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Approximate C^*-ternary ring homomorphisms

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Abstract. In this paper, we establish the generalized Hyers–Ulam–Rassias stability of C^*-ternary ring homomorphisms associated to the Trif functional equation

$$d \cdot C_{d-2}^{l-2} f\left(\frac{x_1 + \cdots + x_d}{d}\right) + C_{d-2}^{l-1} \sum_{j=1}^{d} f(x_j) = l \cdot \sum_{1 \le j_1 < \cdots < j_l \le d} f\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right).$$

Keywords: generalized Hyers-Ulam-Rassias stability, C^*-ternary ring, C^*-ternary homomorphism, Trif's functional equation.

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1 Introduction and preliminaries

A *ternary ring of operators* (TRO) is a closed subspace of the space $B(\mathcal{H}, \mathcal{K})$ of bounded linear operators between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ which is closed under the ternary product $[xyz] := xy^*z$. This concept was introduced by Hestenes [8]. The class of TRO's includes Hilbert C^*-modules via the ternary product $[xyz] := \langle x, y\rangle z$. It is remarkable that every TRO is isometrically isomorphic to a corner $p\mathcal{A}(1-p)$ of a C^*-algebra $\mathcal{A}$, where p is a projection. A closely related structure to TRO's is the so-called JC^*-triple that is a norm closed subspace of $B(\mathcal{H})$ being closed under the triple product $[xyz] = (xy^*z + zy^*x)/2$; cf. [7]. It is also true that a commutative TRO, i.e. a TRO with the property $xy^*z = zy^*x$, is an associative JC^*-triple.

Following [25] a *C*-ternary ring* is defined to be a Banach space $\mathcal{A}$ with a ternary product $(x, y, z) \mapsto [xyz]$ from $\mathcal{A}$ into $\mathcal{A}$ which is linear in the outer variables, conjugate linear in the middle variable, and associative in the sense that $[xy[zts]] = [x[tzy]s] = [[xyz]ts]$, and satisfies $\|[xyz]\| \le \|x\|\|y\|\|z\|$ and $\|[xxx]\| = \|x\|^3$. For instance, any TRO is a C*-ternary ring under the ternary

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product $[xyz] = xy^*z$. A linear mapping φ between C^*-ternary rings is called a *homomorphism* if $\varphi([xyz]) = [\varphi(x)\varphi(y)\varphi(z)]$ for all $x, y, z \in \mathcal{A}$.

The stability problem of functional equations originated from a question of Ulam [24], posed in 1940, concerning the stability of group homomorphisms. In the next year, Hyers [9] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1950, Aoki [2] and in 1978, Th. M. Rassias [20] extended the theorem of Hyers by considering the unbounded Cauchy difference

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p),$$

where $\varepsilon > 0$ and $p \in [0, 1)$ are constants. The result of Th.M. Rassias has provided a lot of influence in the development of what we now call the *Hyers–Ulam–Rassias stability* of functional equations. In 1994, a generalization of Rassias' result, the so-called generalized Hyers–Ulam–Rassias stability, was obtained by Găvruta [6] by following the same approach as in [20]. During the last decades several stability problems of functional equations have been investigated in the spirit of Hyers–Ulam–Rassias-Găvruta. See [5, 10, 12, 21, 14] and references therein for more detailed information on stability of functional equations.

As far as the author knows, [4] is the first paper dealing with stability of (ring) homomorphisms. Another related result is that of Johnson [11] in which he introduced the notion of almost algebra $*$-homomorphism between two Banach $*$-algebras. In fact, so many interesting results on the stability of homomorphisms have been obtained by many mathematicians; see [22] for a comprehensive account on the subject. In [3] the stability of homomorphisms between J^*-algebras associated to the Cauchy equation $f(x+y) = f(x) + f(y)$ was investigated. Some results on stability ternary homomorphisms may be found at [1, 16].

