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Periodic solutions of generalized Liénard equations with a p-Laplacian-like operator*

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Abstract. In this paper, sufficient conditions are established to guarantee the existence of at least one periodic solution to a generalized Liénard equation with a p-Laplacian-like operator. Generalized polar coordinates are employed in our proof.

Keywords: generalized polar coordinates, Liénard equation, Laplacian-like operator, periodic solutions.

Mathematical subject classification: 34C25, 34B15.

1 Introduction

In recent years the existence of periodic solutions for second order Liénard equations of the form

$$u'' + f(u, u')u' + g(u) = e(t, u, u'),$$

have been studied by many authors (see e.g. [1-12] and the references therein). More general equations or systems involving periodic boundary conditions have also been considered. For example, in [15], R. Manásevich and J. Mawhin investigated the existence of periodic solutions to the boundary value problem

$$(\phi(u'))' = f(t, u, u'), \qquad u(0) = u(T), u'(0) = u'(T),$$

where the function $\phi\colon R^N \to R^N$ is quite general and satisfies some monotonicity conditions which ensure that ϕ is an homeomorphism onto R^N. On the other hand, when $\phi = \phi_p\colon R \to R$ is the so-called one-dimensional p-Laplacian operator given by $\phi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\phi_p(0) = 0$, corresponding periodic boundary value problems have also been considered in [13, 14].

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In [16, 17], the authors studied the existence of periodic solutions for nonlinear differential equations of the form

$$(\phi(u'))' + f(t, u, u') = 0,$$

where ϕ is or is a slightly more general than the p-Laplacian operator. In this paper, we will go one step beyond and consider the following generalized Liénard equations with p-Laplacian-like operator

$$(\phi(u'))' + f(u, u')u' + g(u) = e(t, u, u'), \qquad t \in [0, T], p > 1, \qquad (1)$$

where $\phi, g \in \mathbb{C}(\mathbb{R}, \mathbb{R})$, $f \in \mathbb{C}(\mathbb{R}^2, \mathbb{R})$ and $e \in \mathbb{C}([0, T] \times \mathbb{R}^2, \mathbb{R})$. The reason for studying such an equation is as follows, the term $f(u, u')u'$ stands for a damping factor and the term $e(t, u, u')$ stands for a control. Such terms are missing in the previous equation while in the the original Liénard equation, such damping and control factors have been proven to be important in designing devices that generate periodic phenomena.

Our technique is motivated by that used in [18] and employs a polar coordinate transformation to investigate the existence of periodic solutions for (1). That there is a periodic solution of (1) under some conditions may not be surprising, yet to find good and meaingful conditions may be quite challenging. Our goal here is quite modest and is to find one such set of sufficient conditions which can also be illustrated by simple examples. Indeed, by a great deal of efforts, we succeed in finding the following conditions:

(H1) ϕ is continuous and strictly increasing, $y\phi(y) > 0$ for $y \neq 0$, and there exist $p > 2, m_2 \geq m_1 > 0$, such that

$$m_1|y|^{p-1} \leq |\phi(y)| \leq m_2|y|^{p-1}.$$

(H2) $e \in \mathbb{C}([0, T] \times \mathbb{R}^2, \mathbb{R})$ is T-periodic in t and there exist $\alpha_1, \beta_1, \gamma_1, \delta_1, s_1 > 0, p > s_1 + 1$ such that

$$|e(t, x, y)| \leq \alpha_1|x|^{p-1} + \beta_1|x|^{p-s_1-1}|y|^{s_1} + \gamma_1|y|^{p-1} + \delta_1,$$
$$\text{for } (t, x, y) \in [0, T] \times \mathbb{R}^2.$$

(H3) $f \in \mathbb{C}(\mathbb{R}^2, \mathbb{R})$, there exist $\alpha_2, \beta_2, \gamma_2, \delta_2, s_2 > 0, p > s_2 + 2$ such that

$$|f(x, y)| \leq \alpha_2|x|^{p-2} + \beta_2|x|^{p-s_2-2}|y|^{s_2} + \gamma_2|y|^{p-2} + \delta_2,$$
$$\text{for } (x, y) \in \mathbb{R}^2.$$

(H4) There exist $\lambda > 0$, $\mu > 0$; $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \sum_{i=1}^{4}\lambda_i \in (0,1)$; $\mu_1, \mu_2, \mu_3, \mu_4, \sum_{i=1}^{4}\mu_i \in (0,\infty)$ and $n \geq 0$ such that

$$\begin{aligned} & h_1(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2) + \frac{m_2}{m_1}\left(1-\sum_{i=1}^{4}\lambda_i\right)^{1-p}\left(\frac{2n\pi_p}{T}\right)^p \\ & < \lambda \leq \frac{g(x)}{\phi(x)} \leq \mu < -h_2(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2) \\ & + \frac{m_1}{m_2}\left(1+\sum_{i=1}^{4}\mu_i\right)^{1-p}\left(\frac{2(n+1)\pi_p}{T}\right)^p, \end{aligned}$$

where

$$h_1 = \frac{1}{m_1}\left[\alpha_1 + L_1\alpha_2^{\frac{p}{p-1}} + L_2\beta_2^{\frac{p}{p-s_2-1}} + L_3\beta_1^{\frac{p}{p-s_1}} + L_4(\gamma_1+\gamma_2)^p\right],$$

$$h_2 = \frac{1}{m_2}\left[\alpha_1 + P_1\alpha_2^{\frac{p}{p-1}} + P_2\beta_2^{\frac{p}{p-s_2-1}} + P_3\beta_1^{\frac{p}{p-s_1}} + P_4(\gamma_1+\gamma_2)^p\right],$$

$$L_1 = \frac{p-1}{p}(\lambda_1 p(p-1))^{-\frac{1}{p-1}}\left(\frac{m_2}{m_1^p}\right)^{\frac{1}{(p-1)^2}},$$

$$L_2 = \frac{p-s_2-1}{p}\left(\frac{\lambda_2 p(p-1)}{s_2+1}\right)^{-\frac{s_2+1}{p-s_2-1}}\left(\frac{m_2}{m_1^p}\right)^{\frac{s_2+1}{(p-1)(p-s_2-1)}},$$

$$L_3 = \frac{p-s_1}{p}\left(\frac{\lambda_3 p(p-1)}{s_1}\right)^{-\frac{s_1}{p-s_1}}\left(\frac{m_2}{m_1^p}\right)^{\frac{s_1}{(p-1)(p-s_1)}},$$

$$L_4 = \frac{1}{p}(\lambda_4 p)^{1-p}\frac{m_2}{m_1^p},$$

$$P_1 = \frac{p-1}{p}(\mu_1 p(p-1))^{-\frac{1}{p-1}} m_1^{-\frac{1}{p-1}},$$

$$P_2 = \frac{p-s_2-1}{p}\left(\frac{\mu_2 p(p-1)}{s_2+1}\right)^{-\frac{s_2+1}{p-s_2-1}} m_1^{-\frac{s_2+1}{p-s_2-1}},$$

$$P_3 = \frac{p-s_1}{p}\left(\frac{\mu_3 p(p-1)}{s_1}\right)^{-\frac{s_1}{p-s_1}} m_1^{-\frac{s_1}{p-s_1}},$$

$$P_4 = \frac{1}{p}(\mu_4 p)^{1-p} m_1^{1-p},$$

and

$$\pi_p = \frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin(\pi/p)}.$$

(H5) The solutions of Eq. (1) are unique with respect to initial values.

Our main result is then

Theorem 1. *Suppose* $(H1)$-$(H5)$ *hold. Then Eq.* (1) *has at least one* T-*periodic solution.*

The condition (H1) restricts the growth of the function ϕ. The condition (H2) requires the control term to be periodic (for synchronizing purpose) and bounded by mixed powers of $|x|$ and $|y|$. Similarly, the growth of the damping term is restricted by the condition (H3). The condition (H4) imposes bounds for the ratio $g(x)/\phi(x)$.

In the next section, we will transform Eq. (1) into a first order nonlinear system so that polar coordinate transformation can be introduced further. Then estimates for the modulus function and the rate of change of the phase angle function are found in Lemma 1 and Lemma 2 respectively. The number π_p arises since we need to estimate the time for traversing a cycle around the origin (see Lemma 3 below). The main result is then proved by means of degree argument. A concrete example is given in the last section which clarifies our hypotheses.

2 Periodic solutions

In this section, we present some lemmas and prove our main result.

