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# Explicit expressions for moments of gamma order statistics

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**Abstract.** Explicit closed form expressions are derived for the moments of order statistics from the gamma and generalized gamma distributions. The expressions involve the Lauricella functions of type A and type B. The usefulness of the result is illustrated through two quality control data sets.

**Keywords:** gamma distribution, generalized gamma distribution, Lauricella function of type A, Lauricella function of type B, order statistics, quality control.

**Mathematical subject classification:** 33C90, 62E99.

## 1 Introduction

Why do we need to care about moments of order statistics? Actually, moments of order statistics play an important role in such areas as quality control testing and reliability. If the reliability of an item is high, the duration of an "all items fail" life-test can be too expensive in both time and money. This fact prevents a practitioner from knowing enough about the product in a relatively short time. Therefore, a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictions are often based on moments of order statistics.

The gamma distribution is a popular model in such areas as life testing and quality control problems. Some recent applications of the gamma distribution in these areas are Cohen and Whitten (1986), Crowder and Hamilton (1992), Grego (1993), Lunani et al. (1997), Chang and Bai (2001), Nahar et al. (2001), Shapiro and Chen (2001), Stoumbos and Sullivan (2002), Christensen et al. (2003), Jearkpaporn et al. (2003), Pievatolo et al. (2003), Testik et al. (2003),

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Chang and Bai (2004), Phillips (2004), Ghilagaber (2005), Huang et al. (2005) and Robinson et al. (2006).

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the gamma distribution given by the probability density function (pdf):

$$f(x) = \frac{x^{\alpha-1}\exp(-x)}{\Gamma(\alpha)} \tag{1}$$

for $x > 0$ and $\alpha > 0$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. The moments of gamma order statistics are $E(X_{r:n}^k)$ for $k = 1, 2, \ldots$. Note that it is sufficient to consider the single parameter gamma distribution given by (1). This is so because if $Y_{1:n} < Y_{2:n} < \cdots < Y_{n:n}$ are the order statistics for a random sample $Y_1, Y_2, \ldots, Y_n$ from the two parameter gamma pdf

$$f(y) = \frac{y^{\alpha-1}\exp(-y/c)}{c^{\alpha}\Gamma(\alpha)}$$

then $E(Y_{r:n}^k) = c^k E(X_{r:n}^k)$ for all $r$, $n$ and $k$.

There has been a large amount of work relating to the moments of gamma order statistics. As far as we know, there are eight significant papers on the calculation of $E(X_{r:n}^k)$. In the earliest paper, Gupta (1960) derived a recurrence relation for $E(X_{r:n}^k)$ for integer values of the shape parameter $\alpha$. Gupta used this relation to tabulate values of $E(X_{r:n}^k)$ for various combinations of $k$, $n$ and $\alpha$. Gupta also discussed some illustrative applications to life-testing and reliability problems. Joshi (1979) re-derived the recurrence relation of Gupta (1960) and showed that if $E(X_{1:n}^k)$ for $k = -(r-1), \ldots, -1$ are known then one can obtain expressions for all of $E(X_{r:n}^k)$. Krishnaiah et al. (1967) extended the work of Gupta (1960) for the case that $\alpha$ is any positive real number. Breiter and Krishnaiah (1968) tabulated the values of $E(X_{r:n}^k)$, $k = 1, 2, 3, 4$ for various $\alpha$ obtained by using the recurrence relations in Krishnaiah et al. (1967). A Gauss-Legendre quadrature formula was used for the computations. Khan and Khan (1983) derived some recurrence relations for $E(X_{r:n}^k)$ when $f(\cdot)$ is the generalized gamma pdf given by

$$f(x) = \frac{cx^{c\alpha-1}\exp\left(-x^c\right)}{\Gamma(\alpha)} \tag{2}$$

for $x > 0, \alpha > 0$ and $c > 0$. Based on the available recurrence relations, Walter and Stitt (1988) constructed extensive tabulations of $E(X_{r:n}^k)$ for the gamma distribution. Sobel and Wells (1990) showed that $E(X_{r:n}^k)$ can be expressed in terms of Dirichlet integrals (integrals involving gamma functions) and provided a table for reading the Dirichlet integrals. Most recently, Abdelkader (2004) derived some recurrence relations for $E(X_{r:n}^k)$ when $X_1, X_2, \ldots, X_n$ are independent but not identically distributed gamma random variables. Abdelkader also discussed some applications in reliability.

