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On p-nilpotency of finite groups*

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Abstract. Let $\mathcal{F}$ be a class of groups. A subgroup H of a group G is called $\mathcal{F}$-s-supplemented in G, if there exists a subgroup K of G such that $G = HK$ and $K/K \cap H_G$ belongs to $\mathcal{F}$ where H_G is the maximal normal subgroup of G which is contained in H. The main purpose of this paper is to study some subgroups of Fitting subgroup and generalized Fitting subgroup $\mathcal{F}$-s-supplemented and some new criterions of p-nilpotency of finite groups are obtained.

Keywords: $\mathcal{F}$-s-supplement; Fitting subgroup; generalized Fitting subgroup; p-nilpotent group.

Mathematical subject classification: 20D10, 20D15, 20D20.

1 Introduction

The primary subgroups has been studied extensively by many scholars in determining the structure of finite groups. For instance, Kramer [5] shows that a finite solvable group G is supersolvable if and only if, for every maximal subgroup M of G, either $F(G) \leq M$, the Fitting subgroup of G, or $M \cap F(G)$ is a maximal subgroup of $F(G)$. Buckley [2] proved that a group G of odd order is supersolvable if all minimal subgroups of G are normal in G. A. Ballester-Bolinches, Wang and Guo [1] introduced the concept of c-supplementation of a finite group and generalized Buckley's Theorem by replacing normality with c-supplementation. Recently, Wang, Wei and Li [9] extended further the results to a saturated formation containing the class of supersolvable groups by limiting the c-supplementation of maximal or minimal subgroups to the Fitting subgroup of a solvable group. More recently, Miao and Guo [6] propose the new concept of $\mathcal{F}$-s-supplemented subgroup and obtain some new criterions of supersolvability and p-nilpotency. In this paper, we continue to investigate the structure

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of p-nilpotent groups using $\mathcal{F}$-s-supplementation of some subgroups of Fitting subgroups and generalized Fitting subgroups.

Definition 1.1. *Let $\mathcal{F}$ be a class of groups. A subgroup H of G is called $\mathcal{F}$-s-supplemented in G, if there exists a subgroup K of G such that $G = HK$ and $K/K \cap H_G \in \mathcal{F}$ where H_G is the maximal normal subgroup of G which is contained in H. In this case, K is called an $\mathcal{F}$-s-supplement of H in G. Particularly, we say that H is p-nilpotent s-supplemented in G, if there exists a subgroup K of G such that $G = HK$ and $K/K \cap H_G$ is p-nilpotent.*

2 Preliminaries

All groups considered in this paper are finite. Most of the notation is standard and can be found in [3] and [7].

We denote by $F(G)$ the Fitting subgroups G; $F^*(G)$ denotes the generalized Fitting subgroup of G; $O_p(G)$ is the maximal normal p-subgroup of G; $\Phi(G)$ is the intersection of all maximal subgroup of G; $|G|$ denotes the order of a group G; $M < \cdot\, G$ denotes M is a maximal subgroup of group G.

Let π be a set of primes. Then we say that the group $G \in E_\pi$ if G has a $Hall$ π-subgroup. We also say that $G \in C_\pi$ if $G \in E_\pi$ and any two $Hall$ π-subgroups of G are conjugate in G. Moreover, we say that $G \in D_\pi$ if $G \in C_\pi$ and every π-subgroup of G is contained in a $Hall$ π-subgroup of G.

