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Effects of partial slip on the steady Von Kármán flow and heat transfer of a non-Newtonian fluid

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Abstract. The steady Von Kármán flow and heat transfer of a non-Newtonian fluid is extended to the case where the disk surface admits partial slip. The constitutive equation of the non-Newtonian fluid is modeled by that for a Reiner-Rivlin fluid. The momentum equations give rise to highly nonlinear boundary value problem. Numerical solutions for the governing nonlinear equations are obtained over the entire range of the physical parameters. The effects of slip and non-Newtonian fluid characteristics on the velocity and temperature fields have been discussed in detail and shown graphically.

Keywords: Reiner-Rivlin fluid, rotating disk, partial slip, heat transfer, finite difference method.

Mathematical subject classification: 76A05, 76A10, 76M20.

1 Introduction

Von Kármán [24] considered the steady flow of a viscous incompressible fluid due to a rotating disk. He solved the equations of motion by an approximate integral method devised by him and Pohlhausen [14]. There are a few minor inaccuracies in Kármán's analysis which were corrected by Cochran [7]. The later was able to give an exact numerical solution, a remarkable feat at that time. Stuart [23] studied the effects of uniform suction on the flow due to a rotating disk. The most accurate solution so far seems to have been reported by Ackroyed [1]. The classical problem of the flow due to a rotating disk has been generalized in several manners to include diverse physical effects. The heat transfer aspects have been considered by Millsaps and Pohlhausen [11] for variety of Prandtl numbers in the steady state. Sparrow and Gregg [20] studied the steady state heat transfer from a rotating disk maintained at a constant temperature to fluids

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at any Prandtl number. Later, Attia [3] has studied the heat transfer of a viscous fluid near a rotating disk considering different thermal conditions.

In all of the above studies the fluid is assumed to be Newtonian. Many materials such as polymer solutions or melts, drilling mud, clastomers, certain oils and greases and many other emulsions are classified as non-Newtonian fluids. For these kind of fluids, the commonly accepted assumption of a linear relationship between the stress and the rate of strain does not hold. Most of the fluids used in industries are non-Newtonian fluids. The non-Newtonian fluids have been modeled by constitutive equations which vary greatly in complexity. The non-Newtonian fluid considered in the present paper is that for which the stress tensor τ^i_j is related to the rate of strain tensor e^i_j as

$$\tau^i_j = -p\delta^i_j + 2\mu e^i_j + 2\mu_c e^i_k e^k_j, \quad e^j_j = 0 \tag{1}$$

where p is denoting the pressure, μ is the coefficient of viscosity and μ_c is the coefficient of cross viscosity. This model was introduced by Reiner [16] to describe the behavior of wet sand but was at one time considered as a possible model for non-Newtonian fluid behavior [17, 22]. However, the model does not account for the possibility of both normal stress differences [9] or shear-thinning or shear-thickening. One can refer the recent works [6, 21] in which the authors have thoroughly discussed about the Reiner-Rivlin fluid. The Von Kármán flow of different kind of non-Newtonian fluids have been studied by various authors [8, 12] including diverse physical effects. A detailed discussion up to 1991 regarding the flow of non-Newtonian fluids due to rotating disks can be found in the review paper by Rajagopal [15]. Recently Attia [4, 5] has studied the steady and unsteady Von Kármán flow and heat transfer of Reiner-Rivlin fluid with suction or injection at the surface of the disk.

In all the above mentioned studies, no attention has been given to the effect of partial slip on the flow due to a rotating disk. The no-slip boundary condition (the assumption that a liquid adheres to a solid boundary) is one of the central tenets of the Navier-Stokes theory. However there are situations wherein this condition does not hold. The inadequacy of the no-slip condition is evident for most non-Newtonian fluids. For example, polymer melts often exhibit macroscopic wall slip and that in general governed by a nonlinear and monotone relation between the slip velocity and the traction. This may be an important factor in shear skin, spurt and hysteresis effects. Also the fluids that exhibit boundary slip have important technological applications such as in the polishing of artificial heart valves and internal cavities. Navier [13] first proposed the equivalent partial slip condition for rough surfaces, relating the tangential velocity u to the local tangential shear stress τ

$$u = N\tau$$

where N is a slip coefficient to be determined by experiments. The roughness may not be statistically isotropic. For example, it was found that for parallel, grooved surfaces the slip is larger in the direction along the grooves than the direction transverse to the grooves [25]. The very recent work of Miklavčič and Wang [10] takes into consideration of the influence of partial slip on the flow of a viscous fluid due to a rotating disk. They have discussed the existence proof and obtained the solution numerically.