Let $v = \phi(u')$. Since ϕ is strictly increasing by (H1), Eq. (1) is equivalent to the system

$$\begin{cases} u' = \phi^{-1}(v), \\ v' = -g(u) - f(u, \phi^{-1}(v))\phi^{-1}(v) + e(t, u, \phi^{-1}(v)). \end{cases} \tag{2}$$

Given ξ and $\eta \in R$, by (H5), there is a unique solution $u(t, \xi, \eta)$ of Eq. (1) satisfying the initial value condition

$$u(0, \xi, \eta) = \xi, \quad v(0, \xi, \eta) = \eta.$$

Applying generalized polar coordinates

$$\begin{cases} u = p^{\frac{1}{p}} r^{\frac{2}{p}} |\cos\theta|^{\frac{2-p}{p}} \cos\theta, \\ v = \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} r^{\frac{2(p-1)}{p}} |\sin\theta|^{\frac{p-2}{p}} \sin\theta, \end{cases} \tag{3}$$

or

$$\begin{cases} r\cos\theta = \frac{1}{\sqrt{p}}|u|^{\frac{p-2}{2}} u, \\ r\sin\theta = \sqrt{\frac{p-1}{p}}|v|^{\frac{2-p}{2(p-1)}} v, \end{cases} \tag{4}$$

then

$$r^2 = \frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{\frac{p}{p-1}}$$

and

$$\theta = \tan^{-1}\left[\sqrt{p-1}\,\frac{|v|^{\frac{2-p}{2(p-1)}} v}{|u|^{\frac{p-2}{2}} u}\right].$$

Lemma 1. *Suppose* $(H1)$-$(H5)$ *hold. Then for each* $c > 0$, *there exists* $A > 0$ *such that if* $r(0) = A$, *then* $r(t) \geq c$ *for all* $t \in [0, T]$.

Proof. Consider

$$r^2(t) = \frac{1}{p}|u(t)|^p + \frac{p-1}{p}|v(t)|^{\frac{p}{p-1}}.$$

It is clear that (H1) implies

$$\left(\frac{|v|}{m_2}\right)^{\frac{1}{p-1}} \leq |\phi^{-1}(v)| \leq \left(\frac{|v|}{m_1}\right)^{\frac{1}{p-1}}.$$

So we have

$$\begin{aligned} \left|\frac{dr^2(t)}{dt}\right| &= |\,|u(t)|^{p-2}u(t)u'(t) + |v(t)|^{\frac{2-p}{p-1}} v(t)v'(t)| \\ &\leq |u|^{p-1}|\phi^{-1}(v)| + |v|^{\frac{1}{p-1}}| - g(u) - f(u, \phi^{-1}(v))\phi^{-1}(v) \\ &\quad + e(t, u, \phi^{-1}(v))| \\ &\leq |u|^{p-1}|\phi^{-1}(v)| + \mu|v|^{\frac{1}{p-1}}|\phi(u)| + |v|^{\frac{1}{p-1}}(\alpha_2|u|^{p-2} \end{aligned}$$

$$
\begin{aligned}
&+\ \beta_2|u|^{p-s_2-2}|\phi^{-1}(v)|^{s_2}+\gamma_2|\phi^{-1}(v)|^{p-2}+\delta_2)|\phi^{-1}(v)| \\
&+\ |v|^{\frac{1}{p-1}}(\alpha_1|u|^{p-1}+\beta_1|u|^{p-s_1-1}|\phi^{-1}(v)|^{s_1}+\gamma_1|\phi^{-1}(v)|^{p-1}+\delta_1) \\
\le\ & m_1^{-\frac{1}{p-1}}|u|^{p-1}|v|^{\frac{1}{p-1}}+\mu m_2|u|^{p-1}|v|^{\frac{1}{p-1}}+\alpha_2 m_1^{-\frac{1}{p-1}}|u|^{p-2}|v|^{\frac{2}{p-1}} \\
&+\ \beta_2 m_1^{-\frac{s_2+1}{p-1}}|u|^{p-s_2-2}|v|^{\frac{s_2+2}{p-1}}+\frac{\gamma_2}{m_1}|v|^{\frac{p}{p-1}}+\delta_2 m_1^{-\frac{1}{p-1}}|v|^{\frac{2}{p-1}} \\
&+\ \alpha_1|u|^{p-1}|v|^{\frac{1}{p-1}}+\beta_1 m_1^{-\frac{s_1}{p-1}}|u|^{p-s_1-1}|v|^{\frac{s_1+1}{p-1}}+\ \frac{\gamma_1}{m_1}|v|^{\frac{p}{p-1}}+\delta_1|v|^{\frac{1}{p-1}} \\
=\ & l_1|u|^{p-1}|v|^{\frac{1}{p-1}}+l_2|v|^{\frac{p}{p-1}}+l_3|u|^{p-2}|v|^{\frac{2}{p-1}}+l_4|v|^{\frac{2}{p-1}}+\delta_1|v|^{\frac{1}{p-1}} \\
&+\ l_5|u|^{p-s_1-1}|v|^{\frac{s_1+1}{p-1}}+l_6|u|^{p-s_2-2}|v|^{\frac{s_2+2}{p-1}},
\end{aligned}
$$

where

$$
l_1=m_1^{-\frac{1}{p-1}}+\mu m_2+\alpha_1,\quad l_2=\frac{\gamma_2}{m_1}+\frac{\gamma_1}{m_1},\quad l_3=\alpha_2 m_1^{-\frac{1}{p-1}},
$$

$$
l_4=\delta_2 m_1^{-\frac{1}{p-1}},\quad l_5=\beta_1 m_1^{-\frac{s_1}{p-1}},\quad l_6=\beta_2 m_1^{-\frac{s_2+1}{p-1}},
$$

while

$$
\begin{aligned}
& l_1|u|^{p-1}|v|^{\frac{1}{p-1}}\le l_1\left(\frac{1}{p}|v|^{\frac{p}{p-1}}+\frac{p-1}{p}|u|^p\right)\le l_1\left(p-1+\frac{1}{p-1}\right)r^2, \\
& l_2|v|^{\frac{p}{p-1}}\le\frac{pl_2}{p-1}r^2, \\
& l_3|v|^{\frac{2}{p-1}}|u|^{p-2}\le l_3\left(\frac{2}{p}|v|^{\frac{p}{p-1}}+\frac{p-2}{p}|u|^p\right)\le l_3\left(\frac{2}{p-1}+p-2\right)r^2, \\
& l_4|v|^{\frac{2}{p-1}}\le\frac{2}{p}|v|^{\frac{p}{p-1}}+\frac{p-2}{p}l_4^{\frac{p}{p-2}}\le\frac{2}{p-1}r^2+\frac{p-2}{p}l_4^{\frac{p}{p-2}}, \\
& \delta_1|v|^{\frac{1}{p-1}}\le\frac{1}{p}|v|^{\frac{p}{p-1}}+\frac{p-1}{p}\delta_1^{\frac{p}{p-1}}\le\frac{1}{p-1}r^2+\frac{p-1}{p}\delta_1^{\frac{p}{p-1}}, \\
& l_5|u|^{p-s_1-1}|v|^{\frac{s_1+1}{p-1}}\le l_5\left(\frac{p-s_1-1}{p}|u|^p+\frac{s_1+1}{p}|v|^{\frac{p}{p-1}}\right) \\
&\qquad\qquad\le l_5\left(p-s_1-1+\frac{s_1+1}{p-1}\right)r^2,
\end{aligned}
$$

$$
\begin{aligned}
l_6|u|^{p-s_2-2}|v|^{\frac{s_2+2}{p-1}} &\le l_6\left(\frac{p-s_2-2}{p}|u|^p + \frac{s_2+2}{p}|v|^{\frac{p}{p-1}}\right)\\
&\le l_6\left(p-s_2-2+\frac{s_2+2}{p-1}\right)r^2.
\end{aligned}
$$

So,

$$
\left|\frac{dr^2(t)}{dt}\right| \le br^2(t)+a,
$$

where

$$
\begin{aligned}
a &= \frac{p-2}{p}l_4{}^{\frac{p}{p-2}} + \frac{p-1}{p}\delta_1^{\frac{p}{p-1}},\\
b &= l_1\left(p-1+\frac{1}{p-1}\right) + \frac{pl_2}{p-1} + l_3\left(\frac{2}{p-1}+p-2\right) + \frac{3}{p-1}\\
&\quad + l_5\left(p-s_1-1+\frac{s_1+1}{p-1}\right) + l_6\left(p-s_2-2+\frac{s_2+2}{p-1}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left(r^2(0)+\frac{a}{b}\right)e^{-bT} &\le \left(r^2(0)+\frac{a}{b}\right)e^{-bt} \le r^2(t)+\frac{a}{b}\\
&\le \left(r^2(0)+\frac{a}{b}\right)e^{bt} \le \left(r^2(0)+\frac{a}{b}\right)e^{bT}, \qquad 0 \le t \le T.
\end{aligned}
$$

If we let $A = \left[\left(c^2+\frac{a}{b}\right)e^{bT} - \frac{a}{b}\right]^{\frac{1}{2}}$, then $A > c$ and $r(0) = A$ implies $r(t) \ge c$ for all $t \in [0, T]$. The proof is complete.

