$. The discriminant of the linear bundle is given by:

$$\begin{aligned} \Delta(E) &= \frac{1}{\mathrm{rk}(E)^2}\left(2\mathrm{rk}(E)c_2(E) - (\mathrm{rk}(E)-1)c_1(E)^2\right)\cdot H^{n-3} \\ &= \frac{b(a+c)-4ac}{(b-a-c)^2}\,. \end{aligned} \tag{6}$$

We will also be interested in the kernel bundle $K = \ker\beta$; it has the following topological invariants:

$$\mathrm{rk}(K) = b-c\,,\quad c_1(K) = -c\cdot H \quad \text{and} \quad c_2(K) = \frac{1}{2}(c^2+c)\cdot H^2\,, \tag{7}$$

and

$$\begin{aligned} \Delta(K) &= \frac{1}{\mathrm{rk}(K)^2}\left(2\mathrm{rk}(K)c_2(K) - (\mathrm{rk}(K)-1)c_1(K)^2\right)\cdot H^{n-3} \\ &= \frac{bc}{(b-c)^2}\,. \end{aligned} \tag{8}$$

In particular, for the monads as in (2) considered in our Main Theorem, we have $a = c$ and $b = 2+2c$, so the formulas above reduce to:

$$\mathrm{rk}(E) = 2\,,\quad c_1(E) = 0 \quad \text{and} \quad \Delta(E) = c\,, \tag{9}$$

and

$$\mathrm{rk}(K) = c+2\,,\quad c_1(K) = -c\cdot H \quad \text{and} \quad \Delta(K) = \frac{2c(c+1)}{(c+2)^2}\,. \tag{10}$$

Fløystad has proved a very useful existence theorem for linear monads on projective spaces in [8]; it was later adapted to linear monads on quadric hypersurfaces in [4]. Their argument is easily generalizable to hypersurfaces, and one has the following result.

Theorem 1. *Let $X = X_{(d,n)}$ be a non-singular hypersurface of degree d within $\mathbb{P}^n$, $n \geq 3$. There exists a linear monad on X as in* (1) *if and only if*

- $b \geq a + c + n - 2$, *if n is odd;*
- $b \geq a + c + n - 1$, *if n is even.*

The existence of the monad (2) above is guaranteed by this Theorem; however, since we do not give a proof of Theorem 2 in this letter, we will explicitly establish the existence of monads of the form (2) in the next Section 3.

It is worth mentioning that the monad construction does not yield stable rank 2 bundles with odd first Chern class; to construct those, one needs a variation of the usual Serre construction. For example on $\mathbb{P}^3$, this construction provides a 1-1 correspondence between rank 2 bundles and certain codimension 2 subvarieties of $\mathbb{P}^3$; see Hartshorne's paper [9].

3 Existence of linear monads on 3-fold hypersurfaces

Let $X = X_{(d,4)}$ be a generic, non-singular hypersurface of degree d within $\mathbb{P}^4$; let $[x_0 : x_1 : x_2 : x_3 : x_4]$ be homogeneous coordinates in $\mathbb{P}^4$. We will now explicitly establish the existence of linear monads of the form

$$0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0\,, \qquad c \geq 1\,.$$

Consider the $c \times (c+1)$ matrices:

$$B_1 = \begin{pmatrix} x_0 & x_1 & & & \\ & x_0 & x_1 & & \\ & & \ddots & \ddots & \\ & & & x_0 & x_1 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} x_2 & x_3 & & & \\ & x_2 & x_3 & & \\ & & \ddots & \ddots & \\ & & & x_2 & x_3 \end{pmatrix},$$

and the $(c+1) \times c$ matrices:

$$A_1 = \begin{pmatrix} x_1 & & & \\ x_0 & x_1 & & \\ & \ddots & \ddots & \\ & & x_0 & x_1 \\ & & & x_0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} x_3 & & & \\ x_2 & x_3 & & \\ & \ddots & \ddots & \\ & & x_2 & x_3 \\ & & & x_2 \end{pmatrix}.$$

Notice that all four matrices have maximal rank c. It easy to check that:

$$B_1A_2 = B_2A_1 = \begin{pmatrix} \sigma_1 & \sigma_2 & 0 & \cdots & \cdots & 0 \\ \sigma_0 & \sigma_1 & \sigma_2 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & \sigma_0 & \sigma_1 \end{pmatrix},$$

where $\sigma_0 = x_0x_2$, $\sigma_1 = x_0x_3 + x_1x_2$ and $\sigma_2 = x_1x_3$.