As seen above, all of the work except for Sobel and Wells (1990) express $E(X_{r:n}^k)$ in terms of recurrence relations and/or numerical tables. That is, no explicit expressions are available for $E(X_{r:n}^k)$ except for the one given by Sobel and Wells (1990). The representation given in Sobel and Wells (1990) involves the Dirichlet integrals which are not well known and for which no standard routines are available. The use of the various numerical tables can be limited and highly inaccurate.

In this note, we derive explicit expressions for $E(X_{r:n}^k)$ that are finite sums of well known special functions – namely, the Lauricella function of type A (Exton, 1978) defined by

$$F_A^{(n)}(a, b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{a_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!} \tag{3}$$

and, the Lauricella function of type B (Exton, 1978) defined by

$$F_B^{(n)}(a_1, \ldots, a_n, b_1, \ldots, b_n; c; x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a_1)_{m_1}\cdots(a_n)_{m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{c_{m_1+\cdots+m_n}} \frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}, \tag{4}$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. Numerical routines for the direct computation of (3) and (4) are available, see, for example, Exton (1978) and Aarts (2000).

This note is outlined as follows. Section 2 derives explicit expressions for $E(X_{r:n}^k)$ when $X_1, X_2, \ldots, X_n$ is a random sample from (1). The extension of this result to non-identically distributed (NID) gamma random variables is considered in Section 3. Some further extensions when $X_1, X_2, \ldots, X_n$ is a sample from (2) are considered in Section 4. The use of these results for two data sets on quality testing is illustrated in Section 5.

## 2 IID Case

If $X_1, X_2, \ldots, X_n$ is a random sample from (1) then it is well known that the pdf of $Y = X_{r:n}$ is given by

$$f_Y(y) = \frac{n!}{(r-1)!(n-r)!} \{F(y)\}^{r-1} \{1 - F(y)\}^{n-r} f(y)$$

for $r = 1, 2, \ldots, n$, where $F(\cdot)$ is the cumulative distribution function (cdf) corresponding to (1) given by

$$F(y) = \frac{\gamma(\alpha, y)}{\Gamma(\alpha)}, \tag{5}$$

where $\gamma(\cdot, \cdot)$ denotes the incomplete gamma function defined by

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} \exp(-t)\, dt.$$

Thus, the $k$th moment of $X_{r:n}$ can be expressed as

$$\begin{aligned}
E\left(X_{r:n}^k\right) &= \frac{n!}{(r-1)!(n-r)!\{\Gamma(\alpha)\}^n} \\
&\quad \times \int_0^\infty y^{k+\alpha-1} \exp(-y) \{\gamma(\alpha, y)\}^{r-1} \{\Gamma(\alpha) - \gamma(\alpha, y)\}^{n-r} dy \\
&= \frac{n!}{(r-1)!(n-r)!\{\Gamma(\alpha)\}^n} \int_0^\infty y^{k+\alpha-1} \exp(-y) \\
&\quad \times \sum_{\ell=0}^{n-r} \binom{n-r}{\ell} \{\Gamma(\alpha)\}^{n-r-\ell} (-1)^\ell \{\gamma(\alpha, y)\}^{r+\ell-1} dy \\
&= \frac{n!}{(r-1)!(n-r)!} \sum_{\ell=0}^{n-r} (-1)^\ell \binom{n-r}{\ell} \{\Gamma(\alpha)\}^{-r-\ell} \\
&\quad \times \int_0^\infty y^{k+\alpha-1} \exp(-y) \{\gamma(\alpha, y)\}^{r+\ell-1} dy \\
&= \frac{n!}{(r-1)!(n-r)!} \sum_{\ell=0}^{n-r} (-1)^\ell \binom{n-r}{\ell} \{\Gamma(\alpha)\}^{-r-\ell} I(\ell).
\end{aligned} \tag{6}$$