It seems that no attempt is available in the literature which describes the influence of partial slip on the flow and heat transfer of a non-Newtonian fluid due to a rotating disk. Keeping this in mind, we study the influence of partial slip on the flow and heat transfer of a non-Newtonian Reiner-Rivlin fluid due to a rotating disk. The resulting system of highly nonlinear differential equations for the velocity and temperature field are solved by a second order finite difference method.

2 Formulation of the problem

We consider a non-Newtonian Reiner-Rivlin fluid whose rheological behavior is governed by stress-strain rate law (1), occupying the space $z > 0$ over an infinite rotating disk coinciding with the plane $z = 0$. The disk is assumed to be rotating about z-axis with an uniform angular velocity Ω. It is natural to describe the flow in the cylindrical polar coordinates (r, θ, z). In view of the rotational symmetry, $\frac{\partial}{\partial\theta} \equiv 0$. Taking $\mathbf{V} = (u, v, w)$ for the steady flow,the equations of continuity and motion are,

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{2}$$

and

$$\rho\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}\right) = \frac{\partial \tau_r^r}{\partial r} + \frac{\partial \tau_r^z}{\partial z} + \frac{\tau_r^r - \tau_\phi^\phi}{r}, \tag{3}$$

$$\rho\left(u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right) = \frac{\partial \tau_\phi^r}{\partial r} + \frac{\partial \tau_\phi^z}{\partial z} + \frac{2\tau_\phi^r}{r}, \tag{4}$$

$$\rho\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = \frac{\partial \tau_z^r}{\partial r} + \frac{\partial \tau_z^z}{\partial z} + \frac{\tau_z^r}{r} \tag{5}$$

The no-slip boundary conditions for the velocity field are given as

$$z = 0, \quad u = 0, \quad v = r\Omega, \quad w = 0, \tag{6a}$$

$$z \to \infty, \quad u \to 0, \quad v \to 0, p \to p_\infty. \tag{6b}$$

By using the Von Kármán transformations [24]

$$\begin{gathered} u = r\Omega F(\zeta), \quad v = r\Omega G(\zeta), \quad w = \sqrt{\Omega\nu} H(\zeta), \\ z = \sqrt{\frac{\nu}{\Omega}}\zeta, \quad p - p_\infty = -\rho\nu\Omega P \end{gathered} \tag{7}$$

equations (2)-(5) take the form

$$\frac{dH}{d\zeta} + 2F = 0, \tag{8}$$

$$\frac{d^2F}{d\zeta^2} - H\frac{dF}{d\zeta} - F^2 + G^2 - \frac{1}{2}K\left(\left(\frac{dF}{d\zeta}\right)^2 - 3\left(\frac{dG}{d\zeta}\right)^2 - 2F\frac{d^2F}{d\zeta^2}\right) = 0 \tag{9}$$

$$\frac{d^2G}{d\zeta^2} - H\frac{dG}{d\zeta} - 2FG + K\left(\frac{dF}{d\zeta}\frac{dG}{d\zeta} + F\frac{d^2G}{d\zeta^2}\right) = 0, \tag{10}$$

$$\frac{d^2H}{d\zeta^2} - H\frac{dH}{d\zeta} - \frac{7}{2}K\frac{dH}{d\zeta}\frac{d^2H}{d\zeta^2} + \frac{dP}{d\zeta} = 0. \tag{11}$$

where ζ is the non-dimensional distance measured along the axis of rotation, F, G, H and P are non-dimensional functions of ζ, ρ is the density and ν is the kinematic viscosity($\nu = \frac{\mu}{\rho}$) of the fluid. The boundary conditions (6) become,

$$\zeta = 0: \quad F = 0, \quad G = 1, \quad H = 0, \tag{12a}$$

$$\zeta \to \infty: \quad F \to 0, \quad G \to 0 \tag{12b}$$

where $K = \frac{\mu_c\Omega}{\mu}$ is the parameter that describes the non-Newtonian characteristic of the fluid. The above system (8)- (10) with the prescribed boundary conditions (12)are sufficient to solve for the three components of the flow velocity. Equation (11) can be used to solve for the pressure distribution at any point.

