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New examples of Cantor sets in S^1 that are not C^1-minimal

Aldo Portela

Abstract. Although every Cantor subset of the circle (S^1) is the minimal set of some homeomorphism of S^1, not every such set is minimal for a C^1 diffeomorphism of S^1. In this work, we construct new examples of Cantor sets in S^1 that are not minimal for any C^1-diffeomorphim of S^1.

Keywords: minimal sets, Cantor sets, C^1 diffeomorphisms.

Mathematical subject classification: 37E10, 37C45.

1 Introduction and main results

To study the dynamics of a homeomorphism $f\colon S^1 \to S^1$ it is important to study the invariant sets for f. We say that a set K is a minimal set for f if it is compact, non empty, invariant and minimal (relative to the inclusion) with regard to the former three properties. Simple examples of minimal sets are the fixed points and the periodic orbits of a homeomorphism, and in general the w-limit (α-limit) of any point. Zorn's lemma implies that every homeomorphism of S^1 has at least one minimal set. If f has periodic points (for example when f does not preserve orientation) then any minimal set is finite. On the other hand, if f does not have periodic points the minimal set is unique, infinite and it is the set of accumulation points of the past orbit and future orbit of any point $x \in S^1$. In the latter case the minimal set is a Cantor set (intransitive case) or all S^1 (transitive case). The following theorem allows us to state that the intransitive case cannot happen when f is a diffeomorphism of class C^2.

Theorem 1.1 (Denjoy). *If f is a diffeomorphism of class C^1 of S^1 without periodic points and with derivate of bounded variation then f is transitive.*

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We can find a proof of this theorem in [1]. In this work, Denjoy also constructs intransitive diffeomorphisms of class C^1 (so called Denjoy's examples). Also there exist examples of intransitive diffeomorphisms of class $C^{1+\alpha}$ for $\alpha < 1$, constructed by Herman in [3]. From the existence of intransitive diffeomorphisms and since any two Cantor sets of S^1 are homeomorphic, it follows that any Cantor set of S^1 is C^0-minimal (i.e. it is minimal for some homeomorphism). This is not true when f is a diffeomorphism of class C^1. It is easy to verify that any finite subset of S^1 is C^1-minimal (i.e. it is minimal for some diffeomorphism of class C^1), but not every Cantor set of S^1 is C^1-minimal. In [2] Mc Duff proved that the usual ternary Cantor set is not C^1-minimal and in [4] Norton proved that the affine Cantor sets are not C^1-minimal.

Let K be a Cantor set of circle and let $K^c = \bigcup I_j$ where I_j are the connected components of K^c. We define the spectrum of K (E_K) as the ordered set $\{\lambda_i\}$ $(\lambda_{i+1} < \lambda_i)$, with λ_i the lengths of I_j for some j. We call *covering of the spectrum of* K to every separate family of closed intervals $\{\mathcal{J}_i = [\alpha_i, \beta_i]\}$ such that $E_K \subset \cup \mathcal{J}_i$ and $\alpha_{i+1} \leq \beta_{i+1} < \alpha_i$. In this condition each connected component I_j of K^c is associated to an integer $n(I_j)$ such that $|I_j| \in \mathcal{J}_{n(I_j)}$. In [2] Mc Duff conjectured that if $\lambda_n/\lambda_{n+1} \not\to 1$ the Cantor set K is not C^1-minimal (all known C^1-minimal Cantor sets satisfy $\lambda_n/\lambda_{n+1} \to 1$).

Definition 1.1. *We say that the Cantor set K satisfies the p-**separation condition** for a covering $\{\mathcal{J}_i\}$ if there exists a non negative integer p such that for any $N > 0$ there exists $\eta(N) > 0$ such that*

$$\frac{\alpha_{j+n-1}}{\beta_{j+p+n}} \geq (1 + \eta(N)) \frac{\beta_j}{\alpha_{j+p}} \tag{1}$$

for any integer n, $|n| \leq N$, and for all j, sufficiently large.

Adapting the techniques used by Mc Duff in [2], we obtain the following result.

Theorem 1.2. *If the Cantor set K satisfies the p-separation condition then the Cantor set K is not C^1-minimal.*

This theorem is a generalization of the following theorem proved by Mc Duff in [2].