Now form the linear monad:

$$0 \to \mathcal{O}_X(-1)^{\oplus c} \stackrel{\alpha}{\to} \mathcal{O}_X^{\oplus 2+2c} \stackrel{\beta}{\to} \mathcal{O}_X(1)^{\oplus c} \to 0$$

where the maps α and β are given by:

$$\beta = \begin{pmatrix} B_1 & B_2 \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} A_2 \\ -A_1 \end{pmatrix}$$

Clearly, both maps are of maximal rank c for every point $[x_0 : x_1 : x_2 : x_3 : x_4] \in X$, and $\beta\alpha = B_1A_2 - B_2A_1 = 0$.

Example. Setting $c = 1$ in the construction above, one obtains the following linear monad

$$0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0 \tag{11}$$

with maps given by:

$$\alpha = \begin{pmatrix} x_3 \\ x_2 \\ -x_1 \\ -x_0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix}.$$

It follows from the Main Theorem that the cohomology E of the monad (11) is a stable rank 2 bundle with $c_1 = 0$ and $c_2 = H^2$, while its kernel bundle K is a stable rank 3 bundle with $c_1 = -H$ and $c_2 = H^2$. From equations (9) and (10), we have that $\Delta(E) = 1$ and $\Delta(K) = 4/9$, and since both $\Delta(E)$ and $\Delta(K)$ are increasing as functions of c, we conclude that these are actually the minimum values for the discriminant of the bundles considered in our Main Theorem. □

In the next Section, we will use the previous Example to show that it is actually impossible to have an inequality of the form (3) if the underlying variety is allowed to be too general, as claimed at the Introdution.

4 Chern classes of stable rank 2 bundles on 3-fold hypersurfaces

The characterization of all possible cohomology classes that arise as Chern classes of stable bundles on a given Kähler manifold is not only of mathematical interest, but it is also relevant from the point of view of physics: it amounts to describing the set of all possible charges of BPS particles in type IIa superstring theory. Besides the Bogomolov inequality $\Delta(E) \geq 0$ for every stable bundle E, few general results for higher dimensional varieties are available in the literature.

It is not known whether the Bogomolov inequality is actually sharp; however, it is impossible to have an inequality of the form (3) if the underlying variety is allowed to be too general, even if the dimension is fixed. Indeed, as noted in the Example at the end of the previous Section, one can always find a stable rank 3 bundle K over $X_{(d,4)}$ with $\Delta(K) = 4/9$. On the other hand, formula (4) shows that for fixed dimension n, the right hand side of (3) grows quadratically with the degree, since it is proportional to $c_2(TX_{(d,n)})$.

Therefore, one must somehow restrict the type of varieties allowed, e.g. by considering only Fano or Calabi-Yau varieties of a fixed dimension. For instance, for X being a nonsingular complete intersection Calabi-Yau 3-fold, all stable bundles from the Main Theorem satisfy (3) with $\kappa = 2/45$. In particular, we conclude that for Calabi-Yau 3-folds, the universal positive constant κ in the strong Bogomolov inequality must satisfy $\kappa \leq 2/45$.

The integral cohomology ring of a 3-fold hypersurface $X_{(d,4)}$ is simple to describe:

$$H^*(X_{(d,4)}, \mathbb{Z}) = \mathbb{Z}[H, L, T]\big/(L^2 = T^2 = 0, H^2 = dL, HL = T)\,.$$

Notice that $H^3 = dT$ and $H^4 = 0$. Clearly, H is the generator of $H^2(X_{(d,4)}, \mathbb{Z})$, L is the generator of $H^4(X_{(d,4)}, \mathbb{Z})$ and T is the generator of $H^6(X_{(d,4)}, \mathbb{Z})$.

Now let E be a rank r bundle on a 3-fold hypersurface $X_{(d,4)}$. Recall that for any rank r bundle E on a variety X with cyclic Picard group, there is a uniquely determined integer k_E such that $-r+1 \leq c_1(E(k_E)) \leq 0$; the twisted bundle $E(k_E)$ is called the normalization of E. We set $E_{\text{norm}} := E(k_E)$ and we call E normalized if $E = E_{\text{norm}}$. Therefore it is enough to consider the case when $c_1(E) = k \cdot H$ for $-r+1 \leq k \leq 0$, and study the sets $S_{(r,k)}(X_{(d,4)})$ consisting of all integers $\gamma \in \mathbb{Z}$ for with there exists a stable rank r bundle E with $c_1(E) = k \cdot H$ and $c_2(E) = \gamma \cdot L$.

In the simplest possible case, provided by $d = 1$ (so that $X = \mathbb{P}^3$) and $r = 2$, this problem was completely solved by Hartshorne in [9]. He proved that $S_{(2,0)}(\mathbb{P}^3)$ consists of all positive integers, while $S_{(2,-1)}(\mathbb{P}^3)$ consists of all positive even integers. As far as it is known to the author, Hartshorne's result has not been generalized neither for rank 2 bundles over other 3-folds, nor for higher rank bundles on $\mathbb{P}^3$.