Let $\mathcal{F}$ be a class of groups. $\mathcal{F}$ is called Q-closed if $G/N \in \mathcal{F}$ for all normal subgroups N of G whenever $G \in \mathcal{F}$. $\mathcal{F}$ is called S-closed if every subgroup K of G belongs to $\mathcal{F}$ whenever $G \in \mathcal{F}$. $\mathcal{F}$ is said to be a formation if $\mathcal{F}$ is closed under homomorphic image and subdirect product. It is clear that for a formation $\mathcal{F}$, every group G has a smallest normal subgroup (denoted by $G^{\mathcal{F}}$) whose quotient $G/G^{\mathcal{F}}$ is in $\mathcal{F}$. The normal subgroup $G^{\mathcal{F}}$ is called the $\mathcal{F}$-residual of G. A formation $\mathcal{F}$ is said to be saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. It is well known that the class of all supersolvable groups is a saturated formation. (cf. [8])

Lemma 2.1 [6, Lemma 2.1]. *Let $\mathcal{F}$ be a Q-closed and S-closed class of groups and H a subgroup of G. Then the following statements hold.*

(1) *If K is an $\mathcal{F}$-s-supplement of H in G, and $N \trianglelefteq G$, then KN/N is an $\mathcal{F}$-s-supplement of HN/N in G/N.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$. If K/N is an $\mathcal{F}$-s-supplement of H/N in G/N, then K is an $\mathcal{F}$-s-supplement of H in G.*

(3) *If $H \leq D \leq G$ and K is an $\mathcal{F}$-s-supplement of H in G, then $K \cap D$ is an $\mathcal{F}$-s-supplement of H in D.*

Lemma 2.2 [9, Lemma 2.4]. *Let N be a nontrivial solvable normal subgroup of a group G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which is contained in N.*

Lemma 2.3 [4]. *Let G be a group and N a subgroup of G. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Then*

(1) *If N is normal in G, then $F^*(N) \leq F^*(G)$;*

(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G)$;*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$;*

(4) *$C_G(F^*(G)) \leq F(G)$;*

(5) *Let $P \trianglelefteq G$ and $P \leq O_p(G)$; then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$;*

(6) *If K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

Lemma 2.4 [6, Corollary 3.2]. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is p-nilpotent s-supplement in G, then $G/O_p(G)$ is soluble p-nilpotent.*

Lemma 2.5 [9, Lemma 2.8]. *Let M be a maximal subgroup of G, P a normal p-subgroup of G such that $G = PM$, where p a prime. Then*

(1) *$P \cap M$ is a normal subgroup of G.*

(2) *If $p > 2$ and all minimal subgroups of P are normal in G, then M has index p in G.*

Lemma 2.6. *Let G be a group and p a prime dividing the order of G such that $(|G|, p-1) = 1$. Suppose M is a subgroup of G with $|G : M| = p$. Then M is normal in G.*

Proof. We may assume that $M_G = 1$ by Induction. It is trivial that the lemma holds when $p = 2$. So assume that $p > 2$. Then G is solvable by the Odd Order Theorem. Let N be a minimal normal subgroup of G. Then N is elementary abelian and $G = MN$. This implies that $M \cap N$ is normal in G, hence $M \cap N = 1$. Therefore $|N| = [G : M] = p$. Since $|Aut(N)| = p - 1$ and $G/C_G(N)$ is isomorphic to a subgroup of $Aut(N)$, $|G/C_G(N)|$ must divide $(|G|, p-1) = 1$. So $N \leq Z(G)$. Therefore M is normal in G.

3 Main results

Theorem 3.1. *Let G be a group and N a solvable normal subgroup of G such that G/N is p-nilpotent where p is a prime divisor of $|G|$. Then G is p-nilpotent if and only if every maximal subgroups of the Sylow subgroups of $F(N)$ is p-nilpotent s-supplemented in G.*

Proof. The necessity part is obvious. We only need to prove the sufficiency part. Assume that the assertion is false and choose G to be a counterexample of smallest order. Then

1) $N \cap \Phi(G) = 1$.