A generalization of Navier's partial slip condition gives, in the radial direction,

$$u|_{z=0} = N_1\tau_r^z|_{z=0} \tag{13}$$

and in the azimuthal direction

$$v|_{z=0} = N_2\tau_\phi^z|_{z=0} \tag{14}$$

where N_1, N_2 are respectively the slip coefficients. Let

$$\lambda = N_1\sqrt{\frac{\Omega}{\nu}}\mu, \qquad \eta = N_2\sqrt{\frac{\Omega}{\nu}}\mu \tag{15}$$

With the help of transformation (7) and equations (13)- (15), the boundary conditions (12) reduce to

$$\begin{gathered} F(0) = \lambda[F'(0) - KF(0)F'(0)], \\ G(0) - 1 = \eta[G'(0) - 2KF(0)G'(0)], \quad H(0) = 0. \end{gathered} \tag{16a}$$

$$F(\infty) \to 0, \quad G(\infty) \to 0 \tag{16b}$$

The governing equations are still equations (8)- (10). The boundary conditions at infinity are equation (12b) but those on the disk are replaced by equations (16a).

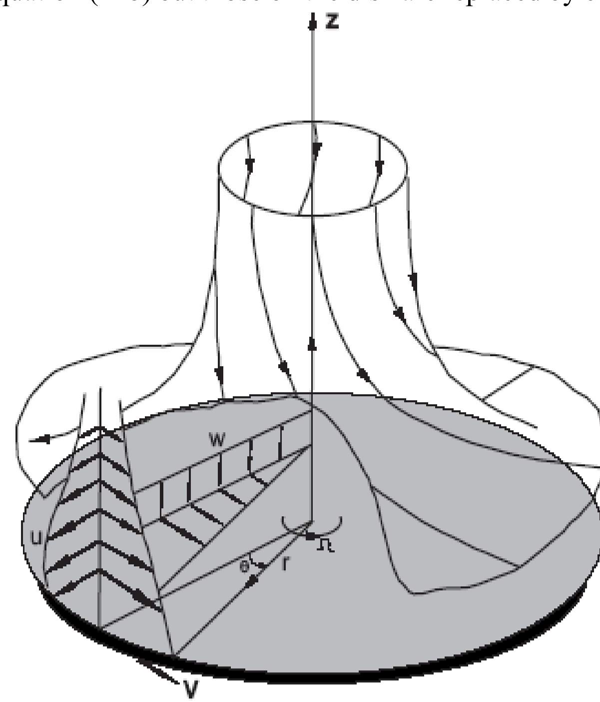

Figure 1: Schematic representation of the flow domain.

2.1 Heat transfer analysis

Due to the temperature difference between the surface of the disk and the ambient fluid, heat transfer takes place. The energy equation, by neglecting the dissipation terms, takes the form,

$$\rho c_p\left(u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}\right)-k\frac{\partial^2 T}{\partial z^2}=0. \tag{17}$$

where c_p is the specific heat at constant pressure and k is the thermal conductivity of the fluid.

Introducing the non-dimensional variable $\theta=\frac{T-T_\infty}{T_w-T_\infty}$ and using the Von Kármán transformations (7), equation (17) becomes,

$$\frac{d^2\theta}{d\zeta^2}-PrH\frac{d\theta}{d\zeta}=0. \tag{18}$$

where T_w is the temperature at the surface of the disk, T_∞ is the temperature of the ambient fluid at large distance from the disk and $Pr=\frac{c_p\mu}{k}$ is the Prandtl number. The boundary conditions in terms of the non-dimensional parameter θ are expressed as

$$\zeta=0:\quad \theta=1;\quad \zeta\to\infty:\quad \theta\to 0. \tag{19}$$

The heat transfer from the disk surface to the fluid is computed by the application of the Fourier's law, $q=-k(\frac{\partial T}{\partial z})_w$. Introducing the transformed variables, the expression for q becomes

$$q=-k(T_w-T_\infty)\sqrt{\frac{\Omega}{\nu}}\frac{d\theta(0)}{d\zeta}. \tag{20}$$

