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Addendum to "Complete rotation hypersurfaces with H_k constant in space forms"*

A. Gervasio Colares[1] and Oscar Palmas[2]

Abstract. Here we study complete rotation hypersurfaces with constant k-th mean curvature H_k in $\mathbb{S}^{n+1}$, k even and $2 < k < n$. We prove the existence of a constant $H_k^0 < 0$ such that there are no such hypersurfaces for $H_k < H_k^0$. We have only one compact hypersurface of this kind with $H_k = H_k^0$. For each $H_k^0 < H_k < 0$ there is a corresponding family of complete immersed rotation hypersurfaces, each family containing two isoparametric hypersurfaces. For $H_k \geq 0$, there is also such a family, now containing only one isoparametric hypersurface. Finally, we prove the existence of compact hypersurfaces with arbitrarily large H_k, neither isometric to a sphere nor to a product of spheres.

Keywords: Rotation hypersurfaces, k-th mean curvature.

Mathematical subject classification: 53C42, 53A10.

1 Introduction

This paper is a continuation of [5], where the second named author classified the hypersurfaces given in the title for ambient spaces of non-positive curvature, while leaving some open questions for an spherical ambient space, namely: we left open some questions about rotation hypersurfaces with H_k constant in $\mathbb{S}^{n+1}$. The aim here is to analyze this last case. (For completeness, we will recall in a moment the definitions and notation used in [5].)

Our work here may be compared with a paper by F. Brito and M.L. Leite [1], where the authors proved that there are no compact minimal embedded rotation hypersurfaces in $\mathbb{S}^{n+1}$ other than Clifford tori and round geodesic spheres. On

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[2]Corresponding author. Partially supported by CNPq, Brazil and DGAPA-UNAM, México.

the other hand, they proved the existence of infinitely many compact embedded rotation hypersurfaces in $\mathbb{S}^{n+1}$ with non-zero constant mean curvature, showing that the situation is radically different in this case. Their method may be resumed as follows: they associate to each curve generating a rotation hypersurface a function which turns out to be periodic. Then, they proved that the only cases where the period is adequate for the profile curve to close well correspond to the hypersurfaces given above.

The study of the period was also developed in [2] for the constant scalar curvature case, where M.L. Leite proved the existence of compact embedded rotation hypersurfaces with constant scalar curvature not isometric to products of spheres.

More recently, H. Li and G. Wei [3] proved a characterization theorem for compact embedded rotation hypersurfaces in $\mathbb{S}^{n+1}$ with $H_k = 0$, showing that there exists only two such kinds of hypersurfaces, namely, Clifford tori and round geodesic spheres. It is worth to observe that H. Otsuki already treated the case of minimal rotation hypersurfaces in [4].

As the cases of constant mean curvature, constant scalar curvature and $H_k = 0$ were treated in the papers cited above, we restrict ourselves to the cases $2 < k < n$, proving the following results.

Theorem 1.1. *Let k be an even integer such that* $2 < k < n$ *and*

$$H_k^0 = -\frac{2}{n}\left(\frac{k-2}{n-k}\right)^{(k-2)/2}. \tag{1}$$

Then,

1. *There are no complete immersed rotation hypersurfaces in* $\mathbb{S}^{n+1}$ *with constant k-th mean curvature* $H_k < H_k^0$*;*

2. *There is only one complete immersed rotation hypersurface in* $\mathbb{S}^{n+1}$ *with constant k-th mean curvature* $H_k = H_k^0$*. Moreover, this hypersurface is compact and thus embedded in* $\mathbb{S}^{n+1}$*;*

3. *If* $H_k^0 < H_k < 0$*, there is a monoparametric family of complete immersed rotation hypersurface in* $\mathbb{S}^{n+1}$ *with constant k-th mean curvature* H_k*. This family contains two isoparametric hypersurfaces.*

For completeness, we have included in the Theorem above the items 1 and 2 already present in [5].