If $N \cap \Phi(G) > 1$,then there exists a minimal normal subgroup R of G which is contained in $N \cap \Phi(G)$. Since N is solvable, we have R is an elementary abelian r-group for some prime r and hence $R \leq F(N)$. Now we shall prove G/R satisfies the hypotheses of the theorem. In fact, $N/R \trianglelefteq G/R$ and $(G/R)/(N/R) \cong G/N$ is p-nilpotent. Let L/R be a maximal subgroup of the Sylow r-subgroup of $F(N/R) = F(N)/R$. Then L is a maximal subgroup of the Sylow r-subgroup of $F(N)$. By the hypotheses of the theorem, L is p-nilpotent s-supplemented in G. By Lemma 2.1, L/R is also p-nilpotent s-supplemented in G/R. Set Q_1/R be a maximal subgroup of the Sylow q-subgroup of $F(N/R) = F(N)/R$, where $q \neq r$. It is clear that $Q_1 = Q_1^* R$, where Q_1^* is a maximal subgroup of Sylow q-subgroup of $F(N)$. By the hypotheses, Q_1^* is p-nilpotent s-supplemented in G. Hence $Q_1^* R/R$ is p-nilpotent s-supplemented in G/R by Lemma 2.1. The minimal choice of G implies that G/R is p-nilpotent. Since $G/\Phi(G) \cong (G/R)/(\Phi(G)/R)$ and the class of all p-nilpotent groups is a saturated formation, it follows that G is p-nilpotent, a contradiction.

2) $F(N) = R_1 \times R_2 \times \cdots \times R_m$, where all $R_i (i = 1.2.\cdots m)$ are minimal normal subgroups of G of prime order.

By 1) and Lemma 2.2, $F(N) = R_1 \times R_2 \times \cdots \times R_m$ where all $R_i (i = 1.2. \dots m)$ are minimal normal subgroups of G which are contained in N. For each i ($i = 1.2. \dots m$), there exists a maximal subgroup M_i of G with $G = R_i M_i$ and $R_i \cap M_i = 1$. Furthermore, $F(N) = F(N) \cap R_i M_i = R_i(F(N) \cap M_i)$. Next we will prove that $F(N) \cap M_i$ is a maximal subgroup of $F(N)$.

Actually, since $F(N) \not\leq M_i$, there at least exists a prime q of $\pi(|N|)$ with $O_q(N) \not\leq M_i$. Then $G = O_q(N)M$ as $O_q(N) \trianglelefteq G$. Let M_q be a Sylow q-subgroup of M_i. Then we know that $G_q = O_q(N)M_q$ is a Sylow q-subgroup of G. Now, let Q_1 be a maximal subgroup of G_q containing M_q and set $Q_2 = Q_1 \cap O_q(N)$. Then $Q_1 = Q_2 M_q$. Moreover, $Q_2 \cap M_q = O_q(N) \cap M_q$, so

$$|O_q(N) : Q_2| = |O_q(N)M_q : Q_2 M_q| = |G_q : Q_1| = q,$$

that is, Q_2 is a maximal subgroup of $O_q(N)$. Hence $Q_2(O_q(N) \cap M_i)$ is a subgroup of $O_q(N)$. By the maximality of Q_2 in $O_q(N)$, we have $Q_2(O_q(N) \cap M_i) = Q_2$ or $O_q(N)$.

a) If $Q_2(O_q(N) \cap M_i) = O_q(N)$, then $G = O_q(N)M_i = Q_2 M_i$. Notice that $O_q(N) \cap M_i = Q_2 \cap M_i$. So $O_q(N) = Q_2$, a contradiction.

b) $Q_2(O_q(N) \cap M_i) = Q_2$, that is, $O_q(N) \cap M_i \leq Q_2$. By Lemma 2.5, $O_q(N) \cap M_i \trianglelefteq G$, so $O_q(N) \cap M_i \leq (Q_2)_G$. On the other hand, since Q_2 is p-nilpotent s-supplemented in G, then there exists a subgroup H of G such that $Q_2 H = G$ and $H/H \cap (Q_2)_G$ is p-nilpotent. Set $K = (Q_2)_G H$, then $G = Q_2 K$ and

$$K/K \cap (Q_2)_G = K/(Q_2)_G = (Q_2)_G H/(Q_2)_G \cong H/H \cap (Q_2)_G$$

is p-nilpotent.