By rephrasing the heat transfer results in terms of the Nusselt number defined as $N_u=\frac{q\sqrt{\frac{\nu}{\Omega}}}{k(T_w-T_\infty)}$, we get

$$N_u=-\frac{d\theta(0)}{d\zeta}. \tag{21}$$

The action of the viscosity in the fluid adjacent to the disk tends to set up tangential shear stress $\overline{\tau}_\varphi$, which opposes the rotation of the disk. There is also a surface shear stress $\overline{\tau}_r$ in the radial direction. In terms of the variables of the analysis, the expressions of $\overline{\tau}_\varphi$ and $\overline{\tau}_r$ are respectively given as

$$\begin{aligned}\frac{\overline{\tau}_\varphi}{\rho r\sqrt{\nu\Omega^3}}&=\tau_\varphi=\frac{dG(0)}{d\zeta}-2KF(0)\frac{dG(0)}{d\zeta};\\ \frac{\overline{\tau}_r}{\rho r\sqrt{\nu\Omega^3}}&=\tau_r=\frac{dF(0)}{d\zeta}-KF(0)\frac{dF(0)}{d\zeta}.\end{aligned} \tag{22}$$

3 Numerical solution of the problem

The system of non-linear differential equations (8)- (10)and (18) is solved under the boundary conditions (16) and (19), respectively. One can see that the initial boundary conditions for F and G in (16a) are unknown contrary to the case of no-slip boundary conditions (12a). Hence, the solution of the system can not proceed numerically using any standard integration routine. Here we have adopted a second order numerical technique which combines the features of the finite difference method and the shooting method. The method is accurate because it uses central differences. A finite value, ζ_∞, large enough, has been substituted for ∞, the numerical infinity to ensure that the solutions are not affected by imposing the asymptotic conditions at a finite distance. The value of ζ_∞ has been kept invariant during the run of the program.

Now suppose we introduce a mesh defined by

$$\zeta_i = ih \ (i = 0, 1, \dots n), \tag{23}$$

h being the mesh size, and discretize equations (8)- (10) and (18) using the central difference approximations for the derivatives, then the following equations are obtained.

$$\begin{aligned} \frac{F_{i+1} - 2F_i + F_{i-1}}{h^2} &- H_i\left(\frac{F_{i+1} - F_{i-1}}{2h}\right) - F_i^2 + G_i^2 \\ &- \frac{1}{2}K\left[\left(\frac{F_{i+1} - F_{i-1}}{2h}\right)^2 - 3\left(\frac{G_{i+1} - G_{i-1}}{2h}\right)^2\right. \\ &\left. - 2F_i\left(\frac{F_{i+1} - 2F_i + F_{i-1}}{h^2}\right)\right] = 0 \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{G_{i+1} - 2G_i + G_{i-1}}{h^2} &- H_i\left(\frac{G_{i+1} - G_{i-1}}{2h}\right) - 2F_iG_i \\ &+ K\left[\left(\frac{F_{i+1} - F_{i-1}}{2h}\right)\left(\frac{G_{i+1} - G_{i-1}}{2h}\right)\right. \\ &\left. + F_i\left(\frac{G_{i+1} - 2G_i + G_{i-1}}{h^2}\right)\right] = 0 \end{aligned} \tag{25}$$

$$\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} - Pr H_i\left(\frac{\theta_{i+1} - \theta_{i-1}}{2h}\right) = 0 \tag{26}$$

$$H_{i+1} = H_i - h(F_i + F_{i+1}) \tag{27}$$

Note that equations (9), (10) and (18), which are written at jth mesh point, the first and second derivatives are approximated by the central differences centered at jth mesh point, while in equation (8), which is written at $(j+\frac{1}{2})th$ mesh point, the first derivative is approximated by the difference quotient at jth and $(j+1)th$ mesh points, and the right hand sides are approximated by the respective averages at the same two mesh points. This scheme ensures that the accuracy of $O(h^2)$ is preserved in the discretization.