Lemma 2. *Suppose the conditions of* $(H1)$-$(H5)$ *are satisfied. There is* $R > 0$ *such that if* $r(0) \ge R$, *then* $\frac{d\theta(t)}{dt} \le 0$ *for all* $t \in [0, T]$.

Proof. We have

$$
\begin{aligned}
\theta' &= \frac{|u|^{\frac{p-2}{2}}|v|^{\frac{2-p}{2(p-1)}}}{2\sqrt{p-1}r^2}\left[uv' - (p-1)u'v\right]\\
&= -\frac{|u|^{\frac{p-2}{2}}|v|^{\frac{2-p}{2(p-1)}}}{2\sqrt{p-1}r^2}\Big[ug(u) + uf(u,\phi^{-1}(v))\phi^{-1}(v)\\
&\qquad + (p-1)v\phi^{-1}(v) - ue(t,u,\phi^{-1}(v))\Big].
\end{aligned}
$$

As

$$\begin{aligned}
& ug(u) + uf(u,\phi^{-1}(v))\phi^{-1}(v) + (p-1)v\phi^{-1}(v) - ue(t,u,\phi^{-1}(v)) \\
& \quad \geq \lambda u\phi(u) + (p-1)v\phi^{-1}(v) - |u|(\alpha_2|u|^{p-2} + \beta_2|u|^{p-s_2-2}|\phi^{-1}(v)|^{s_2} \\
& \qquad + \gamma_2|\phi^{-1}(v)|^{p-2} + \delta_2)|\phi^{-1}(v)| - |u|(\alpha_1|u|^{p-1} + \beta_1|u|^{p-s_1-1}|\phi^{-1}(v)|^{s_1} \\
& \qquad + \gamma_1|\phi^{-1}(v)|^{p-1} + \delta_1) \\
& \quad \geq (\lambda m_1 - \alpha_1)|u|^p + (p-1)m_2^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}} - \alpha_2 m_1^{-\frac{1}{p-1}}|u|^{p-1}|v|^{\frac{1}{p-1}} \\
& \qquad - \beta_2 m_1^{-\frac{s_2+1}{p-1}}|u|^{p-s_2-1}|v|^{\frac{s_2+1}{p-1}} - \left(\frac{\gamma_1}{m_1} + \frac{\gamma_2}{m_1}\right)|u||v| - \delta_2 m_1^{-\frac{1}{p-1}}|u||v|^{\frac{1}{p-1}} \\
& \qquad - \beta_1 m_1^{-\frac{s_1}{p-1}}|u|^{p-s_1}|v|^{\frac{s_1}{p-1}} - \delta_1|u|.
\end{aligned}$$

Let

$$\tau_1 = \frac{1}{6}p(p-1)m_2^{-\frac{1}{p-1}}, \qquad \omega_1 = \frac{6\alpha_2}{p(p-1)}\left(\frac{m_2}{m_1}\right)^{\frac{1}{p-1}},$$

then

$$\begin{aligned}
\alpha_2 m_1^{-\frac{1}{p-1}}|u|^{p-1}|v|^{\frac{1}{p-1}} &= \tau_1\left(|v|^{\frac{1}{p-1}}\omega_1|u|^{p-1}\right) \\
&\leq \tau_1\left(\frac{1}{p}|v|^{\frac{p}{p-1}} + \frac{p-1}{p}(\omega_1|u|^{p-1})^{\frac{p}{p-1}}\right) \\
&\leq \frac{1}{6}p\, m_2^{-\frac{1}{p-1}}\left(\frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{\frac{p}{p-1}}\right) \\
&\quad + \frac{p-1}{p}\tau_1\omega_1^{\frac{p}{p-1}}|u|^p.
\end{aligned}$$

Let

$$\tau_2 = \frac{1}{6}pm_2^{-\frac{1}{p-1}}, \qquad \omega_2 = \frac{6(\gamma_1+\gamma_2)}{pm_1}m_2^{\frac{1}{p-1}},$$

so we have

$$\begin{aligned}
\left(\frac{\gamma_1}{m_1} + \frac{\gamma_2}{m_1}\right)|u||v| &= \tau_2(\omega_2|u|)|v| \\
&\leq \tau_2\left(\frac{1}{p}|\omega_2 u|^p + \frac{p-1}{p}|v|^{\frac{p}{p-1}}\right) \\
&\leq \frac{1}{6}p\, m_2^{-\frac{1}{p-1}}\left(\frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{\frac{p}{p-1}}\right) + \frac{1}{p}\tau_2\omega_2^p|u|^p.
\end{aligned}$$

Let

$$\tau_3 = \frac{1}{6}p(p-1)m_2^{-\frac{1}{p-1}}, \qquad \omega_3 = \frac{6\delta_2}{p(p-1)}\left(\frac{m_2}{m_1}\right)^{\frac{1}{p-1}},$$

then

$$\begin{aligned}
\delta_2 m_1^{-\frac{1}{p-1}}|u||v|^{\frac{1}{p-1}} &= \tau_3|u|\left(|v|^{\frac{1}{p-1}}\omega_3\right)\\
&\le \tau_3|u|\left(\frac{1}{p-1}|v| + \frac{p-2}{p-1}\omega_3^{\frac{p-1}{p-2}}\right)\\
&= \frac{1}{6}p\, m_2^{-\frac{1}{p-1}}|u||v| + \frac{1}{6}p(p-2)m_2^{-\frac{1}{p-1}}\omega_3^{\frac{p-1}{p-2}}|u|\\
&\le \frac{1}{6}p\, m_2^{-\frac{1}{p-1}}\left(\frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{\frac{p}{p-1}}\right)\\
&\quad + \frac{1}{6}p(p-2)m_2^{-\frac{1}{p-1}}\omega_3^{\frac{p-1}{p-2}}|u|.
\end{aligned}$$

Let

$$\tau_4 = \frac{p(p-1)}{6(s_2+1)}m_2^{-\frac{1}{p-1}}, \qquad \omega_4 = \frac{6(s_2+1)\beta_2}{p(p-1)}\left(\frac{m_2}{m_1}\right)^{\frac{1}{p-1}} m_1^{-\frac{s_2}{p-1}},$$

then

$$\begin{aligned}
\beta_2 m_1^{-\frac{s_2+1}{p-1}}|u|^{p-s_2-1}|v|^{\frac{s_2+1}{p-1}} &= \tau_4\left(|v|^{\frac{s_2+1}{p-1}}\omega_4|u|^{p-s_2-1}\right)\\
&\le \tau_4\left(\frac{s_2+1}{p}|v|^{\frac{p}{p-1}} + \frac{p-s_2-1}{p}\omega_4^{\frac{p}{p-s_2-1}}|u|^p\right)\\
&= \frac{1}{6}(p-1)m_2^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}}\\
&\quad + \frac{p-s_2-1}{p}\tau_4\omega_4^{\frac{p}{p-s_2-1}}|u|^p\\
&\le \frac{1}{6}p\, m_2^{-\frac{1}{p-1}}\left(\frac{1}{p}|u|^p + \frac{p-1}{p}|v|^{\frac{p}{p-1}}\right)\\
&\quad + \frac{p-s_2-1}{p}\tau_4\omega_4^{\frac{p}{p-s_2-1}}|u|^p.
\end{aligned}$$

Let

$$\tau_5 = \frac{p(p-1)}{6s_1}m_2^{-\frac{1}{p-1}}, \qquad \omega_5 = \frac{6s_1\beta_1}{p(p-1)}\left(\frac{m_2}{m_1}\right)^{\frac{1}{p-1}} m_1^{\frac{1-s_1}{p-1}},$$

then

$$\begin{aligned}
\beta_1 m_1^{-\frac{s_1}{p-1}} |u|^{p-s_1} |v|^{\frac{s_1}{p-1}} &= \tau_5 (|v|^{\frac{s_1}{p-1}} \omega_5 |u|^{p-s_1}) \\
&\le \tau_5 \left(\frac{s_1}{p} |v|^{\frac{p}{p-1}} + \frac{p-s_1}{p} \omega_5^{\frac{p}{p-s_1}} |u|^p \right) \\
&= \frac{1}{6}(p-1) m_2^{-\frac{1}{p-1}} |v|^{\frac{p}{p-1}} + \frac{p-s_1}{p} \tau_5 \omega_5^{\frac{p}{p-s_1}} |u|^p \\
&\le \frac{1}{6} p\, m_2^{-\frac{1}{p-1}} \left(\frac{1}{p} |u|^p + \frac{p-1}{p} |v|^{\frac{p}{p-1}} \right) \\
&\quad + \frac{p-s_1}{p} \tau_5 \omega_5^{\frac{p}{p-s_1}} |u|^p.
\end{aligned}$$