Theorem 1.3. *If a Cantor set K satisfies the p-separation condition for $p = 0$ then the Cantor set K is not C^1-minimal.*

We say that a covering $\{\mathcal{J}_i\}$ of the spectrum of K is a ϵ-covering (with $\epsilon > 0$) if $\dfrac{\alpha_j}{\beta_{j+1}} = 1 + \epsilon$, for every j. The other result obtained is the following.

Theorem 1.4. *If $\{\mathcal{J}_i\}$ is a ϵ-covering of the spectrum of a Cantor set K and $\beta_i/\alpha_i = k$ then the Cantor set K is not C^1-minimal.*

Finally, in the last section we give the construction of a Cantor set that satisfies the p-separation condition for $p = 1$, but does not satisfy the condition given by Mc Duff in [2] (this is the p-separation condition for $p = 0$).

2 Proof of Theorems 1.2 and 1.4

The following lemmas will be used in the proof of Theorem 1.2.

Lemma 1. *If the Cantor set K is C^1-minimal and $\{\mathcal{J}_i\}$ is a covering of E_K then $\frac{\alpha_i}{\beta_{i+1}}$ is bounded.*

Proof. We can suppose that any interval of the covering of E_K contains some element of E_K. Let f be a diffeomorphism for which K is C^1-minimal. If I is a connected component of K^c and $\{|f^n(I)| : n \in \mathbf{N}\} = \{\gamma_1, \ldots, \gamma_j, \ldots\}$ with $\gamma_{j+1} < \gamma_j$, we have

$$\frac{\gamma_j}{\gamma_{j+1}} \leq \max\big\{M, 1/m\big\}, \tag{2}$$

where M and m are the maximum and minimum of f' respectively. For every i there exists j_i such that $\gamma_{j_i} \in \mathcal{J}_i$ and $\gamma_{j_i+1} \in \mathcal{J}_{i+1}$. Then

$$\frac{\alpha_i}{\beta_{i+1}} \leq \frac{\gamma_{j_i}}{\gamma_{j_i+1}}\,. \tag{3}$$

Therefore using (2) and (3) we have

$$\frac{\alpha_i}{\beta_{i+1}} \leq \max\big\{M, 1/m\big\}.$$

This ends the proof. □

Lemma 2. *If the Cantor set K is C^1-minimal and satisfies the p-separation condition for $\{\mathcal{J}_i\}$ then $\dfrac{\beta_j}{\alpha_j}$ is bounded.*

Proof. Taking $N = n = 1$ in (1) we have

$$\frac{\alpha_j}{\beta_{j+p+1}} \geq \big(1 + \eta(1)\big)\, \frac{\beta_j}{\alpha_{j+p}}$$

for all j sufficiently large. Then

$$\frac{\beta_j}{\alpha_j} \leq \frac{1}{1 + \eta(1)}\, \frac{\alpha_{j+p}}{\beta_{j+p+1}}.$$

The result follows from the previous lemma. $\square$

It is simple to verify the following properties.

1. If the Cantor set K is C^1-minimal for f, then for every $r > 1$ there exists a finite covering of K formed by disjoint closed intervals T_i such that if x, y belong to a same T_i,

$$\frac{1}{r} \leq \frac{f'(x)}{f'(y)} \leq r.$$

2. If the Cantor set K satisfies the p-separation condition for $\{\mathcal{J}_j\}$ then

$$\frac{\alpha_j}{\beta_{j+1}} \geq 1 + \eta(1)$$

for all j, sufficiently large.

Lemma 3. *If the Cantor set K is C^1-minimal for f and satisfies the p-separation condition then for every component I of K^c, $|n(I) - n(f(I))|$ is bounded.*

Proof. If m and M are the minimum and maximum of f' respectively then $m|I| \leq |f(I)| \leq M|I|$. If $n(f(I)) \geq n(I)$, using property 2 we have

$$\begin{aligned}(1 + \eta(1))^{n(f(I))-n(I)} &\leq \frac{\alpha_{n(I)}}{\beta_{n(I)+1}} \cdot \frac{\alpha_{n(I)+1}}{\beta_{n(I)+2}} \cdots \frac{\alpha_{n(f(I))-1}}{\beta_{n(f(I))}} \\ &\leq \frac{\alpha_{n(I)}}{\beta_{n(f(I))}} \leq \frac{|I|}{|f(I)|} \leq \frac{1}{m}.\end{aligned}$$