Using the series expansion

$$\gamma(\alpha, x) = x^\alpha \sum_{m=0}^\infty \frac{(-x)^m}{(\alpha+m)m!},$$

the integral $I(\ell)$ in (6) can be expressed as

$$\begin{aligned}
I(\ell) &= \int_0^\infty y^{k+\alpha-1}\exp(-y)\left\{y^\alpha \sum_{m=0}^\infty \frac{(-y)^m}{(\alpha+m)m!}\right\}^{r+\ell-1} dy = \int_0^\infty \sum_{m_1=0}^\infty \cdots \\
&\quad \cdots \sum_{m_{r+\ell-1}=0}^\infty \frac{(-1)^{m_1+\cdots+m_{r+\ell-1}} y^{k+\alpha(r+\ell)+m_1+\cdots+m_{r+\ell-1}-1}\exp(-y)}{(\alpha+m_1)\cdots(\alpha+m_{r+\ell-1})m_1!\cdots m_{r+\ell-1}!} dy \\
&= \sum_{m_1=0}^\infty \cdots \sum_{m_{r+\ell-1}=0}^\infty \frac{(-1)^{m_1+\cdots+m_{r+\ell-1}}}{(\alpha+m_1)\cdots(\alpha+m_{r+\ell-1})m_1!\cdots m_{r+\ell-1}!} \\
&\quad \times \int_0^\infty y^{k+\alpha(r+\ell)+m_1+\cdots+m_{r+\ell-1}-1}\exp(-y)dy = \sum_{m_1=0}^\infty \cdots \\
&\quad \cdots \sum_{m_{r+\ell-1}=0}^\infty \frac{(-1)^{m_1+\cdots+m_{r+\ell-1}}\Gamma\left(k+\alpha(r+\ell)+m_1+\cdots+m_{r+\ell-1}\right)}{(\alpha+m_1)\cdots(\alpha+m_{r+\ell-1})m_1!\cdots m_{r+\ell-1}!}.
\end{aligned} \tag{7}$$

Using the fact $(f)_k = \Gamma(f+k)/\Gamma(f)$ and the definition in (3), one can reexpress (7) as

$$\begin{aligned}
I(\ell) &= \alpha^{1-r-\ell}\Gamma\big(k+\alpha(r+\ell)\big) \times F_A^{(r+\ell-1)}\big(k+\alpha(r+\ell), \alpha, \ldots, \alpha; \\
&\qquad \alpha+1, \ldots, \alpha+1; -1, \ldots, -1\big).
\end{aligned} \tag{8}$$

Combining (6) and (8), we obtain the expression

$$\begin{aligned}
E\left(X_{r:n}^k\right) &= \frac{n!}{(r-1)!(n-r)!}\sum_{\ell=0}^{n-r}(-1)^\ell \binom{n-r}{\ell}\{\Gamma(\alpha)\}^{-r-\ell}\alpha^{1-r-\ell}\Gamma\left(k+\alpha(r+\ell)\right) \\
&\quad \times F_A^{(r+\ell-1)}\left(k+\alpha(r+\ell), \alpha, \ldots, \alpha; \alpha+1, \ldots, \alpha+1; -1, \ldots, -1\right).
\end{aligned} \tag{9}$$

Note that (9) is a finite sum of the Lauricella function of type A, a function that can be computed directly, see Section 5.

## 3 NID Case

Suppose now that $X_1, X_2, \ldots, X_n$ are independent gamma random variables with the probability density functions (pdfs) given by

$$f_i(x) = \frac{x^{\alpha_i - 1}\exp(-x)}{\Gamma(\alpha_i)}$$

for $x > 0$ and $\alpha_i > 0$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. To find the $k$th moment of $X_{r:n}$, we use the following result due to Barakat and Abdelkader (2004):

$$E\left(X_{r:n}^k\right) = \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{n-r} I_j(k), \tag{10}$$

where

$$I_j(k) = k \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \int_0^\infty x^{k-1} \prod_{t=1}^{j} \left\{1 - F_{i_t}(x)\right\} dx, \tag{11}$$