Equations (24), (25) and (26) are three term recurrence relations in F, G and θ respectively. Hence, in order to start the recursion, besides the values of F_0, G_0 and θ_0, the values of F_1, G_1 and θ_1 are also required. These values can be obtained by Taylor series expansion near $\zeta = 0$ for F, G and θ.

If

$$F'(0) = s_1, \quad G'(0) = s_2 \quad \text{and} \quad \theta'(0) = s_3 \tag{28}$$

we have

$$\begin{aligned} F_1 &= F(0) + hF'(0) + \frac{h^2}{2}F''(0) + O(h^3) \\ G_1 &= G(0) + hG'(0) + \frac{h^2}{2}G''(0) + O(h^3) \\ \theta_1 &= \theta(0) + h\theta'(0) + \frac{h^2}{2}\theta''(0) + O(h^3) \end{aligned} \tag{29}$$

The values of $F(0)$, $G(0)$ and $\theta(0)$ are given as boundary conditions in (16) and (19). The values $F''(0)$, $G''(0)$ and $\theta''(0)$ can be obtained directly from (9), (10) and (18) and using the values in (28). After obtaining the values of F_1, G_1 and θ_1, the integration can now be performed as follows. H_1 can be obtained from (27). Using the values of H_1 in (24), (25) and (26), the values of F_2, G_2 and θ_2 are obtained. At the next cycle, H_2 is computed from (27) and is used in equations (24), (25) and (26) to obtain F_3, G_3 and θ_3 respectively. The order indicated above is followed for the subsequent cycles. The integration is carried out until the values of F, G, H and θ are obtained at all the mesh points.

Note that we need to satisfy the three asymptotic boundary conditions (16) and (19). In fact s_1, s_2 and s_3 must be found by shooting method so as to fulfil the free boundary condition at ζ_∞ in (16) and (19). We have adopted Newton's method as our zero finding algorithm. The fact that the algorithm has an accuracy of only $O(h^2)$ need not concern us unduly as we can easily hike the accuracy to $O(h^4)$ by invoking Richardson's extrapolation. With reasonably close trial values to start the iterations, the convergence to the actual values within an accuracy of $O(10^{-6})$ could be attained in 9-11 iterations.

4 Results and discussion

The method described above was translated into a FORTRAN 90 program and was run on a Pentium IV personal computer. The value of ζ_∞, the numerical infinity has been kept invariant through out the run of the program. To see if the program runs correctly, the results of H_∞, N_u, τ_r and τ_φ for no-slip condition *i.e* for $\lambda = \eta = 0$ are compared with (Table 1) those reported by Attia [4] at selected values of K (and for no suction/injection). The comparison is found to be in good agreement. In order to have insight of the flow and heat transfer characteristics, results are plotted graphically in Figs.(2)-(9) for uniform roughness ($\lambda = \eta$) and different choice of the non-Newtonian parameter K.

Figs. 2 and 3 depict the variations of the radial component of velocity $F(\zeta)$ as a function of ζ for different values of the slip parameter $\lambda(= \eta)$ and the non-Newtonian parameter K respectively. It is clear that as the slip parameter increases in magnitude, permitting more fluid to slip past the disk, the maximum radial velocity decreases and its location moves towards the disk. Moreover, it is observed that the effect of slip decreases $F(\zeta)$ near the disk and increases far away from it. This results in a cross over of the radial velocity profile. The effect of K is opposite to that of λ on the flow.

In Figs. 4 and 5 we plot the dimensionless azimuthal component of velocity $G(\zeta)$ as a function of ζ with $\lambda(= \eta)$ and K respectively. Its value in general decreases as slip is increased and increases with an increase in the value of K. It is found that the slip has a prominent effect on $G(\zeta)$ near the disk. Figs. 6 and 7 show the axial velocity profile $-H(\zeta)$. Slip decreases the axial component throughout the interval and has a prominent effect far away from the disk. The axial component of the velocity increases with an increase in K.