The following two lemmas are simple consequences of the Bogomolov inequality and our Main Theorem, respectively:

Lemma 2. *If $\gamma \in S_{(r,k)}(X_{(d,4)})$, then $\gamma \geq \frac{r-1}{2r} dk^2$.*

Lemma 3. *For every positive integer $c \geq 1$, $cd \in S_{(2,0)}(X_{(d,4)})$.*

Based on Hartshorne's result mentioned above, it seems reasonable to conjecture that $S_{(2,0)}(X_{(d,4)})$ consists exactly of all positive multiples of d.

5 Proof of the Main Theorem

The proof is based on a very useful criterion (due to Hoppe) to decide whether a bundle on a variety with cyclic Picard group is stable.

Proposition 4 **([10, Lemma 2.6]).** *Let E be a rank r holomorphic vector bundle on a variety X with $\mathrm{Pic}(X) = \mathbb{Z}$. If $H^0((\wedge^q E)_{\mathrm{norm}}) = 0$ for $1 \leq q \leq r-1$, then E is stable.*

Our argument follows [1, Theorem 2.8]. Consider the linear monad

$$0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0\,;$$

setting $K = \ker \beta$, one has the sequences:

$$0 \to K \to \mathcal{O}_X^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0 \quad \text{and} \tag{12}$$

$$0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} K \to E \to 0\,. \tag{13}$$

First, we will show that the kernel bundle K is stable. That implies that K is simple, which in turn implies that the cohomology bundle E is simple. Since any simple rank 2 bundle is stable, we conclude that E is also stable.

Recall that one can associate to the short exact sequence of locally-free sheaves $0 \to A \to B \to C \to 0$ two long exact sequences of symmetric and exterior powers:

$$0 \to \wedge^q A \to \wedge^q B \to \wedge^{q-1} B \otimes C \to \cdots \to B \otimes S^{q-1} C \to S^q C \to 0\,, \tag{14}$$

and

$$0 \to S^q A \to S^{q-1} A \otimes B \to \cdots \to A \otimes \wedge^{q-1} B \to \wedge^q B \to \wedge^q C \to 0\,. \tag{15}$$

In what follows, $\mu(F) = c_1(F)/\mathrm{rk}(F)$ is the slope of the sheaf F, as usual.

Claim. *K is stable.*

From the sequence dual to sequence (12), we get that:

$$\mu(K^*) = \frac{c}{c+2}, \quad \mu(\wedge^q K^*) = \frac{qc}{c+2}$$

so that $(\wedge^q K^*)_{\text{norm}} = \wedge^q K^*(k)$ for some $k \leq -1$, and if $H^0(\wedge^q K^*(-1)) = 0$, then $H^0((\wedge^q K^*)_{\text{norm}}) = 0$.

The vanishing of $h^0(K^*(-1))$ (i.e $q = 1$) is obvious from the dual to sequence (12). For the case $q = 2$, start from the dual to (12) and consider the associated sequence

$$0 \to S^2(\mathcal{O}_X(-1)^{\oplus c}) \to \mathcal{O}_X(-1)^{\oplus c} \otimes \mathcal{O}_X^{\oplus 2c+2} \to \wedge^2(\mathcal{O}_X^{\oplus 2c+2}) \to \wedge^2 K^* \to 0\,.$$

Twist it by $\mathcal{O}_X(-1)$ and break it into two short exact sequences to obtain

$$0 \to \mathcal{O}_X(-3)^{\oplus\binom{c+1}{2}} \to \mathcal{O}_X(-2)^{\oplus 2c^2+2c} \to Q \to 0 \quad \text{and}$$

$$0 \to Q \to \mathcal{O}_X(-1)^{\oplus\binom{2c+2}{2}} \to \wedge^2 K^*(-1) \to 0\,.$$

Passing to cohomology, we get $H^0(\wedge^2 K^*(-1)) = H^1(Q) = 0$.

Now set $q = 3 + t$ for $t = 0, 1, \cdots, c-2$ and note that

$$\mu(\wedge^{3+t} K^*(-t-1)) = \frac{(3+t)c}{c+2} - t - 1 = 2\frac{c-t-1}{c+2} > 0\,.$$

Thus $(\wedge^{3+t} K^*)_{\text{norm}} = \wedge^{3+t} K^*(k)$ for some $k \leq -t-2$, and if $H^0(\wedge^{3+t} K^*(-t-2)) = 0$, then $H^0((\wedge^{3+t} K^*)_{\text{norm}}) = 0$.

