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Hypersurfaces which are equiaffine extremal and centroaffine extremal

Huili Liu* and Seoung Dal Jung**

Abstract. For a centroaffine hypersurface, we use the relations between equiaffine and centroaffine invariants to find many new examples of equiaffine extremal hypersurfaces.

Keywords: equiaffine extremal hypersurface, centroaffine extremal hypersurface, Tchebychev form, Tchebychev vector field.

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1 Introduction

In equiaffine differential geometry, the variational problem for the equiaffine area integral leads to equiaffine minimal surfaces. Such surfaces have zero equiaffine mean curvature $H(e) = 0$. These surfaces were called affine minimal by Blaschke and his school ([Bla-II]). Calabi [Cal-2] pointed out that, for locally strongly convex surface with $H(e) = 0$, the second variation of the area integral is negative. So he suggested that the surfaces with $H(e) = 0$ should be called affine maximal surfaces. For a locally strongly convex surface, the equiaffine Euler-Lagrange equation is a nonlinear PDE of fourth order ([V-V]). In the following, we will call the solutions of the equiaffine Euler-Lagrange equations "extremal" surfaces, in case of locally strongly convex surfaces also "affine maximal". To find examples of equiaffine extremal (the equiaffine metric is definite or indefinite) hypersurfaces is a very interesting and important problem. Wang ([Wang]) studied the variation of the centroaffine area integral and introduced centroaffine extremal hypersurfaces. Such hypersurfaces have the property of $\mathrm{trace}_G \widehat{\nabla} \widehat{T} \equiv 0$, where G is the centroaffine metric, $\widehat{\nabla}$ the centroaffine

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Levi-Civita connection and $\widehat{T}$ the centroaffine Tchebychev form. Many examples of centroaffine extremal surfaces and hypersurfaces are given in [Li-W], [Liu-W-1], [Liu-1], etc.. In [Liu-1], the surfaces which are centroaffine extremal and equiaffine extremal are classified. There are few examples of hypersurfaces which are equiaffine extremal.

In this paper, we will give many examples of equiaffine extremal hypersurfaces using the knowledge of centroaffine extremal hypersurfaces. We will prove the following theorem.

Theorem 1.1. *Let $x : \mathbf{M} \to \mathbb{R}^{n+1}$ be a centroaffine extremal hypersurface immersion. Then x is also an equiaffine extremal hypersurface if and only if the centroaffine Tchebychev vector field T of x satisfies*

$$||T||^2 := G_{ij} T^i T^j = \pm \frac{(n+2)^2}{2n^2}.$$

Therefore, we can find examples of equiaffine extremal hypersurfaces within the class of centroaffine extremal hypersurfaces. In this class, many examples are known as the centroaffine normalization is the most natural and simple relative normalization ([S-S-V]).

2 Centroaffine hypersurfaces

Let $x : \mathbf{M} \to \mathbb{R}^{n+1}$ be a hypersurface immersion and $[\ ,\ \dots\ ,\ ]$ the standard determinant in $\mathbb{R}^{n+1}$. x is said to be a centroaffine hypersurface if the position vector of x, denoted also by x, is always transversal to the tangent space $x_*(\mathbf{TM})$ at each point of $\mathbf{M}$ in $\mathbb{R}^{n+1}$. We define a symmetric bilinear form G on $\mathbf{TM}$ by

$$G = \sum_{i,j=1}^{n} \frac{[e_1(x), e_2(x), \dots, e_n(x), e_i e_j(x)]}{[e_1(x), e_2(x), \dots, e_n(x), x]}\, \theta^i \otimes \theta^j, \tag{2.1}$$

where $\{e_1, e_2, \dots, e_n\}$ is a local basis of $\mathbf{TM}$ with the dual basis $\{\theta^1, \theta^2, \dots, \theta^n\}$. Note that G is globally defined. A centroaffine hypersurface x is said to be nondegenerate if G is nondegenerate. We call G the centroaffine metric of x. We say that a hypersurface is definite (or indefinite) if G is definite (or indefinite). Geometrically, a hypersurface x with positive (resp. negative) definate centroaffine metric G is the locally strongly convex hypersurface in $\mathbb{R}^{n+1}$ and such hypersurface is called hyperbolic type (respectively, elliptic type) in [L-L-S].

For a centroaffine hypersurface x, let $\nabla = \{\Gamma_{ij}^k\}$ be the induced connection and $\widehat{\nabla} = \{\widehat{\Gamma}_{ij}^k\}$ the Levi-Civita connection of centroaffine metric G. The coefficients of cubic form C of x are defined by (in the following, we use the Einstein summation convention)

$$\Gamma_{ij}^{k} - \widehat{\Gamma}_{ij}^{k} =: C_{ij}^{k}, \quad C_{ijk} := G_{km} C_{ij}^{m}, \quad i, j, k, m = 1, 2, \dots, n. \tag{2.2}$$

We know that $C = C_{ijk}\theta^i\theta^j\theta^k$ is the centroaffine Fubini-Pick form for x which is totally symmetric. Tchebychev vector field and Tchebychev form of x are defined by

$$T \quad := \quad T^j e_j = \frac{1}{n} G^{ik} C_{ik}^{j} e_j \tag{2.3}$$

and

$$\widehat{T} \quad := \quad T_j \theta^j = G_{ij} T^i \theta^j \tag{2.4}$$

respectively. It is well-known that T and $\widehat{T}$ are centroaffine invariants. The Gauss equation of x is

$$\frac{\partial^2 x}{\partial x^i \partial x^j} = \Gamma_{ij}^{k} e_k(x) + G_{ij} x. \tag{2.5}$$

Definition 2.1. *Let $x : \mathbf{M} \to \mathbb{R}^{n+1}$ be a centroaffine hypersurface. If* $\mathrm{trace}_G \widehat{\nabla}\widehat{T} = G^{ij}\widehat{\nabla}_i T_j \equiv 0$, *$x$ is called a centroaffine extremal hypersurface.* ([Wang])

Remark 2.1. $\widehat{\nabla}\widehat{T}$ is symmetric ([Wang]).