We plot the dimensionless temperature profile $\theta(\zeta)$, as shown in Figs. 8 and 9 for various values of $\lambda(= \eta)$ and K. Clearly the slip increases the value of $\theta(\zeta)$, whereas the non-Newtonian parameter K shows an opposite effect on the temperature profile.

	$K = 0$		$K = 2$	
	Previous result [4]	Current study	Previous result [4]	Current study
H_∞	0.8752	0.875211	0.5056	0.505601
N_u	1.1402	1.140213	0.8398	0.839762
τ_r	0.5104	0.510421	0.1617	0.161703
τ_φ	0.6154	0.615376	0.4879	0.487883

Table 1: Variation of some standard parameters with K for $\lambda = \eta = 0$.

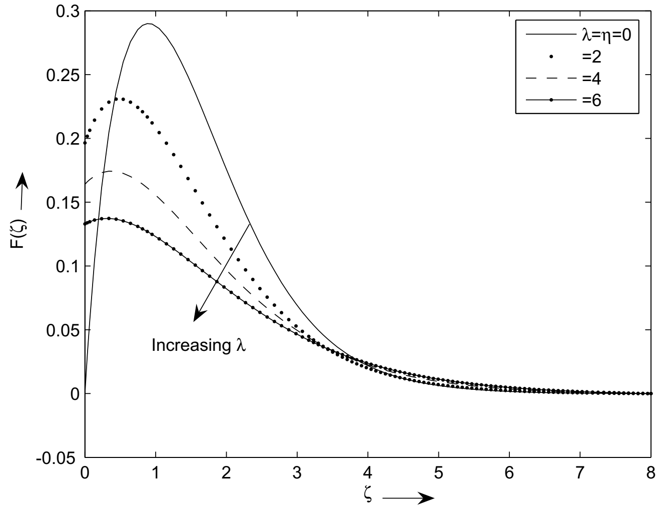

Figure 2: Variation of F with $\lambda(=\eta)$ at $K = 4$.

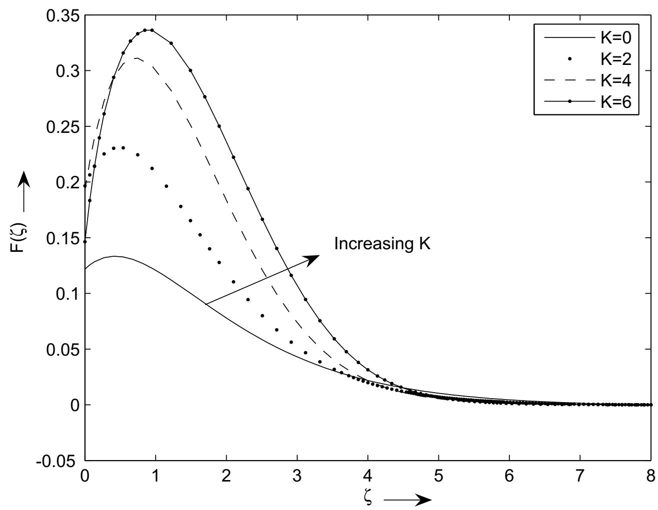

Figure 3: Variation of F with K at $\lambda = \eta = 2$.

5 Conclusions

This work presents the effects of the partial slip on the steady flow and heat transfer of a non-Newtonian Reiner-Rivlin fluid due to a rotating disk. The constitutive equation of the fluid gives rise to momentum equations which, when transformed using the similarity variables, reduce to highly nonlinear system of **BVP**. The new set of slip flow boundary conditions (16) aimed to accommodate

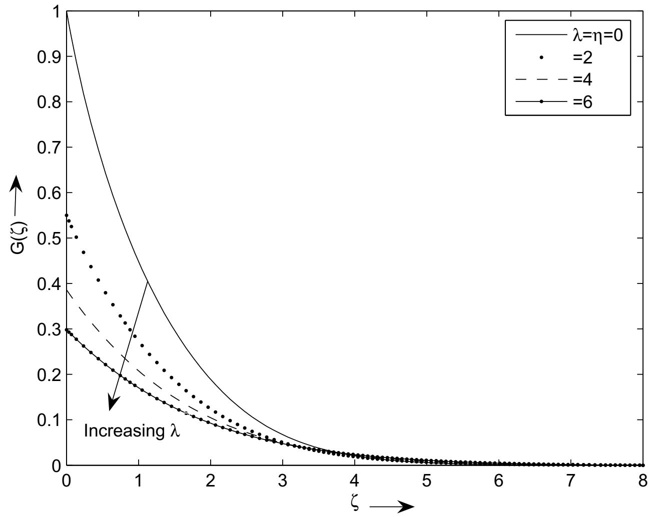

Figure 4: Variation of G with $\lambda(=\eta)$ at $K = 4$.