Trif [23] proved the generalized stability for the so-called Trif functional equation

$$d \cdot C_{d-2}^{l-2} f\left(\frac{x_1 + \cdots + x_d}{d}\right) + C_{d-2}^{l-1} \sum_{j=1}^{d} f(x_j) = l \cdot \sum_{1 \leq j_1 < \cdots < j_l \leq d} f\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right),$$

deriving from an inequality of Popoviciu [19] for convex functions (here, C_r^k denotes $\frac{r!}{k!(r-k)!}$). Hou and Park [17] applied the result of Trif to study $*$-homomorphisms between unital C^*-algebras. Further, Park investigated the stability of Poisson JC^*-algebra homomorphisms associated with Trif's equation (see [18]).

In this paper, using some strategies from [3, 13, 17, 18, 23], we establish the generalized Hyers–Ulam–Rassias stability of C^*-ternary homomorphisms associated to the Trif functional equation. If a C^*-ternary ring $(\mathcal{A}, [\ ])$ has an identity, i.e. an element e such that $x = [xee] = [eex]$ for all $x \in \mathcal{A}$, then it is easy to verify that $x \odot y := [xey]$ and $x^* := [exe]$ make $\mathcal{A}$ into a unital C^*-algebra (due to the fact that $\|x \odot x^* \odot x\| = \|x\|^3$). Conversely, if $(A, \odot)$ is a (unital) C^*-algebra, then $[xyz] := x \odot y^* \odot z$ makes $\mathcal{A}$ into a C^*-ternary ring (with the unit e such that $x \odot y = [xey]$) (see [15]). Thus our approach may be applied to investigate of stability of homomorphisms between unital C^*-algebras.

Throughout this paper, $\mathcal{A}$ and $\mathcal{B}$ denote C^*-ternary rings. In addition, let $q = \frac{l(d-1)}{d-l}$ and $r = -\frac{l}{d-l}$ for positive integers l, d with $2 \le l \le d-1$. By an *approximate C^*-ternary ring homomorphism associated to the Trif equation* we mean a mapping $f : \mathcal{A} \to \mathcal{B}$ for which there exists a certain control function $\varphi : \mathcal{A}^{d+3} \to [0, \infty)$ such that if

$$\begin{aligned} &D_\mu f(x_1, \ldots, x_d, u, v, w) \\ &= \left\| d \cdot C_{d-2}^{l-2} f\left(\frac{\mu x_1 + \cdots + \mu x_d}{d} + \frac{[uvw]}{d \cdot C_{d-2}^{l-2}}\right) + C_{d-2}^{l-1} \sum_{j=1}^{d} \mu f(x_j) \right. \\ &\qquad \left. - l \cdot \sum_{1 \le j_1 < \cdots < j_l \le d} \mu f\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right) - \big[f(u)f(v)f(w)\big] \right\|. \end{aligned}$$

then

$$D_\mu f(x_1, \cdots, x_d, u, v, w) \le \varphi(x_1, \cdots, x_d, u, v, w), \tag{1}$$

for all scalars μ in a subset $\mathbb{E}$ of $\mathbb{C}$ and all $x_1, \cdots, x_d, u, v, w \in \mathcal{A}$.

It is not hard to see that a function $T : X \to Y$ between linear spaces satisfies Trif's equation if and only if there is an additive mapping $S : X \to Y$ such that $T(x) = S(x) + T(0)$ for all $x \in X$. In fact, $S(x) := (1/2)(T(x) - T(-x))$; see [23].

2 Main Results

In this section, we are going to establish the generalized Hyers–Ulam–Rassias stability of homomorphisms in C^*-ternary rings associated with the Trif functional equation. We start our work with investigating the case in which an approximate C^*-ternary ring homomorphism associated to the Trif equation is an exact homomorphism.