Remark 2.2. From the definition of centroaffine extremal hypersurface we know that

1. the proper affine hyperspheres are centroaffine extremal;
2. the centroaffine hypersurfaces with parallel Tchebychev form are centroaffine extremal.

3 Proof of the theorem and the examples

3.1 The proof of the theorem

Let $x : \mathbf{M} \to \mathbb{R}^{n+1}$ be a centroaffine hypersurface. We denote the equiaffine support function and the equiaffine Weingarten operator by $\rho(e)$ and $S(e)$. Then we have the following relation between the equiaffine quantities and the centroaffine quantities of x (cf. [S-S-V], 5.1.3; [L-S-W], 4.3.2; [Liu-1], 3.2)

$$\rho(e)S(e) = \pm\mathrm{id} - \frac{2n}{n+2}\widehat{\nabla}T - \frac{2n}{n+2}\left\{C(T,\circ) - \frac{2n}{n+2}\widehat{T}\otimes T\right\}. \quad (3.1)$$

If x is both equiaffine extremal and centroaffine extremal, from (3.1) we get

$$0 = \pm n - \frac{2n}{n+2}\{n||T||^2 - \frac{2n}{n+2}||T||^2\},$$

that is

$$||T||^2 := G_{ij}T^iT^j = \pm\frac{(n+2)^2}{2n^2}.$$

This proves the theorem. □

3.2 Example one

The hypersurface

$$x_{n+1} = x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}, \qquad \sum_{i=1}^{n}\alpha_i \neq 1; \alpha_i \neq 0, i = 1, 2, \dots, n \quad (3.2.1)$$

is centroaffine extremal. The centroaffine metric of this hypersurface is flat (cf. (3.2.8)) and the coefficients of the cubic form are constant (cf. (3.2.9)). With parameter $(u_1, u_2, \dots, u_n)$, the hypersurface (3.2.1) can be written as

$$x = \big(x_1, x_2, \dots, x_n, x_{n+1}\big) = \big(e^{u_1}, e^{u_2}, \dots, e^{u_n}, e^{\alpha_1u_1+\alpha_2u_2+\cdots+\alpha_nu_n}\big). \quad (3.2.2)$$

In $\mathbb{R}^n$, introducing the canonical scalar product $\langle\alpha, u\rangle := \sum_i \alpha_iu_i$, we have

$$\langle\alpha, u\rangle = \alpha_1u_1 + \alpha_2u_2 + \cdots + \alpha_nu_n,$$

$$\langle(\mathbf{1}+\alpha), u\rangle = (1+\alpha_1)u_1 + (1+\alpha_2)u_2 + \cdots + (1+\alpha_n)u_n,$$

here $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^n$. We denote

$$x_i = \frac{\partial x}{\partial u_i}, \qquad x_{ij} = \frac{\partial^2 x}{\partial u_i\partial u_j}.$$

Then

$$\begin{aligned} x_i &= \big(0, \dots, e^{u_i}, \dots, 0, \alpha_ie^{\alpha_1u_1+\alpha_2u_2+\cdots+\alpha_nu_n}\big) \\ &= \big(0, \dots, e^{u_i}, \dots, 0, \alpha_ie^{\langle\alpha,u\rangle}\big), \end{aligned} \quad (3.2.3)$$

$$\begin{aligned} x_{ij} &= \left(0, \ldots, \delta_{ij} e^{u_i}, \ldots, 0, \alpha_i \alpha_j e^{\alpha_1 u_1 + \cdots + \alpha_n u_n}\right) \\ &= \left(0, \ldots, \delta_{ij} e^{u_i}, \ldots, 0, \alpha_i \alpha_j e^{\langle \alpha, u \rangle}\right). \end{aligned} \tag{3.2.4}$$

Therefore

$$[x_1, x_2, \ldots, x_n, x] = \left(1 - \sum_{t=1}^{n} \alpha_t\right) e^{\langle (\mathbf{1}+\alpha), u \rangle}, \tag{3.2.5}$$

$$\left[x_1, x_2, \ldots, x_n, x_{ij}\right] = (\alpha_i \alpha_j - \alpha_i \delta_{ij}) e^{\langle (\mathbf{1}+\alpha), u \rangle}, \tag{3.2.6}$$

$$[x_1, \ldots, x_{kj}, \ldots, x_n, x] = \begin{cases} \big(1 - \cdots - \alpha_{k-1} - \alpha_k^2 - \alpha_{k+1} \\ \qquad - \cdots - \alpha_n\big) e^{\langle (\mathbf{1}+\alpha), u \rangle}, & j = k \\ -\alpha_j \alpha_k e^{\langle (\mathbf{1}+\alpha), u \rangle}, & j \neq k. \end{cases} \tag{3.2.7}$$

By the definition of the centroaffine metric, we get

$$G_{ij} = \frac{[x_1, x_2, \ldots, x_n, x_{ij}]}{[x_1, x_2, \ldots, x_n, x]} = \frac{\alpha_i \alpha_j - \alpha_i \delta_{ij}}{1 - \sum_{t=1}^{n} \alpha_t}. \tag{3.2.8}$$