Theorem 1.2. *If $H_k \geq 0$, there is a monoparametric family of complete immersed rotation hypersurface in $\mathbb{S}^{n+1}$ with constant k-th mean curvature H_k. This family contains one isoparametric hypersurface. Moreover, for N arbitrarily large, there is a compact hypersurface M in $\mathbb{S}^{n+1}$ with constant k-th mean curvature $H_k > N$, neither isometric to a sphere nor to a product of spheres with constant radii.*

2 Definitions and basic lemmas

We recall that rotation hypersurfaces in $\mathbb{S}^{n+1}$ are constructed taking the orbit of a curve α (called the *profile curve*) under the orthogonal transformations in $\mathbb{R}^{n+2}$ leaving fixed a geodesic γ. Denote by s the arc length parameter of α, by $r(s)$ the distance from $\alpha(s)$ to γ, given by the length of a geodesic ray from $\alpha(s)$ to a point $p \in \gamma$ and by $h(s)$ the *height function* measured from a fixed origin in γ to the point p. The profile curve α is parameterized by arc length if and only if

$$\dot{r}^2 + \left(\frac{df}{dr}\right)^2 \dot{h}^2 = 1. \tag{2}$$

where $f(s) = \sin r(s)$. (Compare with eq. 2.2 in [2].) It follows that

$$\dot{h}^2 = \frac{1-\dot{r}^2}{\cos^2 r} = \frac{1-f^2-\dot{f}^2}{(1-f^2)^2}. \tag{3}$$

It was proved in [5] that a rotation hypersurface constructed in this way has constant k-th mean curvature H_k if and only if $f(s)$ satisfies the following differential equation:

$$nH_k f^k = (n-k)(1-f^2-\dot{f}^2)^{k/2} - k(1-f^2-\dot{f}^2)^{(k-2)/2}(\ddot{f}+f)f, \tag{4}$$

where the derivatives are taken with respect to s. This expression is equivalent (see [5]) to its first integral, given by

$$G_k(f,\dot{f}) = f^{n-k}\left((1-f^2-\dot{f}^2)^{k/2} - H_k f^k\right) = C, \tag{5}$$

where C is constant; that is, f is a solution of (4) if and only if the pair $(f,\dot{f})$ is entirely contained in a level curve of G_k. Moreover, $f(s)$ is defined for all s if and only if the pair $(f(s),\dot{f}(s))$ satisfies

$$f^2+\dot{f}^2 \leq 1 \quad \text{and} \quad f \geq 0.$$

Thus, we are lead to analyze the level curves of

$$G_k(u,v) = u^{n-k}\left((1-u^2-v^2)^{k/2} - H_k u^k\right), \quad \text{where } u^2+v^2 \leq 1.$$

The following lemmas give the basic information about the level curves of G_k. In the next section we will translate this information to describe the rotation hypersurfaces we are interested in.

Lemma 2.1. *The level curve $G_k(u, v) = 0$ consists of the v-axis ($u = 0$) and the conic*

$$\left(1 + H_k^{2/k}\right) u^2 + v^2 = 1,$$

which is contained in the region $u^2 + v^2 \leq 1$ if and only if $H_k \geq 0$.

The proof is straightforward and we omit it.

Lemma 2.2. *The critical point set of $G_k(u, v)$ consists of the v-axis ($u = 0$) and the points $(u, 0)$ on the u-axis satisfying*

$$g(x) = nH_k x_0^k + kx_0^2 - (n - k) = 0, \tag{6}$$

where $x = u/\sqrt{1 - u^2}$.

Proof. Another straightforward calculation gives

$$\begin{aligned} \frac{\partial G_k}{\partial u} &= -ku^{n-k}\left(u(1 - u^2 - v^2)^{k/2-1} + H_k u^{k-1}\right) \\ &\quad + (n-k)u^{n-k-1}\left((1 - u^2 - v^2)^{k/2} - H_k u^k\right), \\ \frac{\partial G_k}{\partial v} &= -ku^{n-k}v(1 - u^2 - v^2)^{k/2-1}. \end{aligned}$$

From $\partial G_k/\partial v = 0$, we have to consider three cases:

1. $u = 0$. Then $\partial G_k/\partial u = 0$ is equivalent to
$$u^{n-k-1}(1 - v^2)^{k/2} = 0.$$
If $n - k - 1 > 0$, then the above expression vanishes and the v-axis is contained in the critical point set of G_k. On the other hand, if $n-k-1 = 0$, then $v = \pm 1$.

2. $1 - u^2 - v^2 = 0$. Then $\partial G_k/\partial u = 0$ reduces to $nH_k u^{n-1} = 0$. Thus $u = 0$ or $H_k = 0$.