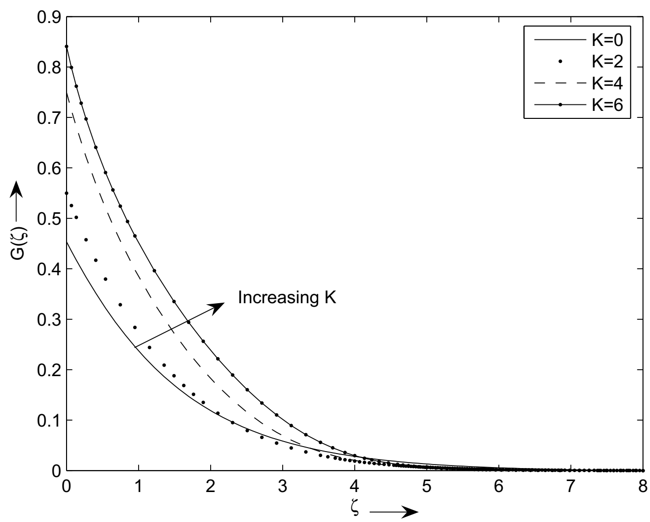

Figure 5: Variation of G with K at $\lambda = \eta = 2$.

for the partial slip effect. A second order numerical scheme, which is a combination of the shooting technique and the finite difference method, has been adopted to solve the resulting system of equations.

The effects of slip and non-Newtonian fluid parameter K on the velocity and temperature distributions have been discussed in detail. The flows have boundary layer character. As the non-Newtonian fluid parameter K, increases in magni-

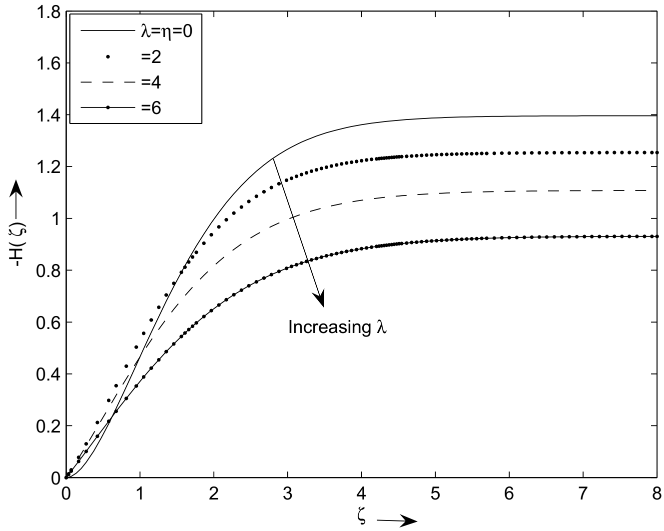

Figure 6: Variation of H with $\lambda(=\eta)$ at $K = 4$.

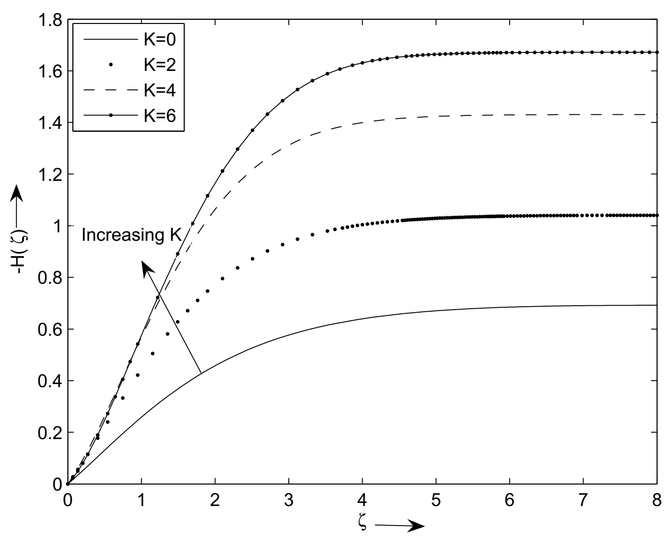

Figure 7: Variation of H with K at $\lambda = \eta = 2$.