If $n(f(I)) < n(I)$ then

$$(1 + \eta(1))^{n(I)-n(f(I))} \leq \frac{\alpha_{n(f(I))}}{\beta_{n(I)}} \leq \frac{|f(I)|}{|I|} \leq M\,.$$

In both cases we conclude that $|n(I) - n(f(I))|$ is bounded. $\square$

2.1 Proof of Theorem 1.2

Proof. Suppose by contradiction that the Cantor set K is C^1-minimal for f and satisfies the p-separation condition for the covering $\{\mathcal{J}_i\}$. From Lemma 3 there exists a non negative integer N_0 such that $|n(I) - n(f(I))| < N_0$ for any connected component I of K^c. Consider a covering of K formed by disjoint open intervals $T_1, \dots, T_s$, such that if x and y belong to a same T_i, then

$$\frac{f'(x)}{f'(y)} < 1 + \frac{\eta(N_0)}{3}. \tag{4}$$

From property 1 we know that such covering exists. Let I and J be two intervals of K^c contained in a same T_i, such that $n(I) - n(J) \le p$ (p is the integer given by the condition of p-separation). We will prove now that $n(f(I)) - n(f(J)) \le p$. Suppose by contradiction that $n(f(J)) < n(f(I)) - p$. Then

$$\frac{|f(J)|}{|f(I)|} \ge \frac{\alpha_{n(f(J))}}{\beta_{n(f(I))}} \ge \frac{\alpha_{n(f(J))}}{\beta_{n(f(J))+p+1}}.$$

Using the p-separation condition and that $|n(J) - n(f(J))| < N_0$, we obtain

$$\frac{|f(J)|}{|f(I)|} \ge (1 + \eta(N_0))\frac{\beta_{n(J)}}{\alpha_{n(J)+p}}.$$

On the other hand, using (4) we obtain

$$\frac{|f(J)|}{|f(I)|} \le \frac{|J|}{|I|}\left(1 + \frac{\eta(N_0)}{3}\right) \le \left(1 + \frac{\eta(N_0)}{3}\right)\frac{\beta_{n(J)}}{\alpha_{n(I)}} \le \left(1 + \frac{\eta(N_0)}{3}\right)\frac{\beta_{n(J)}}{\alpha_{n(J)+p}}$$

and this is a contradiction. Therefore, if I and J are in the same component T_i such that $n(I) - n(J) \le p$ then $n(f(I)) - n(f(J)) \le p$. For each component of the complement of $\cup T_i$ there exists a component of K^c that contains it. Let us denote such components by $L_1, \dots, L_s$. Let I be a component of K^c. As $|f^j(I)| \to 0$ when $j \to \infty$ then there exists j_0 such that for all $j > j_0$,

$$n\big(f^j(I)\big) > p + \max\big\{n(L_i)\colon i = 1, \dots, s\big\}.$$

In these conditions there exists i_0 such that $f^{j_0}(I) = (a_{j_0}, b_{j_0})$ is contained in T_{i_0}. Let c_{j_0} be a point of K contained in T_{i_0} such that $|(c_{j_0}, a_{j_0})| < |f^{j_0}(I)|$. From here, if J is a connected component of K^c contained in (c_{j_0}, a_{j_0}) then

$$n\big(f^{j_0}(I)\big) - n(J) \le p, \quad \text{and} \quad n\big(f^{j_0+1}(I)\big) - n(f(J)) \le p.$$

From the choice of j_0 we have that $n(f(J)) > \max\{n(L_i): i = 1, \dots, s\}$ so $f(J) \neq L_i$ for $i = 1, \dots, s$. This shows that $f^{j_0+1}(I)$ and $f((c_{j_0}, a_{j_0}))$ are in the same T_i. Proceeding inductively we have that for any interval J of K^c contained in (c_{j_0}, a_{j_0}), $f^n(J) \neq L_i$, for all positive integer n and $i = 1, \dots, s$. This is a contradiction because for any interval L_i there exist infinite $n > 0$ such that $f^{-n}(L_i) \subset (c_{j_0}, a_{j_0})$. $\square$

2.2 Proof of Theorem 1.4

Proof. Suppose by contradiction that the Cantor set K is C^1-minimal for a diffeomorphism f.

Claim. There exist connected components, T and I, of K^c such that $|T|$ and $|I|$ belong to the same interval $\mathcal{J}_i$, but $|f(T)|$ and $|f(I)|$ belong to different ones.