We select λ large enough such that

$$\begin{aligned}
\delta &= \lambda m_1 - \alpha_1 - \frac{p-1}{p} \tau_1 \omega_1^{\frac{p}{p-1}} - \frac{1}{p} \tau_2 \omega_2^p \\
&\quad - \frac{p-s_2-1}{p} \tau_4 \omega_4^{\frac{p}{p-s_2-1}} - \frac{p-s_1}{p} \tau_5 \omega_5^{\frac{p}{p-s_1}} - m_2^{-\frac{1}{p-1}} > 0,
\end{aligned}$$

Let $d = \delta_1 + \frac{1}{6} p(p-2) m_2^{-\frac{1}{p-1}} \omega_3^{\frac{p-1}{p-2}}$, we also have

$$d|u| = \delta p |u| \left(\frac{d}{\delta p} \right) \le \delta |u|^p + (p-1)\delta \left(\frac{d}{p\,\delta} \right)^{\frac{p}{p-1}},$$

therefore

$$\begin{aligned}
&ug(u) + uf(u, \phi^{-1}(v))\phi^{-1}(v) + (p-1)v\phi^{-1}(v) - ue(t, u, \phi^{-1}(v)) \\
&\quad \ge \frac{1}{6} p\, m_2^{-\frac{1}{p-1}} \left[\frac{1}{p} |u|^p + \frac{p-1}{p} |v|^{\frac{p}{p-1}} \right] - (p-1)\delta \left(\frac{d}{p\,\delta} \right)^{\frac{p}{p-1}} \\
&\quad = \frac{1}{6} p\, m_2^{-\frac{1}{p-1}} r^2(t) - (p-1)\delta \left(\frac{d}{p\,\delta} \right)^{\frac{p}{p-1}}.
\end{aligned}$$

Lemma 1 implies that there is $R > 0$, such that

$$\frac{1}{6} p\, m_2^{-\frac{1}{p-1}} r^2(t) > (p-1)\delta \left(\frac{d}{p\,\delta} \right)^{\frac{p}{p-1}},$$

when $r(0) > R$, then our assertion is verified.

Lemma 3. *Suppose that conditions* $(H1)$-$(H5)$ *hold. Given an arbitrary positive number* ν, *there exists* $A \gg 1$ *such that if* $r(0) = A$, *then*

$$(u(T,\xi,\eta), v(T,\xi,\eta)) \neq (\nu^{2/p}\xi, \nu^{2(p-1)/p}\eta).$$

Proof. It follows from Lemma 1 that,

$$r(0) = A \Rightarrow r(t) \geq c \quad \text{for} \quad t \in [0,T].$$

On the other hand, when $r(0) \to \infty$, it holds uniformly from (H1)-(H3) that

$$\begin{aligned}
-\theta' &= \frac{|u|^{\frac{p-2}{2}}|v|^{\frac{2-p}{2(p-1)}}}{2\sqrt{p-1}r^2}\big[ug(u) + uf(u,\phi^{-1}(v))\phi^{-1}(v) \\
&\quad + (p-1)v\phi^{-1}(v) - ue(t,u,\phi^{-1}(v))\big] \\
&\geq \frac{|u|^{\frac{p-2}{2}}|v|^{\frac{2-p}{2(p-1)}}}{2\sqrt{p-1}r^2}\big[(\lambda m_1 - \alpha_1)|u|^p + (p-1)m_2^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}} \\
&\quad - \alpha_2 m_1^{-\frac{1}{p-1}}|u|^{p-1}|v|^{\frac{1}{p-1}} - \beta_2 m_1^{-\frac{s_2+1}{p-1}}|u|^{p-s_2-1}|v|^{\frac{s_2+1}{p-1}} \\
&\quad - \left(\frac{\gamma_1}{m_1} + \frac{\gamma_2}{m_1}\right)|u||v| - \delta_2 m_1^{-\frac{1}{p-1}}|u||v|^{\frac{1}{p-1}} \\
&\quad - \beta_1 m_1^{-\frac{s_1}{p-1}}|u|^{p-s_1}|v|^{\frac{s_1}{p-1}} - \delta_1|u|\big].
\end{aligned}$$

Let

$$\begin{aligned}
C_1^p &= \lambda_1 p(p-1)m_2^{-\frac{1}{p-1}}, \quad C_2^{\frac{p}{s_2+1}} = \lambda_2\frac{p(p-1)}{s_2+1}m_2^{-\frac{1}{p-1}}, \\
C_3^{\frac{p}{s_1}} &= \lambda_3\frac{p(p-1)}{s_1}m_2^{-\frac{1}{p-1}}, \quad C_4^{\frac{p}{p-1}} = \lambda_4 p m_2^{-\frac{1}{p-1}}, \\
L_1 &= \frac{p-1}{p}(\lambda_1 p(p-1))^{-\frac{1}{p-1}}\left(\frac{m_2}{m_1^p}\right)^{\frac{1}{(p-1)^2}}, \\
L_2 &= \frac{p-s_2-1}{p}\left(\frac{\lambda_2 p(p-1)}{s_2+1}\right)^{-\frac{s_2+1}{p-s_2-1}}\left(\frac{m_2}{m_1^p}\right)^{\frac{s_2+1}{(p-1)(p-s_2-1)}}, \\
L_3 &= \frac{p-s_1}{p}\left(\frac{\lambda_3 p(p-1)}{s_1}\right)^{-\frac{s_1}{p-s_1}}\left(\frac{m_2}{m_1^p}\right)^{\frac{s_1}{(p-1)(p-s_1)}}, \\
L_4 &= \frac{1}{p}(\lambda_4 p)^{1-p}\frac{m_2}{m_1^p},
\end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \sum_{i=1}^{4}\lambda_i \in (0,1)$, so we have the following inequalities

$$\begin{aligned}
\alpha_2 m_1^{-\frac{1}{p-1}}|u|^{p-1}|v|^{\frac{1}{p-1}} &= \left(C_1^{-1}\alpha_2 m_1^{-\frac{1}{p-1}}|u|^{p-1}\right)\left(C_1|v|^{\frac{1}{p-1}}\right)\\
&\le \frac{p-1}{p}\left(C_1^{-1}\alpha_2 m_1^{-\frac{1}{p-1}}\right)^{\frac{p}{p-1}}|u|^p + \frac{1}{p}C_1^p|v|^{\frac{p}{p-1}}\\
&= L_1\alpha_2^{\frac{p}{p-1}}|u|^p + \lambda_1(p-1)m_2^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}},
\end{aligned}$$

$$\begin{aligned}
\beta_2 m_1^{-\frac{s_2+1}{p-1}}|u|^{p-s_2-1}|v|^{\frac{s_2+1}{p-1}} &= \left(C_2^{-1}\beta_2 m_1^{-\frac{s_2+1}{p-1}}|u|^{p-s_2-1}\right)\left(C_2|v|^{\frac{s_2+1}{p-1}}\right)\\
&\le \frac{p-s_2-1}{p}\left(C_2^{-1}\beta_2 m_1^{-\frac{s_2+1}{p-1}}\right)^{\frac{p}{p-s_2-1}}|u|^p\\
&\quad + \frac{s_2+1}{p}C_2^{\frac{p}{s_2+1}}|v|^{\frac{p}{p-1}}\\
&= L_2\beta_2^{\frac{p}{p-s_2-1}}|u|^p + \lambda_2(p-1)m_2^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}},
\end{aligned}$$

$$\begin{aligned}
\beta_1 m_1^{-\frac{s_1}{p-1}}|u|^{p-s_1}|v|^{\frac{s_1}{p-1}} &= \left(C_3^{-1}\beta_1 m_1^{-\frac{s_1}{p-1}}|u|^{p-s_1}\right)\left(C_3|v|^{\frac{s_1}{p-1}}\right)\\
&\le \frac{p-s_1}{p}\left(C_3^{-1}\beta_1 m_1^{-\frac{s_1}{p-1}}\right)^{\frac{p}{p-s_1}}|u|^p + \frac{s_1}{p}C_3^{\frac{p}{s_1}}|v|^{\frac{p}{p-1}}\\
&= L_3\beta_1^{\frac{p}{p-s_1}}|u|^p + \lambda_3(p-1)m_2^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}},
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_1}\right)|u||v| &= \left[C_4^{-1}\left(\frac{\gamma_1+\gamma_2}{m_1}\right)|u|\right](C_4|v|)\\
&\le \frac{1}{p}\left[C_4^{-1}\left(\frac{\gamma_1+\gamma_2}{m_1}\right)\right]^p|u|^p + \frac{p-1}{p}C_4^{\frac{p}{p-1}}|v|^{\frac{p}{p-1}}\\
&= L_4(\gamma_1+\gamma_2)^p|u|^p + \lambda_4(p-1)m_2^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}}.
\end{aligned}$$