Proposition 2.1. *Let $T : \mathcal{A} \to \mathcal{B}$ be an approximate C^*-ternary ring homomorphism associated to the Trif equation with $\mathbb{E} = \mathbb{C}$ and a control function φ*

satisfying

$$\lim_{n\to\infty} q^{-n}\varphi\left(q^n x_1, \cdots, q^n x_d, q^n u, q^n v, q^n w\right) = 0,$$

for all $x_1, \cdots, x_d, u, v, w \in \mathcal{A}$. *Suppose that* $T(qx) = qT(x)$ *for all* $x \in \mathcal{A}$. *Then* T *is a* C^**-ternary homomorphism.*

Proof. $T(0) = 0$, because $T(0) = qT(0)$ and $q > 1$. We have

$$\begin{aligned} D_1 T(x_1, \cdots, x_d, 0, 0, 0) &= q^{-n} D_1 T(q^n x_1, \cdots, q^n x_d, 0, 0, 0) \\ &\leq q^{-n}\varphi(q^n x_1, \cdots, q^n x_d, 0, 0, 0). \end{aligned}$$

Taking the limit as $n \to \infty$ we conclude that T satisfies Trif's equation. Hence T is additive. It follows from

$$\begin{aligned} D_\mu T(q^n x, \cdots, q^n x, 0, 0, 0) &= q^n \left\| d \cdot C_{d-2}^{l-2}(T(\mu x) - \mu T(x)) \right\| \\ &\leq \varphi(q^n x, \cdots, q^n x, 0, 0, 0), \end{aligned}$$

that T is homogeneous.

Set $x_1 = \cdots = x_d = 0$ and replace u, v, w by $q^n u, q^n v, q^n w$, respectively, in (1). Since T is homogeneous, we have

$$\begin{aligned} \left\| T([uvw]) - [T(u)T(v)T(w)] \right\| &= q^{-3n} \left\| T([q^n u q^n v q^n w]) \right. \\ &\quad \left. - [T(q^n u)T(q^n v)T(q^n w)] \right\| \\ &\leq q^{-n}\varphi(0, \cdots, 0, q^n u, q^n v, q^n w), \end{aligned}$$

for all $u, v, w \in \mathcal{A}$. The right hand side tends to zero as $n \to \infty$. Hence $T([uvw]) = [T(u)T(v)T(w)]$ for all $u, v, w \in \mathcal{A}$. □

Theorem 2.2. *Let* $f : \mathcal{A} \to \mathcal{B}$ *be an approximate* C^**-ternary ring homomorphism associated to the Trif equation with* $\mathbb{E} = \mathbb{T}$ *and a control function* $\varphi : \mathcal{A}^{d+3} \to [0, \infty)$ *satisfying*

$$\widetilde{\varphi}(x_1, \cdots, x_d, u, v, w) := \sum_{j=0}^{\infty} q^{-j}\varphi(q^j x_1, \cdots, q^j x_d, q^j u, q^j v, q^j w) < \infty, \quad (2)$$

for all $x_1, \ldots, x_d, u, v, w \in \mathcal{A}$. *If* $f(0) = 0$*, then there exists a unique* C^**-ternary ring homomorphism* $T : \mathcal{A} \to \mathcal{B}$ *such that*

$$\|f(x) - T(x)\| \leq \frac{1}{l \cdot C_{d-1}^{l-1}} \widetilde{\varphi}(qx, rx, \ldots, rx, 0, 0, 0),$$

for all $x \in \mathcal{A}$.

Proof. Set $u = v = w = 0$, $\mu = 1$ and replace $x_1, \ldots, x_d$ by $qx, rx, \ldots, rx$ in (1). Then

$$\|C_{d-2}^{l-1} f(qx) - l \cdot C_{d-1}^{l-1} f(x)\| \le \varphi(qx, rx, \cdots, rx, 0, 0, 0) \quad (x \in \mathcal{A}).$$