Since the centroaffine metric is flat, we have $C_{ij}^k = \Gamma_{ij}^k$. So we obtain

$$\begin{aligned} C_{jk}^k &= \frac{[x_1, x_2, \ldots, x_{kj}, \ldots, x_n, x]}{[x_1, x_2, \ldots, x_n, x]} \\ &= \begin{cases} \dfrac{(1 - \alpha_1 - \cdots - \alpha_{k-1} - \alpha_k^2 - \alpha_{k+1} - \cdots - \alpha_n)}{1 - \sum_{t=1}^{n} \alpha_t}, & j = k \\ \dfrac{-\alpha_j \alpha_k}{1 - \sum_{t=1}^{n} \alpha_t}, & j \neq k. \end{cases} \end{aligned} \tag{3.2.9}$$

For the Tchebychev form, we have

$$T_i = \sum_{t=1}^{n} C_{it}^t = \frac{1 + \alpha_i}{n}. \tag{3.2.10}$$

We denote the matrix of the centroaffine metric G also by G, then

$$\begin{aligned} G &= (G_{ij}) \\ &= \frac{1}{1 - \sum_{t=1}^{n} \alpha_t} \begin{pmatrix} \alpha_1^2 - \alpha_1 & \alpha_1 \alpha_2 & \ldots & \alpha_1 \alpha_n \\ \alpha_2 \alpha_1 & \alpha_2^2 - \alpha_2 & \ldots & \alpha_2 \alpha_n \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_n \alpha_1 & \alpha_n \alpha_2 & \ldots & \alpha_n^2 - \alpha_n \end{pmatrix}. \end{aligned} \tag{3.2.11}$$

The determinant of G is

$$\begin{aligned} |G| = \det G &= \frac{(-1)^{n-1}(\sum_{t=1}^{n}\alpha_t - 1)\Pi_{t=1}^{n}\alpha_t}{(1-\sum_{t=1}^{n}\alpha_t)^n} \\ &= \frac{(-1)^n \Pi_{t=1}^{n}\alpha_t}{(1-\sum_{t=1}^{n}\alpha_t)^{n-1}}. \end{aligned} \tag{3.2.12}$$

The inverse matrix G^{-1} of G is

$$\begin{aligned} G^{-1} &= (G_{ij}^{-1}) \\ &= -\begin{pmatrix} \frac{1-\sum_{t=1}^{n}\alpha_t+\alpha_1}{\alpha_1} & 1 & \dots & 1 \\ 1 & \frac{1-\sum_{t=1}^{n}\alpha_t+\alpha_2}{\alpha_2} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & \frac{1-\sum_{t=1}^{n}\alpha_t+\alpha_n}{\alpha_n} \end{pmatrix}. \end{aligned} \tag{3.2.13}$$

Then for the Tchebychev vector field T, we obtain

$$||T||^2 = \frac{1}{n^2}\left[n(n+1) + \sum_{t=1}^{n}\frac{1}{\alpha_t} - \sum_{t,s=1,t\neq s}^{n}\frac{\alpha_t}{\alpha_s} + \sum_{t=1}^{n}\alpha_t \right]. \tag{3.2.14}$$

From Theorem 1 and Remark 2.2 we have

Theorem 3.1. *The hypersurface*

$$x_{n+1} = x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}, \qquad \sum_{i=1}^{n}\alpha_i \neq 1; \alpha_i \neq 0, i = 1, 2, \dots, n$$

is an equiaffine extremal hypersurface when $\alpha_1, \dots, \alpha_n$ *satisfy*

$$\frac{1}{n^2}\left[n(n+1) + \sum_{t=1}^{n}\frac{1}{\alpha_t} - \sum_{t,s=1,t\neq s}^{n}\frac{\alpha_t}{\alpha_s} + \sum_{t=1}^{n}\alpha_t \right] = \pm\frac{(n+2)^2}{2n^2}.$$

3.3 Example two

The hypersurface

$$x_{n+1} = \exp\left(\alpha_1 \arctan\frac{x_1}{x_2}\right)(x_1^2+x_2^2)^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4}\dots x_n^{\alpha_n}, \quad \alpha_1^2+\alpha_2^2 \neq 0; \tag{3.3.1}$$

$$2\alpha_2 + \sum_{t=3}^{n}\alpha_t \neq 1; \alpha_t \neq 0, t = 3, \dots, n$$

is centroaffine extremal. The centroaffine metric of this hypersurface is flat (cf. (3.3.29)-(3.3.34)) and the coefficients of the cubic form are constant (cf. (3.3.19)-(3.3.28)). With parameter $(u_1, u_2, \dots, u_n)$, the hypersurface (3.3.1) can be written as

$$\begin{aligned} x &= (x_1, x_2, \dots, x_n, x_{n+1}) \\ &= (e^{u_2}\sin u_1, e^{u_2}\cos u_1, e^{u_3}, \dots, e^{u_n}, e^{\alpha_1 u_1 + 2\alpha_2 u_2 + \cdots + \alpha_n u_n}) \end{aligned} \tag{3.3.2}$$

Let $\beta = (\alpha_1, 2\alpha_2, \dots, \alpha_n)$, $\gamma = (\alpha_1, 2 + 2\alpha_2, \dots, 1 + \alpha_n)$ we have

$$\langle \beta, u\rangle = \alpha_1 u_1 + 2\alpha_2 u_2 + \cdots + \alpha_n u_n$$

and

$$\langle \gamma, u\rangle = (0 + \alpha_1)u_1 + (2 + 2\alpha_2)u_2 + \cdots + (1 + \alpha_n)u_n.$$