So

$$\begin{aligned}
-\theta' \ \ge\ & \frac{|u|^{\frac{p-2}{2}}|v|^{\frac{2-p}{2(p-1)}}}{2\sqrt{p-1}r^2}\Big[(\lambda m_1 - \alpha_1 - L_1\alpha_2^{\frac{p}{p-1}} - L_2\beta_2^{\frac{p}{p-s_2-1}} - L_3\beta_1^{\frac{p}{p-s_1}}\\
&-L_4(\gamma_1+\gamma_2)^p)|u|^p + \left(1-\sum_{i=1}^{4}\lambda_i\right)(p-1)m_2^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}}
\end{aligned}$$

$$
\begin{aligned}
&-\delta_2 m_1^{-\frac{1}{p-1}}|u||v|^{\frac{1}{p-1}} - \delta_1|u|\Big] \\
= \;& \frac{p|\sin\theta|^{\frac{2-p}{p}}|\cos\theta|^{\frac{p-2}{p}}}{2(p-1)^{\frac{1}{p}}}\Big[(\lambda m_1 - \alpha_1 - L_1\alpha_2^{\frac{p}{p-1}} - L_2\beta_2^{\frac{p}{p-s_2-1}} \\
&-L_3\beta_1^{\frac{p}{p-s_1}} - L_4(\gamma_1+\gamma_2)^p))\cos^2\theta + \left(1-\sum_{i=1}^{4}\lambda_i\right)m_2^{-\frac{1}{p-1}}\sin^2\theta\Big] \\
&-\frac{\delta_2 m_1^{-\frac{1}{p-1}} p^{\frac{2}{p}}}{2(p-1)^{\frac{2}{p}} r^{\frac{2(p-2)}{p}}}|\cos\theta||\sin\theta|^{\frac{4-p}{p}} \\
&-\frac{\delta_1 p^{\frac{1}{p}}}{2(p-1)^{\frac{1}{p}} r^{\frac{2(p-1)}{p}}}|\cos\theta||\sin\theta|^{\frac{2-p}{p}} \\
= \;& a_1(b_1\cos^2\theta + \sin^2\theta)|\sin\theta|^{\frac{2-p}{p}}|\cos\theta|^{\frac{p-2}{p}} \\
&-\frac{\delta_2 m_1^{-\frac{1}{p-1}} p^{\frac{2}{p}}}{2(p-1)^{\frac{2}{p}} r^{\frac{2(p-2)}{p}}}|\cos\theta||\sin\theta|^{\frac{4-p}{p}} \\
&-\frac{\delta_1 p^{\frac{1}{p}}}{2(p-1)^{\frac{1}{p}} r^{\frac{2(p-1)}{p}}}|\cos\theta||\sin\theta|^{\frac{2-p}{p}},
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= \frac{p(1-\sum_{i=1}^{4}\lambda_i)}{2(p-1)^{\frac{1}{p}} m_2^{\frac{1}{p-1}}}, \\
b_1 &= \frac{1}{1-\sum_{i=1}^{4}\lambda_i}\Big[\lambda m_1 - \alpha_1 - L_1\alpha_2^{\frac{p}{p-1}} - L_2\beta_2^{\frac{p}{p-s_2-1}} \\
&\qquad\qquad - L_3\beta_1^{\frac{p}{p-s_1}} - L_4(\gamma_1+\gamma_2)^p\Big]m_2^{\frac{1}{p-1}}.
\end{aligned}
$$

Denote $\hat{b} = \min\{b_1, 1\}$. Then we have

$$
\begin{aligned}
-\theta' \;\geq\;& a_1(b_1\cos^2\theta + \sin^2\theta)|\sin\theta|^{\frac{2-p}{p}}|\cos\theta|^{\frac{p-2}{p}} \\
&-\frac{\delta_2 m_1^{-\frac{1}{p-1}} p^{\frac{2}{p}}}{2\hat{b}(p-1)^{\frac{2}{p}} r^{\frac{2(p-2)}{p}}}(b_1\cos^2\theta + \sin^2\theta)|\cos\theta||\sin\theta|^{\frac{4-p}{p}} \\
&-\frac{\delta_1 p^{\frac{1}{p}}}{2\hat{b}(p-1)^{\frac{1}{p}} r^{\frac{2(p-1)}{p}}}(b_1\cos^2\theta + \sin^2\theta)|\sin\theta|^{\frac{2-p}{p}}|\cos\theta|^{\frac{p-2}{p}}
\end{aligned}
$$

$$= \hat{a}_1(b_1\cos^2\theta+\sin^2\theta)|\sin\theta|^{\frac{2-p}{p}}|\cos\theta|^{\frac{p-2}{p}},$$

where

$$\hat{a}_1 = a_1 - \frac{\delta_2 m_1^{-\frac{1}{p-1}} p^{\frac{2}{p}}}{2\hat{b}(p-1)^{\frac{2}{p}} r^{\frac{2(p-2)}{p}}} - \frac{\delta_1 p^{\frac{1}{p}}}{2\hat{b}(p-1)^{\frac{1}{p}} r^{\frac{2(p-1)}{p}}}.$$

Assume that it takes time Δt for the motion $(r(t),\theta(t))(r(0)=A,\theta(0)=\theta_0)$ to complete one cycle around the origin. It follows from above inequality that

$$\Delta t < \int_{\theta_0}^{\theta_0+2\pi} \frac{d\theta}{\hat{a}_1(b_1\cos^2\theta+\sin^2\theta)|\sin\theta|^{\frac{2-p}{p}}|\cos\theta|^{\frac{p-2}{p}}}$$
$$= \frac{4}{\hat{a}_1}\int_0^{\frac{\pi}{2}} \frac{d\theta}{(b_1\cos^2\theta+\sin^2\theta)|\sin\theta|^{\frac{2-p}{p}}|\cos\theta|^{\frac{p-2}{p}}}.$$

Let

$$\eta = \tan^{-1}\frac{1}{\sqrt{b_1}}\tan\theta.$$

Then

$$\Delta t < \frac{4}{\hat{a}_1 b_1^{\frac{1}{p}}}\int_0^{\frac{\pi}{2}} \frac{d\eta}{|\tan\eta|^{\frac{2-p}{p}}} = \frac{2}{\hat{a}_1 b_1^{\frac{1}{p}}} B\left(\frac{1}{p},\frac{p-1}{p}\right) = \frac{2\pi}{\hat{a}_1 b_1^{\frac{1}{p}}\sin\frac{\pi}{p}},$$

from (H4), we have

$$a_1 b_1^{\frac{1}{p}}\sin\frac{\pi}{p}$$
$$= \frac{\pi}{\pi_p}\left(1-\sum_{i=1}^{4}\lambda_i\right)^{\frac{p-1}{p}}\left(\frac{\lambda m_1-\alpha_1-L_1\alpha_2^{\frac{p}{p-1}}-L_2\beta_2^{\frac{p}{p-s_2-1}}-L_3\beta_1^{\frac{p}{p-s_1}}-L_4(\gamma_1+\gamma_2)^p}{m_2}\right)^{\frac{1}{p}}$$
$$> \frac{2n\pi}{T}.$$