One can use induction to show that

$$\begin{aligned} &\left\| q^{-n} f(q^n x) - q^{-m} f(q^m x) \right\| \\ &\le \frac{1}{l \cdot C_{d-1}^{l-1}} \sum_{j=m}^{n-1} q^{-j} \varphi\left(q^j(qx), q^j(rx), \ldots, q^j(rx), 0, 0, 0\right), \end{aligned} \tag{3}$$

for all nonnegative integers $m < n$ and all $x \in \mathcal{A}$. Hence the sequence $\left\{q^{-n} f(q^n x)\right\}_{n \in \mathbb{N}}$ is Cauchy for all $x \in \mathcal{A}$. Therefore we can define the mapping $T : \mathcal{A} \to \mathcal{B}$ by

$$T(x) := \lim_{n \to \infty} \frac{1}{q^n} f(q^n x) \quad (x \in \mathcal{A}). \tag{4}$$

Since

$$\begin{aligned} D_1 T(x_1, \cdots, x_d, 0, 0, 0) &= \lim_{n \to \infty} q^{-n} D_1 f(q^n x_1, \cdots, q^n x_d, 0, 0, 0) \\ &\le \lim_{n \to \infty} q^{-n} \varphi(q^n x_1, \cdots, q^n x_d, 0, 0, 0) \\ &= 0, \end{aligned}$$

we conclude that T satisfies the Trif equation and so it is additive (note that (4) implies that $T(0) = 0$). It follows from (4) and (3) with $m = 0$ that

$$\|f(x) - T(x)\| \le \frac{1}{l \cdot C_{d-1}^{l-1}} \widetilde{\varphi}(qx, rx, \cdots, rx, 0, 0, 0),$$

for all $x \in \mathcal{A}$.

We use the strategy of [23] to show the uniqueness of T. Let T' be another additive mapping fulfilling

$$\|f(x) - T'(x)\| \le \frac{1}{l \cdot C_{d-1}^{l-1}} \widetilde{\varphi}(qx, rx, \cdots, rx, 0, 0, 0),$$

for all $x \in \mathcal{A}$. We have

$$\begin{aligned} \|T(x) - T'(x)\| &= q^{-n} \|T(q^n x) - T'(q^n x)\| \\ &\le q^{-n} \|T(q^n x) - f(q^n x)\| + q^{-n} \|f(q^n x) - T'(q^n x)\| \\ &\le \frac{2q^{-n}}{l \cdot C_{d-1}^{l-1}} \widetilde{\varphi}\left(q^n(qx), q^n(rx), \cdots, q^n(rx), 0, 0, 0\right) \\ &\le \frac{2}{l \cdot C_{d-1}^{l-1}} \sum_{j=n}^{\infty} q^{-j} \varphi\left(q^j(qx), q^j(rx), \cdots, q^j(rx), 0, 0, 0\right), \end{aligned}$$

for all $x \in \mathcal{A}$. Since the right hand side tends to zero as $n \to \infty$, we deduce that $T(x) = T'(x)$ for all $x \in \mathcal{A}$.

Let $\mu \in \mathbb{T}^1$. Setting $x_1 = \cdots = x_d = x$ and $u = v = w = 0$ in (1) we get

$$\|d \cdot C_{d-2}^{l-2}\big(f(\mu x) - \mu f(x)\big)\| \leq \varphi(x, \cdots, x, 0, 0, 0),$$

for all $x \in \mathcal{A}$. So that

$$q^{-n}\|d \cdot C_{d-2}^{l-2}\big(f(\mu q^n x) - \mu f(q^n x)\big)\| \leq q^{-n}\varphi(q^n x, \cdots, q^n x, 0, 0, 0),$$

for all $x \in \mathcal{A}$. Since the right hand side tends to zero as $n \to \infty$, we have

$$\lim_{n\to\infty} q^{-n}\|f(\mu q^n x) - \mu f(q^n x)\| = 0,$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Hence

$$T(\mu x) = \lim_{n\to\infty} \frac{f(q^n \mu x)}{q^n} = \lim_{n\to\infty} \frac{\mu f(q^n x)}{q^n} = \mu T(x),$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$.