Now, we consider the following cases.

Case 1: $K < G$**.** Suppose that K_1 is a maximal subgroup of G containing K. Then $O_q(N) \cap K_1 \trianglelefteq G$ by Lemma 2.5, which implies that $(O_q(N) \cap K_1)M_i$ is a subgroup of G. If $(O_q(N) \cap K_1)M_i = G = O_q(N)M$, then $O_q(N) \cap K_1 = O_q(N)$ since $(O_q(N) \cap K_1) \cap M_i = O_q(N) \cap M_i$. This implies that $O_q(N) \leq K_1$, and hence $G = O_q(N)K_1 = K_1$, which is contrary to the above hypotheses on K_1. Thus $(O_q(N) \cap K_1)M_i = M_i$, $O_q(N) \cap K_1 \leq M_i$. Furthermore, $Q_2 \cap K \leq O_q(N) \cap K \leq O_q(N) \cap M_i \leq (Q_2)_G \leq Q_2 \cap K$, that is, $O_q(N) \cap K = O_q(N) \cap M_i = Q_2 \cap K$. This is contrary to $G = Q_2 K = O_q(N)K$.

Case 2: $K = G$**.** In this case, if $p \neq q$, then we are easy to have G is p-nilpotent, a contradiction. So we may assume that $p = q$. Furthermore, if $(Q_2)_G = 1$, then we have G is p-nilpotent, a contradiction. Set $(Q_2)_G \neq 1$. Thus $(Q_2)_G M_i = M_i$ or $(Q_2)_G M_i = G$. If $(Q_2)_G M_i = M_i$, that is $(Q_2)_G \leq M_i$, then $(Q_2)_G \leq O_q(N) \cap M_i \leq (Q_2)_G$. Therefore $O_q(N) \cap M_i = (Q_2)_G$. By hypotheses, $G/(Q_2)_G$ is p-nilpotent and hence $|G/(Q_2)_G : M_i/(Q_2)_G| = |G : M_i| = |F(N)M_i : M_i| = |F(N) : F(N) \cap M_i| = p$. This means that $F(N) \cap M_i$ is a maximal subgroup of $F(N)$. If $(Q_2)_G M_i = G$, then $G = (Q_2)_G M_i = O_q(N)M_i = Q_2 M_i$. Note that $O_q(N) \cap M_i = Q_2 \cap M_i$, so $O_q(N) = Q_2$, a contradiction.

Therefore $F(N) \cap M_i$ is maximal in $F(N)$ and hence $F(N) \cap M_i$ has prime index in $F(N)$ since $F(N)$ is nilpotent. Observe that $R_i \cap M_i = 1$, so R_i has prime order for $i = 1.2 \cdots m$.

3) $G/F(N)$ is p-nilpotent.

Since $G/C_G(R_i)$ is isomorphic to a subgroup of $Aut(R_i)$, $G/C_G(R_i)$ is cyclic and hence $G/C_G(R_i)$ is p-nilpotent for each i. This implies that $G/\cap_{i=1}^m C_G(R_i)$ is p-nilpotent. Again, $C_G(F(N)) = \cap_{i=1}^m C_G(R_i)$, so we have $G/C_G(F(N)$ is also p-nilpotent. Since both $G/C_G(F(N))$ and G/N are all p-nilpotent, we have $G/N \cap C_G(F(N)) = G/C_N(F(N))$ is p-nilpotent. Since $F(N)$ is abelian, $F(N) \leq C_N(F(N))$. On the other hand, $C_N(F(N)) \leq F(N)$ as N is solvable. Thus $F(N) = C_N(F(N))$ and hence $G/F(N)$ is p-nilpotent.

4) Final contradiction.