where $F_{i_t}(\cdot)$ is the cdf of $X_{i_t}$ given by

$$F_{i_t}(x) = \frac{\gamma(\alpha_{i_t}, x)}{\Gamma(\alpha_{i_t})}.$$

Using the series expansion

$$1 - \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} = x^{\alpha-1} \exp(-x) \sum_{m=0}^{\infty} \frac{x^{-m}}{\Gamma(\alpha - m)},$$

one can express $I_j(k)$ in (11) as

$$\begin{aligned}
I_j(k) &= k \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \int_0^\infty x^{a-1} \exp(-jx) \prod_{t=1}^{j} \sum_{m=0}^{\infty} \frac{x^{-m}}{\Gamma(\alpha_{i_t} - m)} dx \\
&= k \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \int_0^\infty x^{a-1} \exp(-jx) \sum_{m_1=0}^{\infty} \cdots \\
&\quad \cdots \sum_{m_j=0}^{\infty} \frac{x^{-m_1-\cdots-m_j}}{\Gamma(\alpha_{i_1} - m_1) \cdots \Gamma(\alpha_{i_j} - m_j)} dx \\
&= k \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \sum_{m_1=0}^{\infty} \cdots \sum_{m_j=0}^{\infty} \frac{1}{\Gamma(\alpha_{i_1} - m_1) \cdots \Gamma(\alpha_{i_j} - m_j)} \\
&\quad \times \int_0^\infty x^{a-m_1-\cdots-m_j-1} \exp(-jx) dx \\
&= k \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \sum_{m_1=0}^{\infty} \cdots \sum_{m_j=0}^{\infty} \frac{\Gamma\left(a - m_1 - \cdots - m_j\right) j^{m_1+\cdots+m_j-a}}{\Gamma(\alpha_{i_1} - m_1) \cdots \Gamma(\alpha_{i_j} - m_j)},
\end{aligned} \tag{12}$$

where $a = k + (\alpha_{i_1} - 1) + \cdots + (\alpha_{i_j} - 1)$. Noting that

$$\Gamma(\alpha_{i_t} - m_t) = \frac{(-1)^{m_t}\Gamma(\alpha_{i_t})}{(1-\alpha_{i_t})_{m_t}},$$

$$\Gamma(a - m_1 - \cdots - m_j) = \frac{(-1)^{m_1+\cdots+m_j}\Gamma(a)}{(1-a)_{m_1+\cdots+m_j}}$$

and the definition in (4), one can reexpress (12) as

$$I_j(k) = k\frac{\Gamma(a)\, F_B^{(j)}\left(1,\ldots,1,1-\alpha_{i_1},\ldots,1-\alpha_{i_j};1-a;j,\ldots,j\right)}{\Gamma\left(\alpha_{i_1}\right)\cdots\Gamma\left(\alpha_{i_j}\right) j^a}$$

and hence (10) can be rewritten as

$$\begin{aligned} E\left(X_{r:n}^k\right) &= k\sum_{j=n-r+1}^{n}(-1)^{j-n+r-1}\binom{j-1}{n-r}\frac{\Gamma(a)}{\Gamma\left(\alpha_{i_1}\right)\cdots\Gamma\left(\alpha_{i_j}\right) j^a} \\ &\quad\times F_B^{(j)}\left(1,\ldots,1,1-\alpha_{i_1},\ldots,1-\alpha_{i_j};1-a;j,\ldots,j\right). \end{aligned} \tag{13}$$

Note that (13) is a finite sum of the Lauricella function of type B, a function that can be computed directly, see Section 5.

## 4 Generalization

A natural extension of the results in Sections 2 and 3 is to consider the moments of order statistics for the generalized gamma distribution in (2). Similar calculations show that (9) generalizes to

$$\begin{aligned} E\left(X_{r:n}^k\right) &= \frac{n!}{(r-1)!(n-r)!}\sum_{\ell=0}^{n-r}(-1)^\ell\binom{n-r}{\ell}\{\Gamma(\alpha)\}^{-r-\ell}\alpha^{1-r-\ell}\Gamma\left(\frac{k}{c}+\alpha(r+\ell)\right) \\ &\quad\times F_A^{(r+\ell-1)}\left(\frac{k}{c}+\alpha(r+\ell),\alpha,\ldots,\alpha;\alpha+1,\ldots,\alpha+1;-1,\ldots,-1\right) \end{aligned}$$