So there exists $\sigma>0$ such that $(a_1-\sigma)b_1^{\frac{1}{p}}\sin\frac{\pi}{p} > \frac{2n\pi}{T}$. For the $\sigma>0$, there exists $R'>0$ such that

$$0 < \frac{\delta_2 m_1^{-\frac{1}{p-1}} p^{\frac{2}{p}}}{2\hat{b}(p-1)^{\frac{2}{p}} r^{\frac{2(p-2)}{p}}} + \frac{\delta_1 p^{\frac{1}{p}}}{2\hat{b}(p-1)^{\frac{1}{p}} r^{\frac{2(p-1)}{p}}} < \sigma,$$

for $A > R'$ large enough. So we have

$$\begin{aligned}\hat{a}_1 b_1^{\frac{1}{p}} \sin\frac{\pi}{p} &= \left(a_1 - \frac{\delta_2 m_1^{-\frac{1}{p-1}} p^{\frac{2}{p}}}{2\hat{b}(p-1)^{\frac{2}{p}} r^{\frac{2(p-2)}{p}}} - \frac{\delta_1 p^{\frac{1}{p}}}{2\hat{b}(p-1)^{\frac{1}{p}} r^{\frac{2(p-1)}{p}}}\right) b_1^{\frac{1}{p}} \sin\frac{\pi}{p} \\ &> (a_1 - \sigma) b_1^{\frac{1}{p}} \sin\frac{\pi}{p} > \frac{2n\pi}{T}.\end{aligned}$$

Therefore

$$\frac{T}{\Delta t} > n.$$

Let

$$D_1^p = \mu_1 p(p-1) m_1^{-\frac{1}{p-1}}, \quad D_2^{\frac{p}{s_2+1}} = \mu_2 \frac{p(p-1)}{s_2+1} m_1^{-\frac{1}{p-1}},$$

$$D_3^{\frac{p}{s_1}} = \mu_3 \frac{p(p-1)}{s_1} m_1^{-\frac{1}{p-1}}, \quad D_4^{\frac{p}{p-1}} = \mu_4 p m_1^{-\frac{1}{p-1}},$$

$$P_1 = \frac{p-1}{p} (\mu_1 p(p-1))^{-\frac{1}{p-1}} m_1^{-\frac{1}{p-1}},$$

$$P_2 = \frac{p-s_2-1}{p} \left(\frac{\mu_2 p(p-1)}{s_2+1}\right)^{-\frac{s_2+1}{p-s_2-1}} m_1^{-\frac{s_2+1}{p-s_2-1}},$$

$$P_3 = \frac{p-s_1}{p} \left(\frac{\mu_3 p(p-1)}{s_1}\right)^{-\frac{s_1}{p-s_1}} m_1^{-\frac{s_1}{p-s_1}},$$

$$P_4 = \frac{1}{p} (\mu_4 p)^{1-p} m_1^{1-p},$$

where $\mu_1, \mu_2, \mu_3, \mu_4, \sum_{i=1}^{4} \mu_i \in (0, \infty)$. Then we have the following inequalities

$$\begin{aligned}\alpha_2 m_1^{-\frac{1}{p-1}} |u|^{p-1} |v|^{\frac{1}{p-1}} &= \left(D_1^{-1} \alpha_2 m_1^{-\frac{1}{p-1}} |u|^{p-1}\right)\left(D_1 |v|^{\frac{1}{p-1}}\right) \\ &\le \frac{p-1}{p} \left(D_1^{-1} \alpha_2 m_1^{-\frac{1}{p-1}}\right)^{\frac{p}{p-1}} |u|^p + \frac{1}{p} D_1^p |v|^{\frac{p}{p-1}} \\ &= P_1 \alpha_2^{\frac{p}{p-1}} |u|^p + \mu_1 (p-1) m_1^{-\frac{1}{p-1}} |v|^{\frac{p}{p-1}},\end{aligned}$$

$$\beta_2 m_1^{-\frac{s_2+1}{p-1}} |u|^{p-s_2-1} |v|^{\frac{s_2+1}{p-1}} = \left(D_2^{-1} \beta_2 m_1^{-\frac{s_2+1}{p-1}} |u|^{p-s_2-1}\right)\left(D_2 |v|^{\frac{s_2+1}{p-1}}\right)$$

$$
\begin{aligned}
&\le \frac{p-s_2-1}{p}\big(D_2^{-1}\beta_2 m_1^{-\frac{s_2+1}{p-1}}\big)^{\frac{p}{p-s_2-1}}|u|^p \\
&\quad + \frac{s_2+1}{p}D_2^{\frac{p}{s_2+1}}|v|^{\frac{p}{p-1}} \\
&= P_2\beta_2^{\frac{p}{p-s_2-1}}|u|^p + \mu_2(p-1)m_1^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}},
\end{aligned}
$$

$$
\begin{aligned}
\beta_1 m_1^{-\frac{s_1}{p-1}}|u|^{p-s_1}|v|^{\frac{s_1}{p-1}} &= \big(D_3^{-1}\beta_1 m_1^{-\frac{s_1}{p-1}}|u|^{p-s_1}\big)\big(D_3|v|^{\frac{s_1}{p-1}}\big) \\
&\le \frac{p-s_1}{p}\big(D_3^{-1}\beta_1 m_1^{-\frac{s_1}{p-1}}\big)^{\frac{p}{p-s_1}}|u|^p + \frac{s_1}{p}D_3^{\frac{p}{s_1}}|v|^{\frac{p}{p-1}} \\
&= P_3\beta_1^{\frac{p}{p-s_1}}|u|^p + \mu_3(p-1)m_1^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}},
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_1}\right)|u||v| &= \left[D_4^{-1}\left(\frac{\gamma_1+\gamma_2}{m_1}\right)|u|\right](D_4|v|) \\
&\le \frac{1}{p}\left[D_4^{-1}\left(\frac{\gamma_1+\gamma_2}{m_1}\right)\right]^p|u|^p + \frac{p-1}{p}D_4^{\frac{p}{p-1}}|v|^{\frac{p}{p-1}} \\
&= P_4(\gamma_1+\gamma_2)^p|u|^p + \mu_4(p-1)m_1^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
-\theta' &= \frac{|u|^{\frac{p-2}{2}}|v|^{\frac{2-p}{2(p-1)}}}{2\sqrt{p-1}r^2}\big[ug(u)+uf(u,\phi^{-1}(v))\phi^{-1}(v) \\
&\quad + (p-1)v\phi^{-1}(v) - ue(t,u,\phi^{-1}(v))\big] \\
&\le \frac{|u|^{\frac{p-2}{2}}|v|^{\frac{2-p}{2(p-1)}}}{2\sqrt{p-1}r^2}\big[(\mu m_2+\alpha_1)|u|^p + (p-1)m_1^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}} \\
&\quad + \alpha_2 m_1^{-\frac{1}{p-1}}|u|^{p-1}|v|^{\frac{1}{p-1}} + \beta_2 m_1^{-\frac{s_2+1}{p-1}}|u|^{p-s_2-1}|v|^{\frac{s_2+1}{p-1}} \\
&\quad + \left(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_1}\right)|u||v| + \delta_2 m_1^{-\frac{1}{p-1}}|u||v|^{\frac{1}{p-1}} \\
&\quad + \beta_1 m_1^{-\frac{s_1}{p-1}}|u|^{p-s_1}|v|^{\frac{s_1}{p-1}} - \delta_1|u|\big] \\
&\le \frac{|u|^{\frac{p-2}{2}}|v|^{\frac{2-p}{2(p-1)}}}{2\sqrt{p-1}r^2}\big[(\mu m_2+\alpha_1+P_1\alpha_2^{\frac{p}{p-1}} + P_2\beta_2^{\frac{p}{p-s_2-1}} + P_3\beta_1^{\frac{p}{p-s_1}}
\end{aligned}
$$

$$
\begin{aligned}
& + P_4(\gamma_1+\gamma_2)^p)|u|^p + \left(1+\sum_{i=1}^4 \mu_i\right)(p-1)m_1^{-\frac{1}{p-1}}|v|^{\frac{p}{p-1}} \\
& + \delta_2 m_1^{-\frac{1}{p-1}}|u||v|^{\frac{1}{p-1}} + \delta_1|u|\big] \\
= \ & \frac{p|\sin\theta|^{\frac{2-p}{p}}|\cos\theta|^{\frac{p-2}{p}}}{2(p-1)^{\frac{1}{p}}}\big[(\mu m_2+\alpha_1+P_1\alpha_2^{\frac{p}{p-1}}+P_2\beta_2^{\frac{p}{p-s_2-1}} \\
& + P_3\beta_1^{\frac{p}{p-s_1}} + P_4(\gamma_1+\gamma_2)^p)\cos^2\theta + \left(1+\sum_{i=1}^4\mu_i\right)m_1^{-\frac{1}{p-1}}\sin^2\theta\big] \\
& + \frac{\delta_2 m_1^{-\frac{1}{p-1}}p^{\frac{2}{p}}}{2(p-1)^{\frac{2}{p}}r^{\frac{2(p-2)}{p}}}|\cos\theta||\sin\theta|^{\frac{4-p}{p}} \\
& + \frac{\delta_1 p^{\frac{1}{p}}}{2(p-1)^{\frac{1}{p}}r^{\frac{2(p-1)}{p}}}|\cos\theta||\sin\theta|^{\frac{2-p}{p}} \\
= \ & a_2(b_2\cos^2\theta+\sin^2\theta)|\sin\theta|^{\frac{2-p}{p}}|\cos\theta|^{\frac{p-2}{p}} \\
& + \frac{\delta_2 m_1^{-\frac{1}{p-1}}p^{\frac{2}{p}}}{2(p-1)^{\frac{2}{p}}r^{\frac{2(p-2)}{p}}}|\cos\theta||\sin\theta|^{\frac{4-p}{p}} \\
& + \frac{\delta_1 p^{\frac{1}{p}}}{2(p-1)^{\frac{1}{p}}r^{\frac{2(p-1)}{p}}}|\cos\theta||\sin\theta|^{\frac{2-p}{p}},
\end{aligned}
$$