For each i, we shall prove G/R_i satisfies the condition of the theorem. Actually, $(G/R_i)/(F(N)/R_i) \cong G/F(N)$ is p-nilpotent. With the similar discussion of 1), we see that G/R_i satisfies the hypotheses of the theorem. The minimal choice of G implies that G/R_i is p-nilpotent and hence $G/\cap_{i=1}^m R_i$ is p-nilpotent. This indicates that G is p-nilpotent if $m > 1$, a contradiction. So we have $F(N) = R_1$. If $|R_1| \neq p$, then G/R_1 is p-nilpotent implies that G is p-nilpotent, a contradiction. If $|R_1| = p$, then the maximal subgroup of R_1 is identity group. Clearly, in this case G is p-nilpotent by the definition of the p-nilpotent s-supplemented subgroup, a contradiction.

The final contradiction completes our proof.

Corollary 3.2. *Let G be a group and p a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if there exists a solvable normal subgroup N with G/N is p-nilpotent and every maximal subgroups of the Sylow subgroups of $F(N)$ is p-nilpotent s-supplemented in G.*

Theorem 3.3. *Let G be a group and p a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Suppose that there exists a normal subgroup H with G/H is p-nilpotent. Then G is p-nilpotent if and only if every maximal subgroups of the Sylow subgroups of $F^*(H)$ is p-nilpotent s-supplemented in G.*

Proof. The necessity part is obvious. We only need to prove the sufficiency part. By Corollary 3.2, we only need to prove H is solvable. Suppose that the claim is false and choose G to be a counterexample of minimal order.

If $p > 2$, then G is solvable and hence H is solvable. So we may assume that $p = 2$. Then

1) $G = H$ and $F^*(H) = F^*(G) = F(G)$.

By Lemma 2.4, $F^*(H)$ is solvable. By Lemma 2.3, we have $F^*(H) = F(H) \neq 1$. Since H satisfies the hypothesis of the theorem, the minimal choice of G implies that H is p-nilpotent if $H < G$. In this case, H is p-nilpotent implies that H is solvable, a contradiction.

2) For any proper normal subgroup N of G containing $F^*(G)$, N is p-nilpotent.

By Lemma 2.3, $F^*(G) = F^*(F^*(G)) \leq F^*(N) \leq F^*(G)$, so $F^*(N) = F^*(G)$. And all maximal subgroups of all Sylow subgroup of $F^*(N)$ are p-nilpotent s-supplemented in G and hence in N by Lemma 2.1. Therefore N is p-nilpotent by the choice of G.

3) $\Phi(G) < F(G)$.

If it is not so, then $\Phi(G) = F(G)$. Let $O_q(G)$ be a Sylow q-subgroup of $F(G)$ where q is a prime divisor of $|F(G)|$ and Q_1 is a maximal subgroup of $O_q(G)$. By hypotheses, Q_1 is p-nilpotent s-supplemented in G. Then there exists a subgroup K of G such that $G = Q_1K$ and $K/K \cap (Q_1)_G$ is p-nilpotent. Clearly, $Q_1 \leq \Phi(G)$ and so $K = G$, we have G is p-nilpotent since the class of all p-nilpotent groups is a saturated formation, a contradiction.

4) G is solvable.

Since $\Phi(G) < F(G)$, there exists some $O_q(G)$ and a maximal subgroup M of G such that $O_q(G) \nleq M$ and $G = O_q(G)M$. Set Q_1 be a maximal subgroup of $O_q(G)$. If $q > 2$, Q_1 is p-nilpotent s-supplemented in G, then there exists a subgroup K_1 of G such that $G = Q_1K_1$ and $K_1/K_1 \cap (Q_1)_G$ is p-nilpotent. It follows that K_1 is solvable. Furthermore, $G = O_q(G)K_1$ and $G/O_q(G) \cong K_1/K_1 \cap O_q(G)$ is solvable. Therefore G is solvable, a contradiction. If $q = 2 = p$, with the similar argument, we have G is solvable.