and that (13) generalizes to

$$\begin{aligned} E\left(X_{r:n}^k\right) &= \frac{k}{c}\sum_{j=n-r+1}^{n}(-1)^{j-n+r-1}\binom{j-1}{n-r}\frac{\Gamma(a)}{\Gamma\left(\alpha_{i_1}\right)\cdots\Gamma\left(\alpha_{i_j}\right) j^a} \\ &\quad\times F_B^{(j)}\left(1,\ldots,1,1-\alpha_{i_1},\ldots,1-\alpha_{i_j};1-a;j,\ldots,j\right), \end{aligned}$$

where $a = k/c + (\alpha_{i_1} - 1) + \cdots + (\alpha_{i_j} - 1)$. For the derivation of the second expression, we have assumed that $X_1, X_2, \ldots, X_n$ are independent gamma random variables from (2) with non-identical $\alpha_i$, $i = 1, 2, \ldots, n$ and common $c$.

The derivation of an explicit expression for $E(X_{r:n}^k)$ for the case of non-identical $\alpha_i$, $i = 1, 2, \ldots, n$ and non-identical $c_i$, $i = 1, 2, \ldots, n$ is an open problem.

## 5 Application

We illustrate the use of (9) through two published data sets on quality control. The first data set shown in Table 1 is taken from Xie et al. (2006). The data set is from the testing process on a middle-size software project. The data contains fault correction times for seventeen weeks. The second data set shown in Table 2 is taken from Nichols and Padgett (2006). The data set relates to a process producing carbon fibers to be used in constructing fibrous composite materials. The data contains tensile strength of carbon fibers of 50 mm in length.

| Week | Correction time $x$ |
|---|---|
| 1 | 3 |
| 2 | 0 |
| 3 | 9 |
| 4 | 20 |
| 5 | 21 |
| 6 | 25 |
| 7 | 11 |
| 8 | 9 |
| 9 | 9 |
| 10 | 2 |
| 11 | 4 |
| 12 | 7 |
| 13 | 5 |
| 14 | 2 |
| 15 | 0 |
| 16 | 8 |
| 17 | 8 |

Table 1: Fault correction times.

We fitted the gamma distribution given by (1) to both data sets. Because (1) is in the standard form, we transformed the data on $x_i$ and $y_i$ to $\bar{x}x_i/s_x^2$ and $\bar{y}y_i/s_y^2$, respectively, where $(\bar{x}, \bar{y})$ and $(s_x^2, s_y^2)$ the sample means and sample variances, respectively, for the two data sets. The fitting of (1) to the transformed data was performed by the method of maximum likelihood. The following estimates were obtained: $\hat{\alpha} = 1.607$ for data set 1 and $\hat{\alpha} = 6.625$ for data set 2. Note that for

| | | | | |
|---|---|---|---|---|
| 3.70 | 2.74 | 2.73 | 2.50 | 3.60 |
| 3.11 | 3.27 | 2.87 | 1.47 | 3.11 |
| 4.42 | 2.41 | 3.19 | 3.22 | 1.69 |
| 3.28 | 3.09 | 1.87 | 3.15 | 4.90 |
| 3.75 | 2.43 | 2.95 | 2.97 | 3.39 |
| 2.96 | 2.53 | 2.67 | 2.93 | 3.22 |
| 3.39 | 2.81 | 4.20 | 3.33 | 2.55 |
| 3.31 | 3.31 | 2.85 | 2.56 | 3.56 |
| 3.15 | 2.35 | 2.55 | 2.59 | 2.38 |
| 2.81 | 2.77 | 2.17 | 2.83 | 1.92 |
| 1.41 | 3.68 | 2.97 | 1.36 | 0.98 |
| 2.76 | 4.91 | 3.68 | 1.84 | 1.59 |
| 3.19 | 1.57 | 0.81 | 5.56 | 1.73 |
| 1.59 | 2.00 | 1.22 | 1.12 | 1.71 |
| 2.17 | 1.17 | 5.08 | 2.48 | 1.18 |
| 3.51 | 2.17 | 1.69 | 1.25 | 4.38 |
| 1.84 | 0.39 | 3.68 | 2.48 | 0.85 |
| 1.61 | 2.79 | 4.70 | 2.03 | 1.80 |
| 1.57 | 1.08 | 2.03 | 1.61 | 2.12 |
| 1.89 | 2.88 | 2.82 | 2.05 | 3.65 |

Table 2: Breaking stress of carbon fibers $y$ (Gba).

data set 1 two of the observations are zero. These observations were removed before the fitting.