Let $\delta > 0$ be as small as necessary. Let $T_1, \dots, T_s$ be as in the proof of Theorem 1.2 such that if x and y belong to a same T_i, then

$$\frac{1}{1+\delta} \leq \frac{f'(x)}{f'(y)} \leq 1 + \delta. \tag{5}$$

Let I, i_0, a_{j_0} and c_{j_0} be as in the proof of Theorem 1.2. Recall that $f^{j_0}(I) = (a_{j_0}, b_{j_0}) \subset T_{i_0}$. Denote $R = f^{j_0}(I)$. If L is any connected component of K^c contained in (c_{j_0}, a_{j_0}), then

$$\frac{1}{(1+\delta)^q} \frac{|L|}{|R|} \leq \frac{|f^q(L)|}{|f^q(R)|} \leq (1+\delta)^q \frac{|L|}{|R|}$$

while $f^{\widetilde{q}}((c_{j_0}, b_{j_0}))$ is contained in $\cup T_i$ for $0 \leq \widetilde{q} \leq q$. As $\{\mathcal{J}_i\}$ is a ϵ-covering with $\beta_i/\alpha_i = k$, if δ is taken sufficiently small, it follows that

$$\left|\left(n\big(f^{q_1}(L)\big) - n\big(f^{q_1}(R)\big)\right) - \left(n\big(f^{q_1+1}(L)\big) - n\big(f^{q_1+1}(R)\big)\right)\right| \leq 1 \tag{6}$$

for $0 \leq q_1 \leq q$. As remarked at the end of the proof of Theorem 1.2, we can take $L = f^{-q_2}(L_1)$ for an adequate $q_2 > 0$. Then

$$n\big(f^{-q_2}(L_1)\big) - n(R) \geq 0 \quad \text{and} \quad n\big(f^{q_2}(f^{-q_2}(L_1))\big) - n\big(f^{q_2}(R)\big) < 1\,.$$

Then (6) implies that there exist $q_3, q_4 > 0$ and L_j such that

$$n(f^{q_3}(f^{-q_4}(L_j))) - n(f^{q_3}(R)) = 0$$

and
$$n(f^{q_3+1}(f^{-q_4}(L_j))) - n(f^{q_3+1}(R)) = -1.$$
Taking $T = f^{q_3-q_4}(L_j)$ and $I = f^{q_3}(R)$ the proof of the claim is finished.

Note, from the proof of the claim, that the intervals T and I are so close as necessary. Also note that given $\delta' > 0$ there exists $\eta > 0$ such that, if $x, y \in E(z, \eta)$ we have

$$\frac{1}{1+\delta'} < \frac{f'(x)}{f'(y)} < 1 + \delta' \tag{7}$$

for any $z \in K$. Then, given $\delta' > 0$, there exist $\eta > 0$, $z \in K$ and $T, I \subset E(z, \eta)$ as in the claim, such that, if $x, y \in E(z, \eta)$ then x, y satisfy (7). As $|f(T)|$ and $|f(I)|$ do not belong to the same $\mathcal{J}_i$, there exists a 'gap' between $|f(T)|$ and $|f(I)|$. Therefore, as by hypothesis $\beta_i/\alpha_i = k$, this 'gap' produces a new 'gap' for the spectrum of the Cantor set $K \cap E(z, \eta)$ in between each one of the original 'gaps'. Formally, we have that there exists a covering

$$\left\{\mathcal{J}_i^{21} = \left[\alpha_i^{21}, \beta_i^{21}\right]\right\} \cup \left\{\mathcal{J}_i^{22} = \left[\alpha_i^{22}, \beta_i^{22}\right]\right\}.$$

of the spectrum of $K_2 = E(z, \eta) \cap K$ such that $\mathcal{J}_i^{21} \cup \mathcal{J}_i^{22} \subset \mathcal{J}_i$ and $\frac{\beta_i^{2r}}{\alpha_i^{2r}} < k\frac{1+\delta'}{1+\epsilon}$ with $r = 1, 2$ (see figure 1).