The final contradiction completes our proof.

Corollary 3.4. *Let G be a finite group and p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every maximal subgroups of the Sylow subgroups of $F^*(G)$ is p-nilpotent s-supplemented in G.*

Theorem 3.5. *Let p be the prime divisor of $|G|$ such that $G \in C_{p'}$ and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if every maximal subgroup of P is p-nilpotent s-supplemented in G.*

Proof. The necessity part is obvious and we omit the proof.

For the sufficiency part, let P_1 be a maximal subgroup of P and of course, $P_1 \neq 1$. Otherwise, we may easy obtain G is p-nilpotent. By our hypotheses, we see that there exists a subgroup M of G such that $G = P_1M$ and $M/M \cap (P_1)_G$ is p-nilpotent. It follows that $P = P_1(P \cap M)$ and $P \cap M$ is a Sylow p-subgroup of M is a Sylow p-subgroup of M. It is clear that $|(P \cap M)/(P_1 \cap M)| = p$. Now $M/M \cap (P_1)_G$ is p-nilpotent. Let $H/M \cap (P_1)_G$ be the normal Hall p'-subgroup of $M/M \cap (P_1)_G$. Then, we have $H \trianglelefteq M$ and $M \cap (P_1)_G$ is a normal Sylow p-subgroup of H. Also by the well-known Schur-Zassenhaus theorem, there exists a Hall p'-subgroup K of H. Obviously, K is also a Hall p'-subgroup of G.

By using the usual Frattini argument, we get that $M = HN_M(K) = (M \cap (P_1)_G)N_M(K)$ and hence it follows that $G = P_1N_G(K)$. Therefore, $N_P(K)$ is a Sylow p-subgroup of $N_G(K)$. If $|G : N_G(K)| = |P : N_P(K)| \geqslant p$, then we may let P_2 be a maximal subgroup of P such that $N_P(K) \leq P_2$. By repeating the above arguments once again, we can also obtain a subgroup M_1 of G such that $G = P_2M_1$, $M_1/M_1 \cap (P_2)_G$ is p-nilpotent and $M_1 = (M_1 \cap (P_2)_G)N_{M_1}(K_1)$, where K_1 is a Hall p'-subgroup of G. By hypotheses, there exists $g \in P$ such that $K_1{}^g = K$ and consequently $N_G(K_1)^g = N_G(K)$. Observe that P_2 is normal in P and $G = P_2N_G(K_1)$, we have $G = P_2N_G(K_1) = (P_2N_G(K_1))^g = P_2N_G(K)$. It follows that $P = P_2(P \cap N_G(K)) = P_2$, a contradiction. Thus, we obtain $|G : N_G(K)| = |P : N_P(K)| = 1$ and hence K is normal in G. This means that G is p-nilpotent.

Corollary 3.6. *Let p be the smallest prime divisor of the order of G and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if every maximal subgroup of P is p-nilpotent s-supplemented in G.*

Theorem 3.7. *Let p be the prime divisor of $|G|$ such that $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every second maximal subgroups of Sylow p-subgroup P of G is p-nilpotent s-supplemented in G.*

Proof. The necessity part is obvious and we omit the proof.

If P is a cyclic group, then we know that G is p-nilpotent by [7, Theorem 10.1.9] since $(|G|, p-1) = 1$. On the other hand, if P is not cyclic , by hypotheses $|P| \geqslant p^2$, then we may let P_2 be a second maximal subgroup of P and of course, $P_2 \neq 1$. By our hypotheses, we see that there exists a subgroup M of G such that $G = P_2M$ and $M/M \cap (P_2)_G$ is p-nilpotent. It follows that $P = P \cap G = P \cap P_2M = P_2(P \cap M)$ and $P \cap M$ is a Sylow p-subgroup of M. It is clear that $|P \cap M/P_2 \cap M| = p^2$. Since $M/M \cap (P_2)_G$ is p-nilpotent, we may let $H/M \cap (P_2)_G$ be the normal Hall p'-subgroup of $M/M \cap (P_2)_G$. Then, we have $H \trianglelefteq M$ and $M \cap (P_2)_G$ is a normal Sylow p-subgroup of H. Also by the well-known Schur-Zassenhaus theorem, there exists a Hall p'-subgroup K of H. Obviously, K is also a Hall p'-subgroup of G.