Then

$$x_1 = \big(e^{u_2}\cos u_1, -e^{u_2}\sin u_1, 0, \dots, 0, \alpha_1 e^{\langle \beta,u\rangle}\big), \tag{3.3.3}$$

$$x_2 = \big(e^{u_2}\sin u_1, e^{u_2}\cos u_1, 0, \dots, 0, 2\alpha_2 e^{\langle \beta,u\rangle}\big), \tag{3.3.4}$$

$$x_i = \big(0, \dots, e^{u_i}, \dots, 0, \alpha_i e^{\langle \beta,u\rangle}\big), \qquad i \geq 3, \tag{3.3.5}$$

$$x_{11} = \big(-e^{u_2}\sin u_1, -e^{u_2}\cos u_1, 0, \dots, 0, \alpha_1^2 e^{\langle \beta,u\rangle}\big), \tag{3.3.6}$$

$$x_{12} = \big(e^{u_2}\cos u_1, -e^{u_2}\sin u_1, 0, \dots, 0, 2\alpha_1\alpha_2 e^{\langle \beta,u\rangle}\big), \tag{3.3.7}$$

$$x_{1i} = \big(0, \dots\dots, 0, \alpha_1\alpha_i e^{\langle \beta,u\rangle}\big), \qquad i \geq 3, \tag{3.3.8}$$

$$x_{22} = \big(e^{u_2}\sin u_1, e^{u_2}\cos u_1, 0, \dots, 0, 4\alpha_2^2 e^{\langle \beta,u\rangle}\big), \tag{3.3.9}$$

$$x_{2i} = \big(0, \dots\dots, 0, 2\alpha_2\alpha_i e^{\langle \beta,u\rangle}\big), \qquad i \geq 3, \tag{3.3.10}$$

$$x_{ik} = \big(0, \dots, \delta_{ik} e^{u_k}, \dots, 0, \alpha_i\alpha_k e^{\langle \beta,u\rangle}\big), \qquad i, k \geq 3. \tag{3.3.11}$$

Therefore

$$D = [x_1, x_2, \dots, x_n, x] = \left(1 - 2\alpha_2 - \sum_{t=3}^{n} \alpha_t\right) e^{\langle \gamma,u\rangle}, \tag{3.3.12}$$

$$[x_1, x_2, \dots, x_n, x_{11}] = \big(\alpha_1^2 + 2\alpha_2\big) e^{\langle \gamma,u\rangle}, \tag{3.3.13}$$

$$[x_1, x_2, \dots, x_n, x_{12}] = \big(2\alpha_1\alpha_2 - \alpha_1\big) e^{\langle \gamma,u\rangle}, \tag{3.3.14}$$

$$[x_1, x_2, \ldots, x_n, x_{1i}] = \alpha_1\alpha_i e^{\langle\gamma,u\rangle}, \qquad i \geq 3, \tag{3.3.15}$$

$$[x_1, x_2, \ldots, x_n, x_{22}] = \left(4\alpha_2^2 - 2\alpha_2\right)e^{\langle\gamma,u\rangle}, \tag{3.3.16}$$

$$[x_1, x_2, \ldots, x_n, x_{2i}] = 2\alpha_2\alpha_i e^{\langle\gamma,u\rangle}, \qquad i \geq 3, \tag{3.3.17}$$

$$\left[x_1, x_2, \ldots, x_n, x_{ij}\right] = \left(\alpha_i\alpha_j - \alpha_j\delta_{ij}\right)e^{\langle\gamma,u\rangle}, \quad i, j \geq 3. \tag{3.3.18}$$

$$DC_{11}^1 = [x_{11}, x_2, \ldots, x_n, x] = 0, \tag{3.3.19}$$

$$DC_{12}^2 = [x_1, x_{12}, x_3, \ldots, x_n, x] = (\alpha_1 - 2\alpha_1\alpha_2)e^{\langle\gamma,u\rangle}, \tag{3.3.20}$$

$$DC_{1k}^k = [x_1, \ldots, x_{1k}, \ldots, x_n, x] = -\alpha_1\alpha_k e^{\langle\gamma,u\rangle}, \qquad k \geq 3, \tag{3.3.21}$$

$$DC_{21}^1 = [x_{12}, x_2, x_3, \ldots, x_n, x] = \left(1 - 2\alpha_2 - \sum_{t=3}^n \alpha_t\right) e^{\langle\gamma,u\rangle}, \tag{3.3.22}$$

$$DC_{22}^2 = [x_1, x_{22}, x_3, \ldots, x_n, x] = \left(1 - 4\alpha_2^2 - \sum_{t=3}^n \alpha_t\right) e^{\langle\gamma,u\rangle}, \tag{3.3.23}$$

$$DC_{2k}^k = [x_1, \ldots, x_{2k}, \ldots, x_n, x] = -2\alpha_2\alpha_k e^{\langle\gamma,u\rangle}, \qquad k \geq 3, \tag{3.3.24}$$

$$DC_{i1}^1 = [x_{i1}, x_2, \ldots, x_n, x] = 0, \qquad i \geq 3, \tag{3.3.25}$$

$$DC_{i2}^2 = [x_1, x_{i2}, x_3, \ldots, x_n, x] = -2\alpha_2\alpha_i e^{\langle\gamma,u\rangle}, \qquad i \geq 3, \tag{3.3.26}$$

$$\begin{aligned} DC_{kk}^k &= [x_1, \ldots, x_{kk}, \ldots, x_n, x] \\ &= \left(1 - 2\alpha_2 - \alpha_3 - \cdots - \alpha_k^2 - \cdots - \alpha_n\right)e^{\langle\gamma,u\rangle}, \qquad k \geq 3, \end{aligned} \tag{3.3.27}$$

$$DC_{ik}^k = [x_1, \ldots, x_{ik}, \ldots, x_n, x] = -\alpha_i\alpha_k e^{\langle\gamma,u\rangle}, \quad i, k \geq 3, i \neq k. \tag{3.3.28}$$