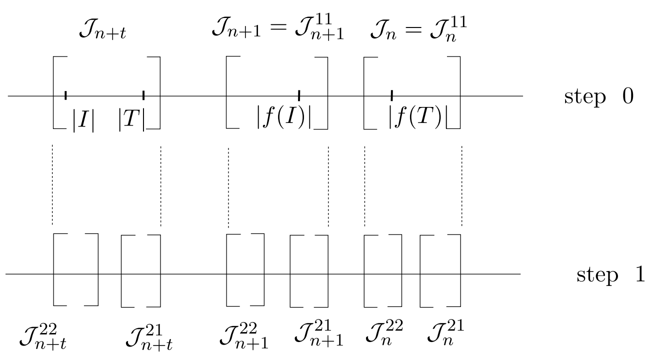

Figure 1

As any C^1-minimal Cantor set is locally C^1-minimal (see [2]), there exists $K_2' \subset K_2$, C^1-minimal with $\{\mathcal{J}_i^{21}\} \cup \{\mathcal{J}_i^{22}\}$ as a covering of its spectrum. Proceeding inductively we obtain a Cantor set K_n', C^1-minimal with

$$\left\{\mathcal{J}_i^{n1}\right\} \cup \left\{\mathcal{J}_i^{n2}\right\} \cup \cdots \cup \left\{\mathcal{J}_i^{nn}\right\}$$

as a covering of its spectrum and such that $1 \leq \frac{\beta_i^{nr}}{\alpha_i^{nr}} < k\left(\frac{1+\delta'}{1+\epsilon}\right)^{n-1}$. As ϵ is fixed and δ' is as small as we want, taking n sufficiently large we obtain a contradiction, and the proof is finished. □

3 Examples of Cantor sets that satisfy the p-separation condition

In this section we will construct a family of Cantor sets that satisfy the p-separation condition for $p = 1$ but does not satisfy the McDuff's condition [2].

3.1 Construction of the Cantor set

First we determine a set of real numbers that will be the spectrum of the Cantor set (here we are not considering the order). Let γ be a positive number such that $\gamma < 3$ and $\gamma^{3/2} > 3$. For each positive integer n we consider the set

$$A(n) = \left\{\eta_{nj} = \frac{\gamma^{\frac{j}{2n}}}{3^{4n+2}} : j = -n, \ldots, n\right\}.$$

If $S(n)$ is the sum of the elements of $A(n)$ we have

$$S(n) = \sum_{j=-n}^{n} \eta_{nj} \leq \frac{2n+1}{3^{4n+2}} \gamma^{1/2} \leq \frac{\gamma^{1/2}}{3^{2n}}.$$

Then $\sum_{n=1}^{\infty} S(n)$ is finite, so the sum of the elements of

$$B = \left\{\eta_i = \frac{1}{3^i} : i \in \mathbf{N}\right\} \cup \bigcup_{i=1}^{\infty} A(i)$$

is finite too. We denote this sum by μ. For the set B we have the figure 2.

Consider the set

$$C = \left\{\frac{2\pi x}{\mu} : x \in B\right\}.$$

The sum of the elements of C is 2π. Let R_θ be a rotation of irrational angle θ in S^1 and x a point in S^1. Let $m : Z \to C$ be a bijection. We define a family of open intervals (a_j, b_j), $j \in Z$ as follows.

$$a_0 = 0, \qquad b_0 = m(0)$$

and for any positive integer j

$$a_j = b_0 + \sum_{R_\theta^k(x) \in (x, R_\theta^j(x))} m(k), \qquad b_j = a_j + m(j).$$

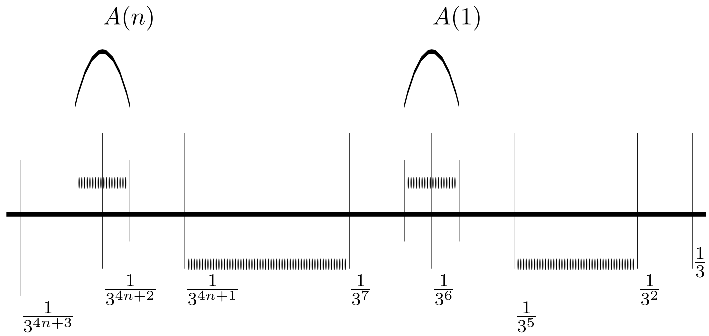

Figure 2

We define

$$K = S^1 \setminus \left(\bigcup_{j \in Z} \left(e^{ia_j}, e^{ib_j}\right) \right).$$

Then K is a Cantor set and C is its spectrum.