We examined the goodness of the fits by plotting the transformed data versus expected order statistics under the gamma distribution, i.e. plot the transformed data from Table 1 versus $E(X_{r:15})$ for $r = 1, 2, \ldots, 15$ (Fig. 1) and the transformed data from Table 2 versus $E(X_{r:100})$ for $r = 1, 2, \ldots, 100$ (Fig. 2). The fits appear reasonable. Thus, the gamma distribution can be used to predict, say, the extremes of the relevant distribution. For instance, the 1% and 99% percentiles (in the case of data set 1, the smallest and the second largest fault correction times if the software had been tested 100 times) can be estimated by $E(X_{1:100})$ and $E(X_{99:100})$, respectively. Tables 3 and 4 give the values of $E(X_{1:10^i})$ and $E(X_{10^i-1:10^i})$ computed using (9) for $\hat{\alpha} = 1.607$ and $\hat{\alpha} = 6.625$, respectively.

The expected order statistics $E(X_{r:15})$ for $r = 1, 2, \ldots, 15$ and $E(X_{r:100})$ for $r = 1, 2, \ldots, 100$ used in Figures 1 and 2 were computed using both (9) and

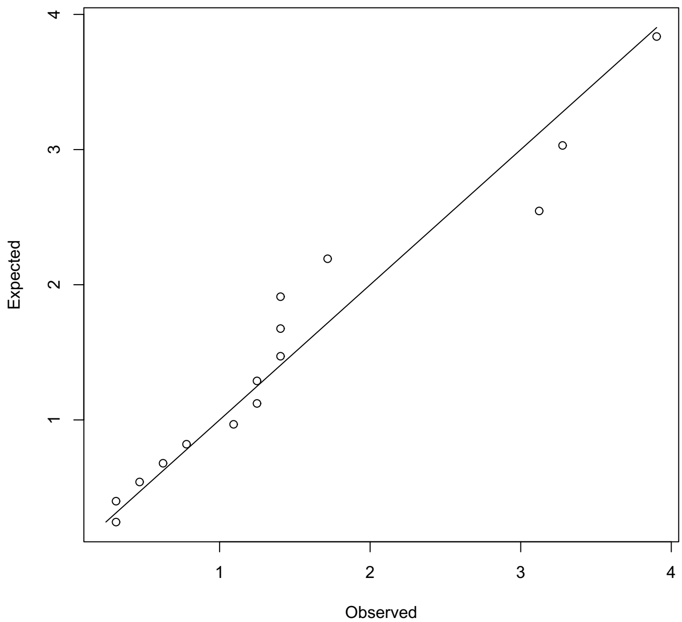


Figure 1: Sorted values of the transformed fault correction time data (in Table 1) versus the expected $E(X_{r:15})$, $r = 1, 2, \ldots, 15$ computed using (9).

| $i$ | $E(X_{1:10^i})$ | $E(X_{10^i-1:10^i})$ |
|---|---|---|
| 1 | 0.300 | 2.833 |
| 2 | 0.066 | 5.419 |
| 3 | 0.015 | 7.921 |
| 4 | 0.004 | 10.372 |
| 5 | 0.001 | 12.792 |
| 6 | 0.000 | 15.193 |

Table 3: Predictions for fault correction time.

| $i$ | $E(X_{1:10^i})$ | $E(X_{10^i-1:10^i})$ |
|---|---|---|
| 1 | 3.285 | 9.217 |
| 2 | 1.959 | 13.329 |
| 3 | 1.261 | 16.842 |
| 4 | 0.842 | 20.069 |
| 5 | 0.574 | 23.128 |
| 6 | 0.396 | 26.075 |

Table 4: Predictions for breaking stress.