3. $v = 0$. $\partial G_k/\partial u = 0$ gives $u = 0$ or
$$-ku^2(1 - u^2)^{k/2-1} - kH_k u^k + (n-k)\big((1 - u^2)^{k/2} - H_k u^k\big) = 0$$
Dividing by $(1 - u^2)^{k/2}$ and simplifying, we get the equation (6). □

Next we study in detail the equation $g(x) = 0$ given by (6) in the above lemma.

Lemma 2.3. *If $H_k \geq 0$, equation* (6) *has one positive solution. On the other hand, if $H_k < 0$, then equation* (6) *has zero, one or two non-negative solutions. respectively, whenever H_k is less than, equal to, or greater than the constant H_k^0 given by* (1).

Proof. Just for reference, we calculate the first and second derivatives of $g(x)$:

$$g'(x) = knH_k x^{k-1} + 2kx, \quad \text{and} \quad g''(x) = k(k-1)nH_k x^{k-2} + 2k.$$

Suppose first that $H_k > 0$. Then $g'(x) \geq 0$ for $x \geq 0$, $g'(0) = 0$ and $g''(0) \geq 0$. Moreover,

$$g(0) = -(n-k) < 0 \quad \text{and} \quad g(x) \to +\infty \qquad \text{when} \quad x \to +\infty.$$

These facts imply that $g(x) = 0$ has only one positive solution. The case $H_k = 0$ is similar, but easier, since in this case $g(x) = kx^2 - (n-k)$.

It remains to analyze the case $H_k < 0$. Again, g has a local minimum at $x = 0$ and $g(0) < 0$. The function g has another critical point x_0 given by $knH_k x_0^{k-2} + 2k = 0$, or

$$x_0^{k-2} = -\frac{2}{nH_k}. \tag{7}$$

As it is easily verified, $g'(x_0) < 0$ so g has a local maximum at x_0. Also, note that $g(x) \to -\infty$ when $x \to \infty$. Thus the equation $g(x) = 0$ has zero, one or two solutions depending, respectively, on whether $g(x_0)$ is negative, zero or positive. From $g(x_0) = 0$ and (7) we obtain the value of H_k^0 given in (1). □

Corolary 2.4. *If $H_k \geq 0$, the function $G_k(u, v)$ has one critical point of the form $(u, 0)$, $0 < u < 1$. On the other hand, if $H_k < 0$, the function $G_k(u, v)$ has zero, one or two critical points of the form $(u, 0)$, $0 < u < 1$, whenever, respectively, H_k is less than, equal to or greater than the constant H_k^0 given in* (1).

3 Proof of the theorems

As in [2] and [3], we will describe the rotation hypersurfaces by studying the level sets of the function G_k. Let us analyze first the case $G_k = 0$.

Proposition 3.1. *The level curve $G_k(u, v) = 0$ is associated to a hypersphere orthogonal to a geodesic.*

Proof. The condition $G_k(u, v) = 0$ may be translated as

$$\left(1 + H_k^{2/k}\right) f^2 + \dot{f}^2 = 1.$$

We integrate this equation with initial condition $f(0) = 0$ to get

$$f(s) = \frac{\sin\left(\sqrt{1 + H_k^{2/k}}\, s\right)}{\sqrt{1 + H_k^{2/k}}}$$

and thus

$$h(s) = -\arctan \frac{\cos\left(\sqrt{1 + H_k^{2/k}}\, s\right)}{H_k^{1/k}}$$

generating a semicircle of radius $1/\sqrt{1 + H_k^{2/k}}$ in the orbit space, which in turn generates a hypersphere orthogonal to a geodesic. (For details, see [2].) □

Now we analyze the critical point set of G_k.

Proposition 3.2. *Every critical point* $(u, 0)$ *of* $G_k(u, v)$, $0 < u < 1$, *corresponds to a curve equidistant to a geodesic, which in turn generates a hypersurface with constant principal curvatures isometric to a product of spheres of constant radii.*

Proof. At a critical point $(u, 0)$ with $0 < u < 1$, the distance r of the profile curve to a fixed geodesic is constant. As the principal curvatures are given by functions of r, they are constant. Thus the corresponding hypersurfaces are given as a product of spheres of constant radii. □

Now we are in position to prove our main results.

Proof of Theorem 1.1. For $H_k < H_k^0$, the region inside the unit circle contains no critical points of G_k. In fact, it may be seen that $\partial G_k/\partial u \neq 0$, so that every level curve can be seen as a graph over the v-axis and so must leave the unit circle. Thus, there are no corresponding complete rotation hypersurfaces in this case.