By the definition of the centroaffine metric, we get

$$G_{11} = \frac{[x_1, x_2, \ldots, x_n, x_{11}]}{[x_1, x_2, \ldots, x_n, x]} = \frac{\alpha_1^2 + 2\alpha_2}{1 - 2\alpha_2 - \sum_{t=3}^n \alpha_t}, \tag{3.3.29}$$

$$G_{12} = \frac{[x_1, x_2, \ldots, x_n, x_{12}]}{[x_1, x_2, \ldots, x_n, x]} = \frac{2\alpha_1\alpha_2 - \alpha_1}{1 - 2\alpha_2 - \sum_{t=3}^n \alpha_t}, \tag{3.3.30}$$

$$G_{1i} = \frac{[x_1, x_2, \ldots, x_n, x_{1i}]}{[x_1, x_2, \ldots, x_n, x]} = \frac{\alpha_1\alpha_i}{1 - 2\alpha_2 - \sum_{t=3}^{n}\alpha_t}, \quad i \geq 3, \tag{3.3.31}$$

$$G_{22} = \frac{[x_1, x_2, \ldots, x_n, x_{22}]}{[x_1, x_2, \ldots, x_n, x]} = \frac{4\alpha_2^2 - 2\alpha_2}{1 - 2\alpha_2 - \sum_{t=3}^{n}\alpha_t}, \tag{3.3.32}$$

$$G_{2i} = \frac{[x_1, x_2, \ldots, x_n, x_{2i}]}{[x_1, x_2, \ldots, x_n, x]} = \frac{2\alpha_2\alpha_i}{1 - 2\alpha_2 - \sum_{t=3}^{n}\alpha_t}, \quad i \geq 3, \tag{3.3.33}$$

$$G_{ij} = \frac{[x_1, x_2, \ldots, x_n, x_{ij}]}{[x_1, x_2, \ldots, x_n, x]} = \frac{\alpha_i\alpha_j - \alpha_j\delta_{ij}}{1 - 2\alpha_2 - \sum_{t=3}^{n}\alpha_t}, \quad i, j \geq 3. \tag{3.3.34}$$

For the Tchebychev form, we have

$$T_1 = \frac{C_{11}^1 + C_{12}^2 + \sum_{t=3}^{n} C_{1t}^t}{n} = \frac{\alpha_1}{n}, \tag{3.3.35}$$

$$T_2 = \frac{C_{21}^1 + C_{22}^2 + \sum_{t=3}^{n} C_{2t}^t}{n} = \frac{2 + 2\alpha_2}{n}, \tag{3.3.36}$$

$$T_i = \frac{C_{i1}^1 + C_{i2}^2 + \sum_{t=3}^{n} C_{it}^t}{n} = \frac{1 + \alpha_i}{n}, \qquad i \geq 3. \tag{3.3.37}$$

We denote $d := 1 - 2\alpha_2 - \sum_{t=3}^{n}\alpha_t$. The matrix of the centroaffine metric G is

$$G = \frac{1}{d}\begin{pmatrix} \alpha_1^2 + 2\alpha_2 & 2\alpha_1\alpha_2 - \alpha_1 & \alpha_1\alpha_3 & \alpha_1\alpha_4 & \ldots & \alpha_1\alpha_n \\ 2\alpha_1\alpha_2 - \alpha_1 & 4\alpha_2^2 - 2\alpha_2 & 2\alpha_2\alpha_3 & 2\alpha_2\alpha_4 & \ldots & 2\alpha_2\alpha_n \\ \alpha_3\alpha_1 & 2\alpha_3\alpha_2 & \alpha_3^2 - \alpha_3 & \alpha_3\alpha_4 & \ldots & \alpha_3\alpha_n \\ \alpha_4\alpha_1 & 2\alpha_4\alpha_2 & \alpha_4\alpha_3 & \alpha_4^2 - \alpha_4 & \ldots & \alpha_4\alpha_n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_n\alpha_1 & 2\alpha_n\alpha_2 & \alpha_n\alpha_3 & \alpha_n\alpha_4 & \ldots & \alpha_n^2 - \alpha_n \end{pmatrix}. \tag{3.3.38}$$

The determinant of G is

$$\begin{aligned} |G| &= \frac{(-1)^{n+1}\left(4\alpha_2^2+\alpha_1^2\right)\left(1-2\alpha_2-\sum_{t=3}^{n}\alpha_t\right)\prod_{t=3}^{n}\alpha_t}{\left(1-2\alpha_2-\sum_{t=3}^{n}\alpha_t\right)^{n}} \\ &= \frac{(-1)^{n+1}\left(4\alpha_2^2+\alpha_1^2\right)\prod_{t=3}^{n}\alpha_t}{\left(1-2\alpha_2-\sum_{t=3}^{n}\alpha_t\right)^{n-1}}. \end{aligned} \tag{3.3.39}$$