Obviously, $T(0x) = 0 = 0T(x)$. Next, let $\lambda \in \mathbb{C}$ $(\lambda \neq 0)$, and let M be a natural number greater than $|\lambda|$. By an easily geometric argument, one can conclude that there exist two numbers $\mu_1, \mu_2 \in \mathbb{T}$ such that $2\frac{\lambda}{M} = \mu_1 + \mu_2$. By the additivity of T we get $T\left(\frac{1}{2}x\right) = \frac{1}{2}T(x)$ for all $x \in \mathcal{A}$. Therefore

$$\begin{aligned}
T(\lambda x) &= T\left(\frac{M}{2} \cdot 2 \cdot \frac{\lambda}{M}x\right) = MT\left(\frac{1}{2} \cdot 2 \cdot \frac{\lambda}{M}x\right) = \frac{M}{2}T\left(2 \cdot \frac{\lambda}{M}x\right) \\
&= \frac{M}{2}T(\mu_1 x + \mu_2 x) = \frac{M}{2}\left(T(\mu_1 x) + T(\mu_2 x)\right) \\
&= \frac{M}{2}(\mu_1 + \mu_2)T(x) = \frac{M}{2} \cdot 2 \cdot \frac{\lambda}{M} \\
&= \lambda T(x),
\end{aligned}$$

for all $x \in \mathcal{A}$, so that T is a $\mathbb{C}$-linear mapping.

Set $\mu = 1$ and $x_1 = \cdots = x_d = 0$, and replace u, v, w by $q^n u, q^n v, q^n w$, respectively, in (1) to get

$$\begin{aligned}
&\frac{1}{q^{3n}}\left\|d \cdot C_{d-2}^{l-2} f\left(\frac{q^{3n}}{d \cdot C_{d-2}^{l-2}}[uvw]\right) - \big[f(q^n u)f(q^n v)f(q^n w)\big]\right\| \\
&\qquad \leq q^{-3n}\varphi(0, \cdots, 0, q^n u, q^n v, q^n w),
\end{aligned}$$

for all $u, v, w \in \mathcal{A}$. Then by applying the continuity of the ternary product $(x, y, z) \mapsto [xyz]$ we deduce

$$\begin{aligned} T([uvw]) &= d \cdot C_{d-2}^{l-2} T\left(\frac{1}{d \cdot C_{d-2}^{l-2}}[uvw]\right) \\ &= \lim_{n\to\infty} \frac{d \cdot C_{d-2}^{l-2}}{q^{3n}} f\left(\frac{q^{3n}}{d \cdot C_{d-2}^{l-2}}[uvw]\right) \\ &= \lim_{n\to\infty} \left[\frac{f(q^n u)}{q^n} \frac{f(q^n v)}{q^n} \frac{f(q^n w)}{q^n}\right] \\ &= [T(u)T(v)T(w)], \end{aligned}$$

for all $u, v, w \in \mathcal{A}$. Thus T is a C^*-ternary homomorphism. $\square$

Example 2.3. Let $S : \mathcal{A} \to \mathcal{A}$ be a (bounded) C^*-ternary homomorphism, and let $f : \mathcal{A} \to \mathcal{A}$ be defined by

$$f(x) = \begin{cases} S(x) & \|x\| < 1 \\ 0 & \|x\| \geq 1 \end{cases} \qquad \text{and} \qquad \varphi(x_1, \ldots, x_d, u, v, w) := \delta,$$

where $\delta := d \cdot C_{d-2}^{l-2} + d \cdot C_{d-2}^{l-1} + l \cdot C_d^l + 1$. Then

$$\widetilde{\varphi}(x_1, \ldots, x_d, u, v, w) = \sum_{n=0}^{\infty} q^{-n} \cdot \delta = \frac{\delta q}{q-1},$$

and

$$D_\mu f(x_1, \ldots, x_d, u, v, w) \leq \varphi(x_1, \ldots, x_d, u, v, w),$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \ldots, x_d, u, v, w \in \mathcal{A}$. Note also that f is not linear. It follows from Theorem 2 that there is a unique C^*-ternary ring homomorphism $T : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{l \cdot C_{d-1}^{l-1}} \widetilde{\varphi}(qx, rx, \ldots, rx, 0, 0, 0) \qquad (x \in \mathcal{A}).$$