If $H_k = H_k^0$, Corollary 2.4 gives one critical point of G_k inside the unit circle. By a continuity argument, this critical point must be degenerate. Again, every level curve leaves the unit circle.

For $H_k^0 < H_k < 0$, Corollary 2.4 gives two critical points of G_k inside the unit circle. To see that G_k attains a maximum at one of them, note that for $H_k = 0$, G_k attains a maximum at $(\sqrt{(n-k)/n}, 0)$; moreover, for $H_k = 0$ it is easy to see that the second partial derivatives of G_k satisfy

$$\frac{\partial^2 G_k}{\partial f^2}, \frac{\partial^2 G_k}{\partial \dot{f}^2} < 0, \qquad \frac{\partial^2 G_k}{\partial f \partial \dot{f}} = 0,$$

so $(\sqrt{(n-k)/n}, 0)$ is a non-degenerate critical point where G_k attains its maximum. A continuity argument completes the proof of the existence of a maximum of G_k for H_k close to 0. It is easily seen that the level curves near this point are closed, thus generating complete immersed rotation hypersurfaces with constant k-th mean curvature H_k. □

In the case $H_k > 0$, two values of C must be pointed out. The first one is 0, which gives a hypersphere, as stated in Proposition 3.1. The second one is C_0, the maximum value of G_k at the only critical point given by Corollary 2.4. This particular value gives a hypersurface with constant principal curvatures isometric to a product of spheres of constant radii, as stated in Proposition 3.2.

Now we will study the behavior of level sets corresponding to values of C in the interval $(0, C_0)$. For every such C, the level set $G_k(f, \dot{f}) = C$ is a closed curve contained in the region given by $f^2 + \dot{f}^2 < 1$. Let $a(H_k, C)$ and $b(H_k, C)$ be points of the domain of f where it attains its minimum and maximum values, respectively, with no critical points of f in the interval $(a(H_k, C), b(H_k, C))$. Let $\alpha(s)$ be the corresponding profile curve and $h(s)$ its height function. Thus we may use (3) and (5) to write

$$\dot{h}^2 = \frac{1}{(1-f^2)^2} \left(\frac{C + H_k f^n}{f^{n-k}} \right)^{2/k}.$$

Dividing by the expression for $\dot{f}^2$ obtained from (5) and simplifying, we have

$$\left(\frac{dh}{df}\right)^2 = \frac{1}{(1-f^2)^2} \frac{(C + H_k f^n)^{2/k}}{f^{2(n-k)/k}(1-f^2) - (C + H_k f^n)^{2/k}}. \tag{8}$$

We will see that h is periodic with respect to f, its period $P(H_k, C)$ being calculated as follows:

$$\begin{aligned} \frac{1}{2} P(H_k, C) &= h(b(H_k, C)) - h(a(H_k, C)) \\ &= \int_{a(H_k,C)}^{b(H_k,C)} \frac{dh}{df} \, df, \end{aligned}$$

so that

$$\frac{1}{2}P(H_k, C) = \int_{a(H_k,C)}^{b(H_k,C)} \frac{1}{1-f^2}\frac{(C+H_k f^n)^{1/k}}{\sqrt{f^{2(n-k)/k}(1-f^2)-(C+H_k f^n)^{2/k}}}\,df. \qquad (9)$$

The profile curves generate an embedded rotation hypersurface if this period $P(H_k, C)$ has the form $2\pi/m$, $m = 1, 2, \ldots$. In the following lemmas we will study the behavior of this period function.

Lemma 3.3. *Fix $H_k \geq 0$ and let $a(H_k, C)$, $b(H_k, C)$ and $P(H_k, C)$ be defined as above. Then $a(H_k, C)$ is a decreasing function of C, while $b(H_k, C)$ and $P(H_k, C)$ are increasing functions of C.*

Proof. The properties of $a(H_k, C)$ and $b(H_k, C)$ follow from the fact that G_k has a maximum at the only critical point. □