3.2 p-separation condition for K

We will show that the Cantor set K satisfies the p-separation condition for $p = 1$. The elements of C are of the form

$$\omega_i = \frac{2\pi}{\mu 3^i}, \qquad \omega_{ij} = \frac{2\pi \gamma^{\frac{j}{2i}}}{\mu 3^{4i+2}}$$

with $i \in \mathbf{N}$ and $j = -i, \ldots, i$. Therefore

$$\frac{2\pi \gamma^{-\frac{1}{2}}}{\mu 3^{4i+2}} \leq \omega_{ij} = \frac{2\pi \gamma^{\frac{j}{2i}}}{\mu 3^{4i+2}} \leq \frac{2\pi \gamma^{\frac{1}{2}}}{\mu 3^{4i+2}}.$$

Now we construct a covering $\{\mathcal{J}_j\}$ of C, $\mathcal{J}_j = [\alpha_j, \beta_j]$, $j > 0$. If $j = 4k+2$ for some $k > 0$ then we define

$$\alpha_j = \frac{2\pi \gamma^{-\frac{1}{2}}}{\mu 3^j}, \qquad \beta_j = \frac{2\pi \gamma^{\frac{1}{2}}}{\mu 3^j},$$

if not

$$\alpha_j = \beta_j = \frac{2\pi}{\mu 3^j}.$$

So, for all integer n we have

$$\frac{\alpha_{j+n-1}}{\beta_{j+n+1}} \geq \frac{9}{\gamma^{\frac{1}{2}}} \quad \text{and} \quad \frac{\beta_j}{\alpha_{j+1}} \leq 3\gamma^{\frac{1}{2}}.$$

As $\gamma < 3$, then K satisfies a p-separation condition for $p = 1$. Note that from Theorem 1.2 we know that the Cantor set K is not C^1-minimal.

3.3 The Cantor set K does not satisfy McDuff's condition

Suppose that K satisfies the McDuff's condition (the 0-separation condition) for a covering $\{L_i\}$, $L_i = [\alpha_i, \beta_i]$. Note that the McDuff's condition implies that every 'gap' $\dfrac{\alpha_i}{\beta_{i+1}}$ is greater than every 'non gap' β_i/α_i. For a fixed k we have

$$\frac{\omega_{kj}}{\omega_{k,j-1}} = \frac{\gamma^{\frac{j}{2k}}}{\gamma^{\frac{j-1}{2k}}} = \gamma^{\frac{1}{2k}}$$

and it limits is 1 when $i \to \infty$. Then, for a sufficiently large k, every ω_{kj} belongs to the same interval $L_{i_k} = [\alpha_{i_k}, \beta_{i_k}]$, so $\dfrac{\beta_{i_k}}{\alpha_{i_k}} \geq \gamma$.

1. If $\beta_{i_k} < \dfrac{2\pi}{\mu 3^{4k+1}}$ then there exists α_r with $r < i_k$ such that

$$\beta_{i_k} < \alpha_r \leq \frac{2\pi}{\mu 3^{4k+1}},$$

so

$$\frac{\alpha_r}{\beta_{i_k}} \leq \frac{\frac{2\pi}{\mu 3^{4k+1}}}{\frac{2\pi\gamma^{1/2}}{\mu 3^{4k+2}}} = \frac{3}{\gamma^{1/2}} < \gamma \leq \frac{\beta_{i_k}}{\alpha_{i_k}}.$$

Then there exists a 'gap' smaller than α_r/β_{i_k}, which is smaller than a 'non gap' β_{i_k}/α_{i_k}, and this is a contradiction.

2. If $\alpha_{i_k} > \dfrac{2\pi}{\mu 3^{4k+3}}$ a contradiction is proved in a similar way.

3. If $\beta_{i_k} \geq \dfrac{2\pi}{\mu 3^{4k+1}}$ and $\alpha_{i_k} \leq \dfrac{2\pi}{\mu 3^{4k+3}}$ then $\dfrac{\beta_{i_k}}{\alpha_{i_k}} \geq 9$. Then we have that the 'gap' β_{i_k}/α_{i_k} is greater than every 'non gap' (every non 'gap' is equal or smaller than 3) and this is a contradiction.

Then the Cantor set K do not satisfy the McDuff's condition.

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Aldo Portela
Instituto de Matemática y Estadística "Prof. Ing. Rafael Laguardia"
Facultad de Ingeniería
CC30, CP 11300
Universidad de la Republica
URUGUAY

E-mail: aldo@fing.edu.uy