The inverse matrix G^{-1} of G is

$$G^{-1}=\begin{pmatrix} \frac{2\alpha_2 d}{4\alpha_2^2+\alpha_1^2} & \frac{-\alpha_1 d}{4\alpha_2^2+\alpha_1^2} & 0 & \dots & 0 \\ \frac{-\alpha_1 d}{4\alpha_2^2+\alpha_1^2} & \frac{-\left(\alpha_1^2+2\alpha_2-2\alpha_2\sum_{t=3}^{n}\alpha_t\right)}{4\alpha_2^2+\alpha_1^2} & -1 & \dots & -1 \\ 0 & -1 & \frac{-\alpha_3-d}{\alpha_3} & \dots & -1 \\ 0 & -1 & -1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & -1 & -1 & \dots & \frac{-\alpha_n-d}{\alpha_n} \end{pmatrix}. \tag{3.3.40}$$

Then for the Tchebychev vector field T, we obtain

$$\begin{aligned} ||T||^2 &= G^{ij}T_iT_j \\ &= \frac{-2\alpha_1^2 d+(2+2\alpha_2)\left(4\alpha_2\sum_{t=3}^{n}\alpha_t-4\alpha_2-(n+1)\alpha_1^2-4(n-1)\alpha_2^2\right)}{n^2\left(4\alpha_2^2+\alpha_1^2\right)} \\ &- \sum_{t=3}^{n}\frac{(1+\alpha_t)(d+(n+1)\alpha_t)}{n^2\alpha_t}. \end{aligned} \tag{3.3.41}$$

From Theorem 1 and Remark 2.2 we have

Theorem 3.1. *The hypersurface*

$$x_{n+1} = \exp\left(\alpha_1 \arctan\frac{x_1}{x_2}\right)\left(x_1^2 + x_2^2\right)^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4} \dots x_n^{\alpha_n}, \qquad \alpha_1^2 + \alpha_2^2 \neq 0;$$

$$2\alpha_2 + \sum_{t=3}^{n} \alpha_t \neq 1;\ \alpha_t \neq 0,\ t = 3, \dots, n$$

is an equiaffine extremal hypersurface when $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ satisfy

$$\frac{-2\alpha_1^2 d + (2 + 2\alpha_2)\left(4\alpha_2 \sum_{t=3}^{n} \alpha_t - 4\alpha_2 - (n+1)\alpha_1^2 - 4(n-1)\alpha_2^2\right)}{n^2\left(4\alpha_2^2 + \alpha_1^2\right)}$$

$$- \sum_{t=3}^{n} \frac{(1+\alpha_t)(d + (n+1)\alpha_t)}{n^2 \alpha_t} = \pm\frac{(n+2)^2}{2n^2}.$$

3.4 Example three

The hypersurface

$$\begin{aligned} x_{n+1} &= \frac{1}{2x_1}\left(x_2^2 + \dots + x_{v-1}^2\right) - \left(\alpha_2 x_2 + \dots + \alpha_{v-1} x_{v-1}\right) \\ &\quad - x_1\left(\alpha_1 \log x_1 + \alpha_v \log x_v + \dots + \alpha_n \log x_n\right), \end{aligned} \tag{3.4.1}$$

$$\alpha_1 + \sum_{t=v}^{n} \alpha_t \neq 0;\ \alpha_t \neq 0,\ t = v, (v+1), \dots, n$$

is centroaffine extremal. The centroaffine metric of this hypersurface is flat (cf. (3.4.28)-(3.4.33)) and the coefficients of the cubic form are constant (cf. (3.4.19)-(3.4.27)). With parameter $(u_1, u_2, \dots, u_n)$, the hypersurface (3.4.1) can be written as

$$\begin{aligned} x &= (x_1, x_2, \dots, x_n, x_{n+1}) \\ &= \Bigg(e^{u_1}, e^{u_1} u_2, \dots, e^{u_1} u_{v-1}, e^{u_v}, e^{u_{v+1}}, \dots, \\ &\qquad e^{u_n}, e^{u_1}\left(\frac{1}{2}\sum_{\lambda=2}^{v-1} u_\lambda^2 - \sum_{\mu=1}^{n} \alpha_\mu u_\mu\right)\Bigg). \end{aligned} \tag{3.4.2}$$

Then

$$x_1 = \left(e^{u_1}, e^{u_1}u_2, \ldots, e^{u_1}u_{v-1}, 0, \ldots, 0, e^{u_1}\left(\frac{1}{2}\sum_{\lambda=2}^{v-1} u_\lambda^2 - \sum_{\mu=1}^{n}\alpha_\mu - \alpha_1\right)\right), \quad (3.4.3)$$

$$x_i = (0, 0, \ldots, e^{u_1}, \ldots, 0, 0, \ldots, 0, e^{u_1}(u_i - \alpha_i)), \quad 2 \le i \le v-1, \quad (3.4.4)$$

$$x_k = (0, 0, \ldots, 0, 0, \ldots, e^{u_k}, \ldots, 0, e^{u_1}(-\alpha_k)), \quad v \le k \le n, \quad (3.4.5)$$

$$x_{11} = \left(e^{u_1}, e^{u_1}u_2, \ldots, e^{u_1}u_{v-1}, 0, \ldots, 0, e^{u_1}\left(\frac{1}{2}\sum_{\lambda=2}^{v-1} u_\lambda^2 - \sum_{\mu=1}^{n}\alpha_\mu - 2\alpha_1\right)\right), \quad (3.4.6)$$

$$x_{1i} = (0, 0, \ldots, e^{u_1}, \ldots, 0, 0, \ldots, 0, e^{u_1}(u_i - \alpha_i)), \quad 2 \le i \le v-1, \quad (3.4.7)$$

$$x_{1k} = (0, 0, \ldots, 0, 0, \ldots, 0, e^{u_1}(-\alpha_k)), \quad v \le k \le n, \quad (3.4.8)$$

$$x_{ij} = (0, 0, \ldots, 0, 0, \ldots, 0, e^{u_1}(\delta_{ij})), \quad 2 \le i, j \le v-1, \quad (3.4.9)$$

$$x_{ik} = (0, 0, \ldots, 0, 0, \ldots, 0, 0), \quad 2 \le i \le v-1, v \le k \le n, \quad (3.4.10)$$

$$x_{kl} = (0, 0, \ldots, 0, 0, \ldots, \delta_{kl}e^{u_k}, \ldots, 0, 0), \quad v \le k, l \le n. \quad (3.4.11)$$