Lemma 3.4. *With the above notation, the period function satisfies*

$$P(H_k, 0) = 2\arctan\frac{1}{H_k^{1/k}}. \qquad (10)$$

Proof. Recall that in this case we have

$$h(s) = -\arctan\frac{\cos\left(\sqrt{1+H_k^{2/k}}\,s\right)}{H_k^{1/k}}.$$

Note also that

$$\frac{1}{2}P(H_k, 0) = h(s_1) - h(s_0).$$

We choose $s_0 = 0$, so that

$$h(0) = -\arctan\frac{1}{H_k^{1/k}}.$$

The other limit value s_1 satisfies

$$\frac{1}{\sqrt{1+H_k^{2/k}}} = f(s_1) = \frac{\sin\left(\sqrt{1+H_k^{2/k}}\,s_1\right)}{\sqrt{1+H_k^{2/k}}}.$$

Thus, $\cos\left(\sqrt{1+H_k^{2/k}}\,s_1\right) = 0$ and $h(s_1) = 0$, proving the lemma. □

Remark 3.5. It is clear from (10) that $P(H_k, 0)$ is a decreasing function of H_k, satisfying

$$\lim_{H_k \to 0} P(H_k, 0) = \pi.$$

In fact, H. Li and G. Wei studied in [3] the case $H_k = 0$ and proved that $\pi \leq P(0, C) < 2\pi$ for every C in the image $[0, C_0]$ of G_k, where as before C_0 is the value of G_k at the only critical point $(f_0, 0)$,

$$f_0 = \sqrt{\frac{n-k}{n}}.$$

We may calculate the precise limit of $P(0, C)$ when $f \to f_0$ (or equivalently $C \to C_0$) following [2]. First observe that

$$C_0 = \lim_{f \to f_0} G_k(u, 0) = \left(\frac{n-k}{n}\right)^{(n-k)/2} \left(\frac{k}{n}\right)^{k/2}.$$

In (9), we calculate the Taylor polynomial of the term inside the square root around f_0, so that

$$\lim_{f \to f_0} P(0, C) = \lim_{f \to f_0} \frac{2C^{1/k}}{1 - f^2} \int_{f_0 - \sqrt{\epsilon/A}}^{f_0 + \sqrt{\epsilon/A}} \frac{df}{\sqrt{\epsilon - A(f - f_0)^2}},$$

where

$$A = \frac{1}{k} \left(\frac{n-k}{n}\right)^{(n-2k)/k} (n-k).$$

As the above integral converges to $\pi/\sqrt{A}$, we have (as in the case $k = 2$ analyzed in [2]) that

$$\lim_{f \to f_0} P(0, C) = \sqrt{2}\pi.$$

Proof of Theorem 1.2. The statements about the existence of immersed and isoparametric hypersurfaces follow from the analysis of the corresponding function G_k, similarly to the case $H_k < 0$.

On the other hand, the existence of compact hypersurfaces follows from continuity of $P(H_k, C)$: Since $P(H_k, 0)$ is given by (10), this function varies from π to 0 when $H_k \to \infty$, so that for each $N > 0$ there is a $H'_k > N$ such that $P(H'_k, 0) = 2\pi/m$, $m \in \mathbb{N}$. By continuity, there is a $H_k > N$ and $C \neq 0, C_0$ sufficiently near 0 such that $P(H_k, C) = 2\pi/m$. Thus, the corresponding hypersurface is compact and neither isometric to a sphere nor to a product of spheres. □

References

[1] F. Brito and M.L. Leite. *A remark on rotational hypersurfaces of* $\mathbb{S}^n$. Bull. Soc. Math. Belg., **42** (1990), ser. B, 303–317.

[2] M.L. Leite. *Rotational hypersurfaces of space forms with constant scalar curvature*. Manuscripta Mathematica, **67** (1990), 285–304.

[3] H. Li and G. Wei. *Compact embedded rotation hypersurfaces of* $\mathbb{S}^{n+1}$, Bull. Braz. Math. Soc., **38**(1) (2007), 81–99.

[4] H. Otsuki. *Minimal hypersurfaces in a Riemannian manifold of constant curvature*. Amer. J. Math., **92** (1970), 145–173.

[5] O. Palmas. *Complete rotation hypersurfaces with* H_k *constant in space forms*. Bull. Braz. Math. Soc., **30**(2) (1999), 139–161.

A. Gervasio Colares
Departamento de Matemática
Universidade Federal do Ceará
60455-760 Fortaleza, CE
BRAZIL

E-mail: gcolares@mat.ufc.br

Oscar Palmas
Departamento de Matemáticas
Facultad de Ciencias, UNAM
04510 México, DF
MÉXICO

E-mail: opv@hp.fciencias.unam.mx