By using the usual Frattini argument, we get that $M = HN_M(K) = (M \cap (P_2)_G N_M(K)$ and hence it follows that $G = P_2N_G(K)$. Therefore, $N_P(K)$ is a Sylow p-subgroup of $N_G(K)$. If $|G : N_G(K)| = |P : N_P(K)| \geqslant p^2$, then we may let P_3 be a maximal subgroup of P_1 and P_1 a maximal subgroup of P such that $N_P(K) \leq P_3$. By repeating the above arguments once again, we can also obtain a subgroup M_1 of G such that $G = P_3M_1$, $M_1/M_1 \cap (P_2)_G$ is p-nilpotent and $M_1 = (M_1 \cap (P_3)_G)N_{M_1}(K_1)$, where K_1 is a Hall p'-subgroup of G. By the hypotheses and Lemma 2.4, we see that G is solvable and hence there exists $g \in P$ such that ${K_1}^g = K$ and consequently $N_G(K_1)^g = N_G(K)$. Observe that P_1 is normal in P and $G = P_3N_G(K_1) = P_1N_G(K_1)$, we have $G = P_1N_G(K_1) = (P_1N_G(K_1))^g = P_1N_G(K)$. It follows that $P = P_1(P \cap N_G(K)) = P_1$, a contradiction. Thus, we obtain

$$|G : N_G(K)| = |P : N_P(K)| = p \quad \text{or} \quad |G : N_G(K)| = |P : N_P(K)| = 1.$$

If $|G : N_G(K)| = |P : N_P(K)| = p$, we have $N_G(K) \trianglelefteq G$ since $(|G|, p-1) = 1$ by Lemma 2.6. It follows from K char $N_G(K) \trianglelefteq G$ that K is normal in G, contrary to $|G : N_G(K)| = p$. Therefore, $N_G(K) = G$, this means that G is p-nilpotent.

Corollary 3.8. *Let p be the smallest prime divisor of the order of G and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if every second maximal subgroup of P (if exist) is p-nilpotent s-supplemented in G.*

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References

[1] A. Ballester-Bolinches, Y. Wang and X. Guo, *C-supplemented subgroups of finite groups*. Glasgow Math. J., **42** (2000), 383–389.

[2] J. Buckley, *Finite groups of whose minimal subgroups are normal*. Math. Z., **116** (1970), 15–27.

[3] W. Guo, "The Theory of Classes of Groups". Science Press-Kluwer Academic Publishers, Beijing-New York-Dordrecht-Boston-London, 2000.

[4] B. Huppert, "Endliche Gruppen I". Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[5] O. Kramer, *Über Durchschmitte von untergruppen endlicher auflösbarer gruppen.* Math. Z., **148** (1976), 89–97.

[6] L. Miao and W. Guo, *Finite groups with some primary subgroups $\mathcal{F}$-s-supplemented*. Comm. Algebra, **33**(8) (2005), 2789–2800.

[7] D.J. Robinson, "A Course in the Theory of Groups". Springer-Verlag, New York-Heidelberg-Berlin, 1982.

[8] L.A. Shemetkov, *Formations of Finite Groups*. Nauka, Moscow, 1978.

[9] Y. Wang, H. Wei and Y. Li, *A generalization of Kramer's theorem and its applications*. Bull. Austral. Math. Soc., **65** (2002), 467–475.

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