Further, $T(0) = \lim_{n\to\infty} \frac{f(0)}{q^n} = 0$ and for $x \neq 0$ we have

$$T(x) = \lim_{n\to\infty} \frac{f(q^n x)}{q^n} = \lim_{n\to\infty} \frac{0}{q^n} = 0,$$

since for sufficiently large n, $\|q^n x\| \geq 1$. Thus T is identically zero.

Corollary 2.4. *Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$ and there exist constants $\varepsilon \geq 0$ and $p \in [0, 1)$ such that*

$$D_\mu f(x_1, \cdots, x_d, u, v, w) \leq \varepsilon \left(\sum_{j=1}^{d} \|x_j\|^p + \|u\|^p + \|v\|^p + \|w\|^p \right),$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_d, u, v, w \in \mathcal{A}$. Then there exists a unique C^*-ternary ring homomorphism $T : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - T(x)\| \leq \frac{q^{1-p}(q^p + (d-1)r^p)\varepsilon}{l \cdot C_{d-1}^{l-1}(q^{1-p} - 1)} \|x\|^p,$$

for all $x \in \mathcal{A}$.

Proof. Define

$$\varphi(x_1, \cdots, x_d, u, v, w) = \varepsilon \left(\sum_{j=1}^{d} \|x_j\|^p + \|u\|^p + \|v\|^p + \|w\|^p \right),$$

and apply Theorem 2.2. □

The following corollary can be applied in the case that our ternary algebra is linearly generated by its 'idempotents', i.e. elements u with $u^3 = u$.

Proposition 2.5. *Let $\mathcal{A}$ be linearly spanned by a set $S \subseteq \mathcal{A}$ and let $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $f(q^{2n}[s_1 s_2 z]) = [f(q^n s_1) f(q^n s_2) f(z)]$ for all sufficiently large positive integers n, and all $s_1, s_2 \in S, z \in \mathcal{A}$. Suppose that there exists a control function $\varphi : \mathcal{A}^d \to [0, \infty)$ satisfying*

$$\widetilde{\varphi}(x_1, \ldots, x_d) := \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \ldots, q^j x_d) < \infty \quad (x_1, \ldots, x_d \in \mathcal{A}).$$

If $f(0) = 0$ and

$$\Bigg\| d \cdot C_{d-2}^{l-2} f\left(\frac{\mu x_1 + \cdots + \mu x_d}{d}\right) + C_{d-2}^{l-1} \sum_{j=1}^{d} \mu f(x_j) - l \cdot \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu f\left(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\right) \Bigg\| \leq \varphi(x_1, \cdots, x_d),$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_d \in \mathcal{A}$, then there exists a unique C^*-ternary ring homomorphism $T : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{l \cdot C_{d-1}^{l-1}} \widetilde{\varphi}(qx, rx, \ldots, rx),$$

for all $x \in \mathcal{A}$.

Proof. Applying the same argument as in the proof of Theorem 2.2, there exists a unique linear mapping $T : \mathcal{A} \to \mathcal{B}$ given by

$$T(x) := \lim_{n\to\infty} \frac{1}{q^n} f(q^n x) \quad (x \in \mathcal{A})$$

such that

$$\|f(x) - T(x)\| \leq \frac{1}{l \cdot C_{d-1}^{l-1}} \widetilde{\varphi}(qx, rx, \ldots, rx),$$

for all $x \in \mathcal{A}$. We have

$$\begin{aligned} T([s_1 s_2 z]) &= \lim_{n\to\infty} \frac{1}{q^{2n}} f\left(\left[(q^n s_1)(q^n s_2) z\right]\right) \\ &= \lim_{n\to\infty} \left[\frac{f(q^n s_1)}{q^n} \frac{f(q^n s_2)}{q^n} f(z)\right] \\ &= \left[T(s_1) T(s_2) f(z)\right]. \end{aligned}$$