tude the flow gets accelerated, whereas increasing K decreases the heat transfer rate throughout the domain of integration, as a result of favoring the incoming flow at near-ambient temperature towards the disk.

The slip increases with the slip factor λ. It is readily seen that λ has a substantial effect on the solution. As the slip parameter increases in magnitude, permitting

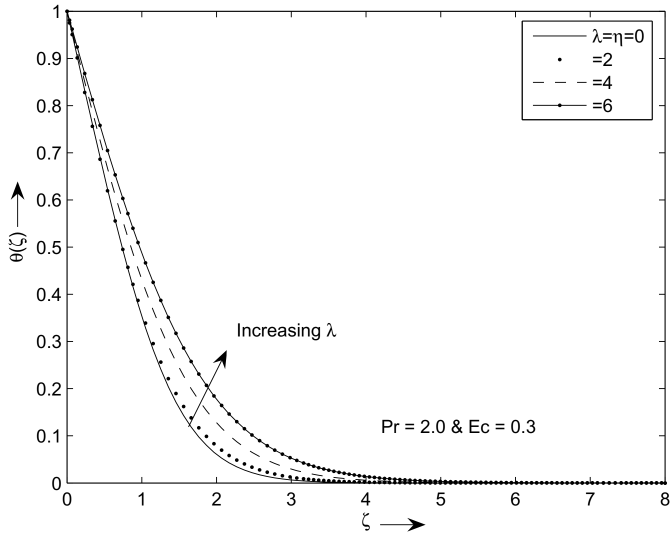

Figure 8: Variation of θ with $\lambda(=\eta)$ at $K = 4$.

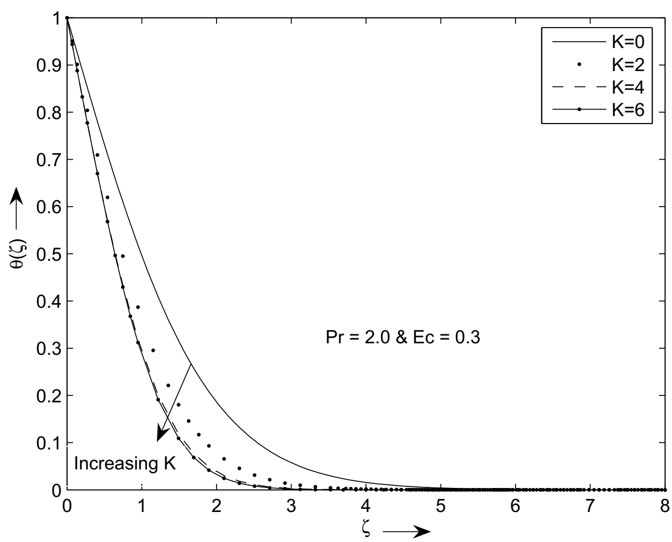

Figure 9: Variation of θ with K at $\lambda = \eta = 2$.

more fluid to slip past the disk surface, the flow slows down for distances close to the disk, or in other words, the boundary layer thickness [2, 18, 19] turns out to be increasing function of λ. The gradual reduction of the peak of the $F(\zeta)$ profiles in Fig. 2 with increasing values of λ is reflected in the distributions of the axial velocity component $-H(\zeta)$ in Fig. 6. This is a consequence of the direct coupling between the radial and the axial velocity components through the continuity constraint (8). The reduction of the axial velocity with increasing λ automatically gives rise to a reduced axial inflow, which in turn becomes the cause for the increase in the heat transfer for all values of ζ. Thus, it is observed that the effects of slip is opposite to that of the non-Newtonian fluid parameter K on the flow and heat transfer due to a rotating disk.

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