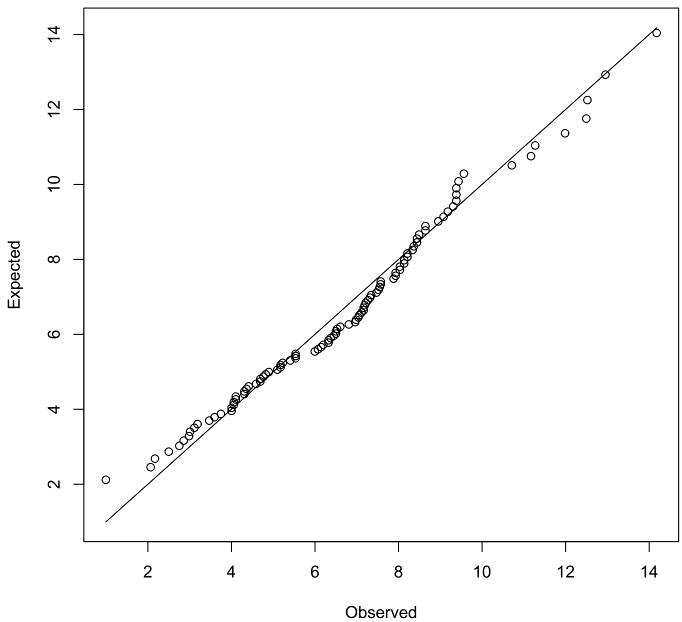


Figure 2: Sorted values of the transformed breaking stress data (in Table 2) versus the expected $E(X_{r:100})$, $r = 1, 2, \ldots, 100$ computed using (9).

the integration formula given by

$$E\left(X_{r:n}\right) = \frac{n!}{(r-1)!(n-r)!} \int_0^\infty y \left\{F(y)\right\}^{r-1} \left\{1 - F(y)\right\}^{n-r} f(y), \qquad (14)$$

where $f(\cdot)$ and $F(\cdot)$ are given by (1) and (5), respectively. The CPU times in seconds taken for 100 computations of (9) and (14) are shown in Tables 5 and 6 (so, 0.110 is the time taken for 100 computations of $E(X_{1:15})$ using (14), 0.080 is the time taken for 100 computations of $E(X_{1:15})$ using (9), and so on). The computations were performed using a software supplied by Exton under a Windows XP 2000 operating system. It is evident that the time taken to compute (9) is consistently smaller. Thus, besides being explicit, the use of (9) is more efficient.

## 6 Conclusions

We have derived explicit expressions for moments of gamma order statistics and generalized gamma order statistics as finite sums of well known special functions.

| $r$ | using (14) | using (9) |
|---|---|---|
| 1 | 0.110 | 0.080 |
| 2 | 0.080 | 0.055 |
| 3 | 0.110 | 0.090 |
| 4 | 0.110 | 0.107 |
| 5 | 0.140 | 0.073 |
| 6 | 0.100 | 0.075 |
| 7 | 0.140 | 0.081 |
| 8 | 0.110 | 0.038 |
| 9 | 0.140 | 0.018 |
| 10 | 0.140 | 0.071 |
| 11 | 0.140 | 0.017 |
| 12 | 0.110 | 0.038 |
| 13 | 0.150 | 0.141 |
| 14 | 0.140 | 0.051 |
| 15 | 0.140 | 0.114 |

Table 5: CPU times to compute $E(X_{r:15})$.

We have illustrated the efficiency of these expressions by means of two quality control data sets.