Therefore

$$D = [x_1, x_2, \ldots, x_n, x] = \left(\alpha_1 + \sum_{t=v}^{n}\alpha_t\right)e^{vu_1+u_v+\cdots+u_n}, \quad (3.4.12)$$

$$[x_1, x_2, \ldots, x_n, x_{11}] = -\alpha_1 e^{vu_1+u_v+\cdots+u_n}, \quad (3.4.13)$$

$$[x_1, x_2, \ldots, x_n, x_{1i}] = 0, \quad 2 \le i \le v-1, \quad (3.4.14)$$

$$[x_1, x_2, \ldots, x_n, x_{1k}] = -\alpha_k e^{vu_1+u_v+\cdots+u_n}, \quad v \le k \le n, \quad (3.4.15)$$

$$\left[x_1, x_2, \ldots, x_n, x_{ij}\right] = \delta_{ij} e^{vu_1+u_v+\cdots+u_n}, \quad 2 \le i, j \le v-1, \quad (3.4.16)$$

$$[x_1, x_2, \ldots, x_n, x_{ik}] = 0, \quad 2 \le i \le v-1, v \le k \le n, \quad (3.4.17)$$

$$[x_1, x_2, \ldots, x_n, x_{kl}] = \delta_{kl}\alpha_k e^{vu_1+u_v+\cdots+u_n}, \quad v \le k, l \le n. \quad (3.4.18)$$

$$DC_{11}^1 = [x_{11}, x_2, \ldots, x_n, x] = \left(2\alpha_1 + \sum_{t=v}^{n}\alpha_t\right)e^{vu_1+u_v+\cdots+u_n}, \quad (3.4.19)$$

$$\begin{aligned} DC_{1i}^i &= [x_1, \ldots, x_{1i}, \ldots, x_n, x] \\ &= \left(\alpha_1 + \sum_{t=v}^{n}\alpha_t\right)e^{vu_1+u_v+\cdots+u_n}, \quad 2 \le i \le v-1, \end{aligned} \quad (3.4.20)$$

$$DC_{1k}^{k} = [x_1, \ldots, x_{1k}, \ldots, x_n, x] = \alpha_k e^{vu_1+u_v+\cdots+u_n}, \quad v \le k \le n, \tag{3.4.21}$$

$$DC_{i1}^{1} = [x_{i1}, x_2, \ldots, x_n, x] = 0, \quad 2 \le i \le v-1, \tag{3.4.22}$$

$$DC_{ij}^{j} = \left[x_1, \ldots, x_{ij}, \ldots, x_n, x\right] = 0, \quad 2 \le i, j \le v-1, \tag{3.4.23}$$

$$DC_{ik}^{k} = [x_1, \ldots, x_v, \ldots, x_{ik}, \ldots, x_n, x] = 0,$$

$$2 \le i \le v-1, v \le k \le n, \tag{3.4.24}$$

$$DC_{k1}^{1} = [x_{k1}, x_2, \ldots, x_n, x] = \alpha_k e^{vu_1+u_v+\cdots+u_n}, \quad v \le k \le n, \tag{3.4.25}$$

$$DC_{ki}^{i} = \left[x_1, \ldots, x_{ki}, \ldots, x_{v-1}, \ldots, x_n, x\right] = 0,$$

$$2 \le i \le v-1, v \le k \le n, \tag{3.4.26}$$

$$\begin{aligned} DC_{kl}^{l} &= [x_1, \ldots, x_v, \ldots, x_{kl}, \ldots, x_n, x] \\ &= \delta_{kl}\left(\alpha_1 + \sum_{t=v}^{n} \alpha_t - \alpha_k\right) e^{vu_1+u_v+\cdots+u_n}, \quad v \le k, l \le n. \end{aligned} \tag{3.4.27}$$

By the definition of the centroaffine metric, we get

$$G_{11} = \frac{[x_1, x_2, \ldots, x_n, x_{11}]}{[x_1, x_2, \ldots, x_n, x]} = \frac{-\alpha_1}{\alpha_1 + \sum_{t=v}^{n} \alpha_t}, \tag{3.4.28}$$

$$G_{1i} = \frac{[x_1, x_2, \ldots, x_n, x_{1i}]}{[x_1, x_2, \ldots, x_n, x]} = 0, \quad 2 \le i \le v-1, \tag{3.4.29}$$

$$G_{1k} = \frac{[x_1, x_2, \ldots, x_n, x_{1k}]}{[x_1, x_2, \ldots, x_n, x]} = \frac{-\alpha_k}{\alpha_1 + \sum_{t=v}^{n} \alpha_t}, \quad v \le k \le n, \tag{3.4.30}$$

$$G_{ij} = \frac{[x_1, x_2, \ldots, x_n, x_{ij}]}{[x_1, x_2, \ldots, x_n, x]} = \frac{\delta_{ij}}{\alpha_1 + \sum_{t=v}^{n} \alpha_t}, \quad 2 \le i, j \le v-1, \tag{3.4.31}$$

$$G_{ik} = \frac{[x_1, x_2, \ldots, x_n, x_{ik}]}{[x_1, x_2, \ldots, x_n, x]} = 0, \quad 2 \le i \le v-1, v \le k \le n, \tag{3.4.32}$$

$$G_{kl} = \frac{[x_1, x_2, \ldots, x_n, x_{kl}]}{[x_1, x_2, \ldots, x_n, x]} = \frac{\delta_{kl}\alpha_k}{\alpha_1 + \sum_{t=v}^{n} \alpha_t}, \quad v \le k, l \le n. \tag{3.4.33}$$