where

$$
\begin{aligned}
a_2 &= \frac{p(1+\sum_{i=1}^4\mu_i)}{2(p-1)^{\frac{1}{p}}m_1^{\frac{1}{p-1}}}, \\
b_2 &= \frac{1}{1+\sum_{i=1}^4\mu_i}\big[\mu m_2+\alpha_1+P_1\alpha_2^{\frac{p}{p-1}}+P_2\beta_2^{\frac{p}{p-s_2-1}} \\
&\qquad + P_3\beta_1^{\frac{p}{p-s_1}}+P_4(\gamma_1+\gamma_2)^p\big]m_1^{\frac{1}{p-1}}.
\end{aligned}
$$

with the similar argument, we also get

$$\frac{T}{\Delta t} < n+1.$$

Therefore

$$n < \frac{T}{\Delta t} < n+1.$$

To complete our proof, we show that if $n < \frac{T}{\Delta t} < n+1$, then

$$(u(T,\xi,\eta), v(T,\xi,\eta)) \neq \left(\nu^{\frac{2}{p}}\xi, \nu^{\frac{2(p-1)}{p}}\eta\right).$$

Indeed, if there is $\nu > 0$ such that $(u(T,\xi,\eta), v(T,\xi,\eta)) = \left(\nu^{\frac{2}{p}}\xi, \nu^{\frac{2(p-1)}{p}}\eta\right)$, then

$$\left(p^{\frac{1}{p}} r(T)^{\frac{2}{p}} |\cos\theta(T)|^{\frac{2-p}{p}} \cos\theta(T), \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} r(T)^{\frac{2(p-1)}{p}} |\sin\theta(T)|^{\frac{p-2}{p}} \sin\theta(T)\right)$$

$$= \left(\nu^{\frac{2}{p}} p^{\frac{1}{p}} r(0)^{\frac{2}{p}} |\cos\theta(0)|^{\frac{2-p}{p}} \cos\theta(0), \nu^{\frac{2(p-1)}{p}} \left(\frac{p}{p-1}\right)^{\frac{p-1}{p}} r(0)^{\frac{2(p-1)}{p}} |\sin\theta(0)|^{\frac{p-2}{p}} \sin\theta(0)\right).$$

So

$$r(T)^{\frac{2}{p}} |\cos\theta(T)|^{\frac{2-p}{p}} \cos\theta(T) = \nu^{\frac{2}{p}} r(0)^{\frac{2}{p}} |\cos\theta(0)|^{\frac{2-p}{p}} \cos\theta(0), \tag{5}$$

$$r(T)^{\frac{2(p-1)}{p}} |\sin\theta(T)|^{\frac{p-2}{p}} \sin\theta(T) = \nu^{\frac{2(p-1)}{p}} r(0)^{\frac{2(p-1)}{p}} |\sin\theta(0)|^{\frac{p-2}{p}} \sin\theta(0). \tag{6}$$

From (5) we have

$$r(T)^{\frac{2}{p}} |\cos\theta(T)|^{\frac{2}{p}} \operatorname{sgn}\cos\theta(T) = (\nu r(0))^{\frac{2}{p}} |\cos\theta(0)|^{\frac{2}{p}} \operatorname{sgn}\cos\theta(0),$$

so, $\operatorname{sgn}\cos\theta(T) = \operatorname{sgn}\cos\theta(0)$, therefore,

$$r(T)^{\frac{2}{p}} |\cos\theta(T)|^{\frac{2}{p}} = (\nu r(0))^{\frac{2}{p}} |\cos\theta(0)|^{\frac{2}{p}},$$

moreover,

$$r(T)\cos\theta(T) = \nu r(0)\cos\theta(0). \tag{7}$$

Similarly from (6) one has

$$r(T)\sin\theta(T) = \nu r(0)\sin\theta(0). \tag{8}$$

So, from (7) and (8) we have

$$r(T) = \nu r(0), \qquad (\cos\theta(T), \sin\theta(T)) = (\cos\theta(0), \sin\theta(0)).$$

Therefore,

$$\theta(T) = \theta(0) + 2k\pi, \qquad \text{or} \qquad \theta(T) - \theta(0) = 2k\pi.$$

However, from $n\Delta t < T < (n+1)\Delta t$, we have

$$\theta(T) - \theta(0) < \theta(n\Delta t) - \theta(0) = -2n\pi,$$

$$\theta(T) - \theta(0) > \theta((n+1)\Delta t) - \theta(0) = -2(n+1)\pi,$$

since $\theta' < 0$. So there is no integer k such that $\theta(T) - \theta(0) = 2k\pi$. This contradiction completes our proof.

Proof Theorem 1. By Lemma 3 we know that there exists $A \gg 1$ such that, if

$$\frac{1}{p}|\xi|^p + \frac{p-1}{p}|\eta|^{\frac{p}{p-1}} = A^2,$$

then

$$(u(T,\xi,\eta), v(T,\xi,\eta)) \neq \left(\lambda^{\frac{2}{p}}\xi, \lambda^{\frac{2(p-1)}{p}}\eta\right) \quad \text{for} \quad \lambda > 0.$$

Assume that

$$\xi_1 = u(T,\xi,\eta), \quad \eta_1 = v(T,\xi,\eta).$$

Consider a two-dimensional open region D_A bounded by

$$D_A = \left\{(\xi,\eta) \colon \frac{1}{p}|\xi|^p + \frac{p-1}{p}|\eta|^{\frac{p}{p-1}} = A^2\right\}.$$

Define the topological mapping

$$H \colon D_A \longmapsto R^2, \quad (\xi,\eta) \longmapsto (\xi_1,\eta_1).$$

It follows from Lemma 3 that

$$(\xi_1,\eta_1) \neq \left(\lambda^{\frac{2}{p}}\xi, \lambda^{\frac{2(p-1)}{p}}\eta\right), \quad (\xi,\eta) \in \partial D_A.$$

Now we define a homotopy $h \colon \overline{D}_A \times [0,1] \to R^2$ by

$$\begin{aligned} h(\xi,\eta,\mu) &= -\left(\mu^{\frac{2}{p}}\xi, \mu^{\frac{2(p-1)}{p}}\eta\right) + \left((1-\mu)^{\frac{2}{p}}\xi_1, (1-\mu)^{\frac{2(p-1)}{p}}\eta_1\right) \\ &= -\begin{pmatrix} \mu^{\frac{2}{p}} & 0 \\ 0 & \mu^{\frac{2(p-1)}{p}} \end{pmatrix} I(\xi,\eta) + \begin{pmatrix} (1-\mu)^{\frac{2}{p}} & 0 \\ 0 & (1-\mu)^{\frac{2(p-1)}{p}} \end{pmatrix} H(\xi,\eta), \end{aligned}$$

for $\mu \in [0,1]$. It is easy to see that $h(\xi,\eta,0), h(\xi,\eta,1) \neq 0$ for $(\xi,\eta) \in \partial D_A$. Then we show that $h(\xi,\eta,\mu) \neq 0$ for $(\xi,\eta) \in \partial D_A$, where $\mu \in (0,1)$. If not, there is $\mu_0 \in (0,1)$, $(\xi,\eta) \in \partial D_A$ such that $h(\xi,\eta,\mu_0) = 0$. i.e.,

$$(\xi_1,\eta_1) = \left(\left(\frac{\mu_0}{1-\mu_0}\right)^{\frac{2}{p}}\xi, \left(\frac{\mu_0}{1-\mu_0}\right)^{\frac{2(p-1)}{p}}\eta\right),$$

which is impossible. So $h(\xi,\eta,\mu) \neq 0$ for $\mu \in [0,1]$.