By the linearity of T we have $T([xyz]) = [T(x)T(y)f(z)]$ for all $x, y, z \in \mathcal{A}$. Therefore $q^n T([xyz]) = T([xy(q^n z)]) = [T(x)T(y)f(q^n z)]$, and so

$$\begin{aligned} T[xyz]) &= \lim_{n\to\infty} \frac{1}{q^n} \left[T(x)T(y)f(q^n z)\right] \\ &= \left[T(x)T(y) \lim_{n\to\infty} \frac{f(q^n z)}{q^n}\right] \\ &= \left[T(x)T(y)T(z)\right], \end{aligned}$$

for all $x, y, z \in \mathcal{A}$. □

Theorem 2.6. *Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an approximate C^*-ternary ring homomorphism associated to the Trif equation with $\mathbb{E} = \{1, \mathbf{i}\}$ and a control function $\varphi : A^{d+3} \to [0, \infty)$ fulfilling (2). If $f(0) = 0$ and for each fixed $x \in \mathcal{A}$ the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$, then there exists a unique C^*-ternary homomorphism $T : \mathcal{A} \to \mathcal{B}$ such that*

$$\|f(x) - T(x)\| \leq \widetilde{\varphi}(qx, rx, \ldots, rx, 0, 0, 0),$$

for all $x \in \mathcal{A}$.

Proof. Put $u = v = w = 0$ and $\mu = 1$ in (1). Using the same argument as in the proof of Theorem 2 we deduce that there exists a unique additive mapping $T : \mathcal{A} \to \mathcal{B}$ given by

$$T(x) = \lim_{n\to\infty} \frac{f(q^n x)}{q^n} \quad (x \in \mathcal{A}).$$

By the same reasoning as in the proof of the main theorem of [20], the mapping T is $\mathbb{R}$-linear.

Putting $x_1 = \cdots = x_d = x$, $\mu = \mathbf{i}$ and $u = v = w = 0$ in (1) we get

$$\|d \cdot C_{d-2}^{l-2}(f(\mathbf{i}x) - \mathbf{i}f(x))\| \leq \varphi(x, \ldots, x, 0, 0, 0) \quad (x \in \mathcal{A}).$$

Hence

$$q^{-n}\|f(q^n\mathbf{i}x) - \mathbf{i}f(q^n x)\| \leq q^{-n}\varphi(q^n x, \ldots, q^n x, 0, 0, 0) \quad (x \in \mathcal{A}).$$

The right hand side tends to zero as $n \to \infty$, hence

$$T(\mathbf{i}x) = \lim_{n\to\infty} \frac{f(q^n\mathbf{i}x)}{q^n} = \lim_{n\to\infty} \frac{\mathbf{i}f(q^n x)}{q^n} = \mathbf{i}T(x) \quad (x \in \mathcal{A}).$$

For every $\lambda \in \mathbb{C}$ we can write $\lambda = \alpha_1 + \mathbf{i}\alpha_2$ in which $\alpha_1, \alpha_2 \in \mathbb{R}$. Therefore

$$\begin{aligned} T(\lambda x) &= T(\alpha_1 x + \mathbf{i}\alpha_2 x) = \alpha_1 T(x) + \alpha_2 T(\mathbf{i}x) \\ &= \alpha_1 T(X) + \mathbf{i}\alpha_2 T(x) = (\alpha_1 + \mathbf{i}\alpha_2)T(x) \\ &= \lambda T(x), \end{aligned}$$

for all $x \in \mathcal{A}$. Thus T is $\mathbb{C}$-linear. □

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