We show that $H^0(\wedge^{3+t} K^*(-t-2)) = 0$ by induction on t. From the dual to sequence (13) twisted by $\mathcal{O}_X(-2)$ we get:

$$0 \to \wedge^3 K^*(-2) \to \wedge^2 K^*(-1)^{\oplus c} \to \cdots$$

since $\wedge^3 E^* = 0$ because E has rank 2. Passing to cohomology, we get that $H^0(\wedge^3 K^*(-2)) = 0$, since, as we have seen above, $H^0(\wedge^2 K^*(-1)) = 0$. This proves the statement for $t = 0$.

By the same token, we get from the dual to sequence (13) after twisting by $\mathcal{O}_X(-2-t)$:

$$0 \to \wedge^{3+t} K^*(-2-t) \to \wedge^{2+t} K^*(-t-1)^{\oplus c} \to \cdots\,.$$

Passing to cohomology, we get

$$H^0(\wedge^{2+t}K^*(-t-1)) = 0 \quad \Rightarrow \quad H^0(\wedge^{3+t}K^*(-t-2)) = 0$$

which is the induction step we needed.

In summary, we have shown that $H^0((\wedge^q K^*)_{\text{norm}}) = 0$ for $1 \leq q \leq c+1$. Thus, by Proposition 5, we have completed the proof of the claim.

Claim. *E is simple, hence stable.*

Applying $\text{Ext}^*(\cdot, E)$ to the sequence (13) we get

$$0 \to \text{Ext}^0(E, E) \to \text{Ext}^0(K, E) \to \cdots . \tag{16}$$

Now applying $\text{Ext}^*(K, \cdot)$ we get:

$$\text{Ext}^0(K, \mathcal{O}_X(-1))^{\oplus c} \to \text{Ext}^0(K, K) \to \text{Ext}^0(K, E) \to \text{Ext}^1(K, \mathcal{O}_X(-1))^{\oplus c}.$$

However, it follows from the dual of sequence (12) twisted by $\mathcal{O}_X(1)$ that $h^0(K^*(-1)) = h^1(K^*(-1)) = 0$, thus

$$\dim \text{Ext}^0(K, E) = \dim \text{Ext}^0(K, K) = 1$$

because K is simple. It then follows from (16) that E is also simple. But E has rank 2, thus E is stable, as desired.

This completes the proof of the Main Theorem. □

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References

[1] V. Ancona and G. Ottaviani, *Stability of special instanton bundles on* $\mathbb{P}^{2n+1}$. Trans. Am. Math. Soc., **341** (1994), 677–693.

[2] M.C. Brambilla, *Simplicity of generic Steiner bundles.* Boll. Unione Mat. Ital. B(8), **8** (2005), 723–735.

[3] R. Bott, *On a theorem of Lefschetz*. Michigan Math. J., **6** (1959), 211–216.

[4] L. Costa and R.M. Miró-Roig, *Monads and instanton bundles over smooth projective varieties*, to appear in Math. Nachr. (2007).

[5] I. Dolgachev and M. Kapranov, *Arrangements of hyperplanes and vector bundles on* $\mathbb{P}^n$. Duke Math. J., **73** (1993), 633–664.

[6] M.R. Douglas, R. Reinbacher and S.-T. Yau, *Branes, bundles and attactors: Bogomolov and beyond*. Preprint math.AG/0604597v2.

[7] L. Ein, *Some stable vector bundles on* $\mathbb{P}^4$ *and* $\mathbb{P}^5$. J. Reine Agnew. Math., **337** (1982), 142–153.

[8] G. Floystad, *Monads on projective spaces*. Comm. Algebra, **28** (2000), 5503–5516.

[9] R. Hartshorne, *Stable vector bundles of rank 2 on* $\mathbb{P}^3$. Math. Ann., **238** (1978), 229–280.

[10] H. Hoppe, *Generischer spaltungstyp und zweite Chernklasse stabiler Vektorraumbündel vom rang 4 auf* $\mathbb{P}^4$. Math. Z., **187** (1984), 345–360.

[11] M. Jardim, *Instanton sheaves on complex projective spaces*. Collect. Math., **57** (2006), 69–91.

[12] M. Jardim and R.V. Martins, *Linear and Steiner bundles on projective varieties*. Preprint math/0702048.

[13] M. Jardim and R.M. Miró-Roig, *On the semistability of instanton sheaves over certain projective varieties*, to appear in Comm. Algebra (2008).

[14] H. Soares, *Steiner bundles on the hyperquadric* $Q_n \subset \mathbb{P}^{n+1}$, to appear in Comm. Algebra (2007).

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