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| $r$ | using (14) | using (9) | $r$ | using (14) | using (9) |
|---|---|---|---|---|---|
| 1 | 0.160 | 0.145 | 38 | 0.210 | 0.017 |
| 2 | 0.150 | 0.072 | 39 | 0.190 | 0.169 |
| 3 | 0.180 | 0.019 | 40 | 0.220 | 0.199 |
| 4 | 0.190 | 0.086 | 41 | 0.220 | 0.000 |
| 5 | 0.220 | 0.075 | 42 | 0.260 | 0.158 |
| 6 | 0.190 | 0.098 | 43 | 0.250 | 0.121 |
| 7 | 0.220 | 0.076 | 44 | 0.290 | 0.003 |
| 8 | 0.220 | 0.200 | 45 | 0.280 | 0.153 |
| 9 | 0.170 | 0.029 | 46 | 0.280 | 0.245 |
| 10 | 0.250 | 0.232 | 47 | 0.310 | 0.294 |
| 11 | 0.230 | 0.159 | 48 | 0.270 | 0.221 |
| 12 | 0.200 | 0.043 | 49 | 0.280 | 0.139 |
| 13 | 0.200 | 0.169 | 50 | 0.280 | 0.231 |
| 14 | 0.250 | 0.106 | 51 | 0.290 | 0.060 |
| 15 | 0.220 | 0.171 | 52 | 0.290 | 0.006 |
| 16 | 0.200 | 0.007 | 53 | 0.270 | 0.025 |
| 17 | 0.180 | 0.157 | 54 | 0.280 | 0.208 |
| 18 | 0.220 | 0.078 | 55 | 0.320 | 0.281 |
| 19 | 0.190 | 0.147 | 56 | 0.330 | 0.031 |
| 20 | 0.220 | 0.159 | 57 | 0.280 | 0.112 |
| 21 | 0.220 | 0.052 | 58 | 0.280 | 0.041 |
| 22 | 0.190 | 0.005 | 59 | 0.330 | 0.046 |
| 23 | 0.210 | 0.148 | 60 | 0.310 | 0.063 |
| 24 | 0.220 | 0.011 | 61 | 0.330 | 0.214 |
| 25 | 0.220 | 0.123 | 62 | 0.280 | 0.263 |
| 26 | 0.180 | 0.179 | 63 | 0.300 | 0.112 |
| 27 | 0.200 | 0.000 | 64 | 0.320 | 0.166 |
| 28 | 0.190 | 0.033 | 65 | 0.300 | 0.211 |
| 29 | 0.190 | 0.007 | 66 | 0.300 | 0.079 |
| 30 | 0.180 | 0.113 | 67 | 0.300 | 0.104 |
| 31 | 0.200 | 0.076 | 68 | 0.280 | 0.116 |
| 32 | 0.190 | 0.010 | 69 | 0.300 | 0.161 |
| 33 | 0.200 | 0.055 | 70 | 0.280 | 0.191 |
| 34 | 0.200 | 0.168 | 71 | 0.300 | 0.113 |
| 35 | 0.190 | 0.019 | 72 | 0.290 | 0.053 |
| 36 | 0.200 | 0.085 | 73 | 0.290 | 0.183 |
| 37 | 0.180 | 0.167 | 74 | 0.280 | 0.263 |

Table 6: CPU times to compute $E(X_{r:100})$.

| $r$ | using (14) | using (9) | $r$ | using (14) | using (9) |
|---|---|---|---|---|---|
| 75 | 0.280 | 0.197 | 88 | 0.250 | 0.172 |
| 76 | 0.270 | 0.242 | 89 | 0.250 | 0.196 |
| 77 | 0.280 | 0.017 | 90 | 0.270 | 0.097 |
| 78 | 0.310 | 0.212 | 91 | 0.250 | 0.005 |
| 79 | 0.300 | 0.267 | 92 | 0.220 | 0.006 |
| 80 | 0.220 | 0.076 | 93 | 0.220 | 0.067 |
| 81 | 0.270 | 0.102 | 94 | 0.220 | 0.100 |
| 82 | 0.270 | 0.247 | 95 | 0.250 | 0.133 |
| 83 | 0.270 | 0.058 | 96 | 0.270 | 0.029 |
| 84 | 0.230 | 0.006 | 97 | 0.250 | 0.183 |
| 85 | 0.260 | 0.010 | 98 | 0.250 | 0.007 |
| 86 | 0.250 | 0.182 | 99 | 0.250 | 0.198 |
| 87 | 0.270 | 0.189 | 100 | 0.220 | 0.112 |

Table 6 (continuation): CPU times to compute $E(X_{r:100})$.

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