For the Tchebychev form, we have

$$T_1 = \frac{C_{11}^1 + \sum_{j=2}^{v-1} C_{1j}^j + \sum_{l=v}^{n} C_{1l}^l}{n} = \frac{v}{n}, \tag{3.4.34}$$

$$T_i = \frac{C_{i1}^1 + \sum_{j=2}^{v-1} C_{ij}^j + \sum_{l=v}^{n} C_{il}^l}{n} = 0, \quad 2 \le i \le v-1, \tag{3.4.35}$$

$$T_k = \frac{C_{k1}^1 + \sum_{j=2}^{v-1} C_{kj}^j + \sum_{l=v}^{n} C_{kl}^l}{n} = \frac{1}{n}, \quad v \le k \le n. \tag{3.4.36}$$

The matrix of the centroaffine metric G is

$$G = \frac{1}{\alpha_1 + \sum_{t=v}^{n} \alpha_t} \begin{pmatrix} -\alpha_1 & 0 & 0 & \ldots & 0 & -\alpha_v & -\alpha_{v+1} & \ldots & -\alpha_n \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ -\alpha_v & 0 & 0 & \ldots & 0 & \alpha_v & 0 & \ldots & 0 \\ -\alpha_{v+1} & 0 & 0 & \ldots & 0 & 0 & \alpha_{v+1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -\alpha_n & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \alpha_n \end{pmatrix}. \tag{3.4.37}$$

The determinant of G is

$$\begin{aligned} |G| &= \frac{-\left(\alpha_1 + \sum_{t=v}^{n} \alpha_t\right) \Pi_{t=v}^{n} \alpha_t}{\left(\alpha_1 + \sum_{t=v}^{n} \alpha_t\right)^n} \\ &= \frac{-\Pi_{t=v}^{n} \alpha_t}{\left(\alpha_1 + \sum_{t=v}^{n} \alpha_t\right)^{n-1}}. \end{aligned} \tag{3.4.38}$$

The inverse matrix G^{-1} of G is

$$G^{-1} = \begin{pmatrix} -1 & 0 & \dots & 0 & -1 & \dots & -1 \\ 0 & \alpha_1 + \sum_{t=v}^{n} \alpha_t & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_1 + \sum_{t=v}^{n} \alpha_t & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & \frac{\alpha_1 + \sum_{t=v}^{n} \alpha_t - \alpha_v}{\alpha_v} & \dots & -1 \\ -1 & 0 & \dots & 0 & -1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & \dots & 0 & -1 & \dots & \frac{\alpha_1 + \sum_{t=v}^{n} \alpha_t - \alpha_n}{\alpha_n} \end{pmatrix}. \qquad (3.4.39)$$

Then for the Tchebychev vector field T, we obtain

$$||T||^2 = G^{ij} T_i T_j = \frac{1}{n^2} \left(\left(\alpha_1 + \sum_{t=v}^{n} \alpha_t \right) \sum_{t=v}^{n} \frac{1}{\alpha_t} - (n+1)^2 \right). \qquad (3.4.40)$$

From Theorem 1 and Remark 2.2 we have

Theorem 3.1. *The hypersurface*

$$\begin{aligned} x_{n+1} &= \frac{1}{2x_1}\big(x_2^2 + \cdots + x_{v-1}^2\big) - \big(\alpha_2 x_2 + \cdots + \alpha_{v-1} x_{v-1}\big) \\ &\quad - x_1\big(\alpha_1 \log x_1 + \alpha_v \log x_v + \cdots + \alpha_n \log x_n\big), \\ &\quad \alpha_1 + \sum_{t=v}^{n} \alpha_t \neq 0;\ \alpha_t \neq 0,\ t = v, (v+1), \dots, n \end{aligned}$$

is an equiaffine extremal hypersurface when $\alpha_1, \alpha_v, \alpha_{v+1}, \dots, \alpha_n$ *satisfy*

$$\frac{1}{n^2} \left(\left(\alpha_1 + \sum_{t=v}^{n} \alpha_t \right) \sum_{t=v}^{n} \frac{1}{\alpha_t} - (n+1)^2 \right) = \pm \frac{(n+2)^2}{2n^2}.$$

4 Remark and problem.

Remark. From Theorem 3.2, Theorem 3.3 and Theorem 3.4, at least locally, we can easily get the examples of equiaffine extremal hypersurfaces with definite (or indefinite) equiaffine metric.

Problem. Classify all hypersurfaces which are both equiaffine extremal and centroaffine extremal.

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Huili Liu
Department of Mathematics
Northeastern University
Shenyang 110004
P.R. CHINA

E-mail: liuhl@mail.neu.edu.cn

Seoung Dal Jung
Department of Mathematics
Cheju National University
Jeju 690-756
REPUBLIC OF KOREA

E-mail: sdjung@cheju.ac.kr