Then, $\deg\{h(\xi,\eta,0), D_A, 0\} = \deg\{h(\xi,\eta,1), D_A, 0\}$, i.e.,

$$\deg\{H, D_A, 0\} = \deg\{-I, D_A, 0\} \neq 0.$$

Therefore, H has at least one fixed point $(\xi^*,\eta^*) \in D_A$. It is easy to see that $u(t) = u(t,\xi^*,\eta^*)$ is a T-periodic solution of the Eq. (1).

3 Example

In this section, we present an example to illustrate our main result. Consider the following differential equation

$$(\phi(u'))' + f(u, u')u' + g(u) = e(t, u, u'), \qquad t \in [0, T]. \tag{9}$$

where

$$\begin{aligned} \phi(x) &= \phi_5(x) = |x|^3 x, \\ f(x, y) &= |x|^3 + \frac{1}{2}|y|^3 + a, \quad a > 0, \\ g(x) &= 20\phi_5(x), \\ e(t, x, y) &= -\frac{3}{5}|x|^3 x - |x|^3 y - \frac{1}{2}|y|^3 y + b\cos 2\pi t, \quad b > 0. \end{aligned}$$

Here $\alpha_1 = \frac{3}{5}$, $\alpha_2 = \beta_1 = 1$, $\beta_2 = 0$, $\gamma_1 = \gamma_2 = \frac{1}{2}$, $\delta_2 = a$, $\delta_1 = b$, $s_1 = s_2 = 1$. Let $\lambda_1 = \mu_1 = \lambda_3 = \mu_3 = \frac{1}{20}$, $\lambda_2 = \mu_2 = \frac{1}{10}$, $\lambda_4 = \mu_4 = \frac{1}{5}$, $n = 0$, $T = 1$. Then

$$3 < \lambda \le \frac{g(x)}{\phi_5(x)} \le \mu < -3 + \left(\frac{5}{7}\right)^4 (2\pi_5)^5,$$

and hence conditions (H1)-(H4) are satisfied.

Now, we check the condition (H5) is satisfied. Suppose $x_1(t)$ and $x_2(t)$ are two different solutions to equation (9) satisfying

$$x_1(t_0) = x_2(t_0) = x_0, \quad x_1'(t_0) = x_2'(t_0) = x_0'.$$

Let $y = \phi(x')$, then $(x_i(t), y_i(t)) = (x_i(t), \phi(x_i'(t)))$, $i = 1, 2$, are two different solutions to the system

$$\begin{cases} x' = \phi^{-1}(y), \\ y' = -g(x) - f(x, \phi^{-1}(y))\phi^{-1}(y) + e(t, x, \phi^{-1}(y)), \end{cases} \tag{10}$$

satisfying $(x_i(t_0), y_i(t_0)) = (x_0, \phi(x'(t_0)))$, $i = 1, 2$. Without loss of generality, we assume that there exists $t_1 > t_0$ such that

$$x_2(t) > x_1(t), \quad t \in (t_0, t_1].$$

As $x_1(t_0) = x_2(t_0) = x_0$, $\quad x_1'(t_0) = x_2'(t_0) = x_0'$, and $x_i \in C^2[t_0, t_1]$, so there exists $t^* \in (t_0, t_1)$ such that

$$x_2'(t) > x_1'(t), \quad t \in (t_0, t^*].$$

So, for $t \in (t_0, t^*]$,

$$\int_{t_0}^{t} \big(|x_2(s)|^3 x_2'(s) - |x_1(s)|^3 x_1'(s)\big)ds = \frac{1}{4}\big(|x_2(t)|^3 x_2(t) - |x_1(t)|^3 x_1(t)\big) > 0.$$

Therefore, for $t \in (t_0, t^*]$, we have

$$\begin{aligned} y_2(t) - y_1(t) &= -\int_{t_0}^{t} \Big\{\big[g(x_2(s)) - g(x_1(s))\big] + \big[f(x_2(s), x_2'(s))x_2'(s) \\ &\quad - f(x_1(s), x_1'(s))x_1'(s)\big] - \big[e(s, x_2(s), x_2'(s)) \\ &\quad - e(s, x_1(s), x_1'(s))\big]\Big\}ds \\ &= -\int_{t_0}^{t} \Big\{20\big[\phi_5(x_2(s)) - \phi_5(x_1(s))\big] + 2\big[|x_2(s)|^3 x_2'(s) \\ &\quad - |x_1(s)|^3 x_1'(s)\big] + \big[|x_2'(s)|^3 x_2'(s) - |x_1'(s)|^3 x_1'(s)\big] \\ &\quad + a(x_2'(s) - x_1'(s)) + \frac{3}{5}\big[|x_2(s)|^3 x_2(s) - |x_1(s)|^3 x_1(s)\big]\Big\}ds \\ &< 0. \end{aligned}$$

i.e.,

$$\phi(x_2'(t)) - \phi(x_1'(t)) < 0, \quad t \in (t_0, t^*].$$

So, $x_2'(t) < x_1'(t), \quad t \in (t_0, t^*]$. This is a contradiction.

Therefore, by Theorem 1, we can conclude that equation (9) has at least one 1-periodic solution.

References

[1] Z. Wang. *Periodic solutions of the second-order forced Liénard equation via time maps*. Nonlinear Anal., **48** (2002), 445–460.

[2] P. Omari, G. Villari and F. Zanolin. *Periodic solutions of the Liénard equation with one-sided growth restrictions*. J. Differential Equations, **67** (1987), 278–293.

[3] G. Villari. *Existence of periodic solutions for Liénard's equation*. Nonlinear Anal., **7** (1983), 71–78.

[4] J. Mawhin and J. Ward. *Periodic solutions of second order forced Liénard differential equations at resonance*. Arch. Math., **41** (1983), 337–351.

[5] T.A. Burton and C.G. Townsend. *On the generalized Liénard equation with forcing term*. J. Differential Equations, **4** (1968), 620–633.

[6] A. Sandqvist and K.M. Andersen. *A necessary and sufficient conditions for the existence of a unique nontrivial periodic solution to a class of equation of Liénards type*. J. Differential Equations, **46** (1982), 356–378.

[7] P.N. Savel'ev. *The dissipativity of the generalized Liénard equation*. Differential Equations, **28** (1992), 794–800.

[8] M.A. Del Pino, R. Manásevich and A. Murua. *On the number of* 2π *periodic solutions for* $u'' + g(u) = s(1 + h(t))$ *using the Poincaré-Birkhoff theorem*. J. Differential Equations, **95** (1992), 240–258.

[9] C. Fabry, J. Mawhin and N. Nkashama. *A multiplicity result for periodic solution of forced nonlinear second order ordinary differential equations*. Bull. London Math. Soc., **18** (1986), 173–180.

[10] T. Ding, R. Iannacci and F. Zanolin. *Existence and multiplicity results for periodic solutions of semilinear Duffing equation*. J. Differential Equations, **105** (1993), 364–409.

[11] J.P. Gossez and P. Omari. *Periodic solutions of a second order ordinary differential equation: a necessary and sufficient condition for nonresonance*. J. Differential Equations, **94** (1991), 67–82.

[12] R. Reissig. *Contraction mappings and periodically perturbed nonconservative system*. Atti Accad. Naz. Lincei RC, **58** (1975), 696–702.

[13] M. Del Pino, R. Manásevich and A. Murua. *Existence and multiplicity of solutions with prescribed period for a second order quasilinear ordinary differential equation*. Nonlinear Anal., **18** (1992), 79–92.

[14] C. Fabry and D. Fayyad. *Periodic solutions of second order differential equations with a p-Laplacian and asymmetric nonlinearities*. Rend. Istit. Mat. Univ. Trieste, **24** (1992), 207–227.

[15] R. Manásevich and J. Mawhin. *Periodic solutions for nonlinear systems with p-Laplacian-like operators*. J. Differential Equations, **145** (1998), 367–393.

[16] Y. Wang and W. Ge. *Existence of periodic solutions for nonlinear differential equations with a p-Laplacian-like operator*. Appl. Math. Lett., **19** (2006), 241–259.

[17] M. del Pino, M. Elgueta and R. Manásevich. *A homotopic deformation along p of a Leray-Schauder degree result and existence for* $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$. J. Differential Equations, **80** (1989), 1–13.

[18] W. Sun and W. Ge. *The existence of solutions to Sturm-Liouville boundary value problems with Laplacian-like operator*. Acta Math. Appl. Sinica, **18** (2002), 314–348.

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