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Generators for semigroup of Lipman

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Abstract. There is a correspondence between functions in the maximal ideal of the local ring of a rational singularity and certain positive divisors supported on the exceptional fiber of a resolution of the singularity. Here we give an algorithm to obtain a generating set over $\mathbb{Z}$ of these divisors.

Keywords: rational surface singularities, positive divisors, toric varieties.

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1 Introduction

An isolated singularity of a complex surface is called rational if the geometric genus of the surface does not change under a resolution of the singularity. Here we are interested in the functions in the maximal ideal of the local ring of a rational singularity. There is a correspondence between these functions and certain positive divisors supported on the exceptional fiber of a resolution of the singularity (see [2] or [7]). The set of such divisors forms a semigroup, called the *semigroup of Lipman*.

In this work, we give an algorithm to obtain a minimal generating set over $\mathbb{Z}$ of the semigroup of Lipman by using the construction of a toric variety corresponding to a given semigroup of Lipman. This algorithm works not only for rational singularities but also for other type of singularities. The motivation for such a work comes from a deep connection between the elements of the semigroup of Lipman and topological invariants of the corresponding singularity. For instance, these divisors can be used to calculate Seiberg–Witten invariants of the plumbed manifold corresponding to a singularity (see [8]), also to read the open book structure of this manifold (see [3], [1]).

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2 Semigroup of Lipman

Let S be a sufficiently small representative of a germ $(S, 0)$ of a normal analytic surface singularity embedded in $\mathbb{C}^N$. A resolution of S is a complex analytic surface X and a proper holomorphic map $\pi : X \longrightarrow S$ such that its restriction to $X - \pi^{-1}(0)$ is a biholomorphic map and $X - \pi^{-1}(0)$ is dense in X. By the Main Theorem of Zariski, the exceptional fiber $E := \pi^{-1}(0)$ is a connected curve. Let us denote by $E_1, \cdots, E_n$ its irreducible components and by $M(E) = (a_{ij})$ its associated intersection matrix, where a_{ii} is the self-intersection of E_i, and a_{ij} is the number of the intersection points of E_i and E_j. It is well known that $M(E)$ is negative definite.

Let $\mathcal{G}$ denotes the set of divisors supported on E with integer coefficients:

$$\mathcal{G} = \left\{ \sum_{i=1}^{n} m_i E_i \mid m_i \in \mathbb{Z} \right\}.$$

When $m_i \in \mathbb{N}$ the elements of $\mathcal{G}$ are called *positive divisors* supported on E. As in [7], consider the set

$$\mathcal{E}^+(E) = \left\{ Y \in \mathcal{G} \mid (Y \cdot E_i) \leq 0 \quad \text{for all } i \right\}.$$

In [11], Zariski proves that the subset $\mathcal{E}^+(E)$ of $\mathcal{G}$ is not empty. Furthermore, for any element $Y = \sum m_i E_i$ in $\mathcal{E}^+(E)$, it is easy to see that the inequality $(Y \cdot E_i) \leq 0$ implies that $m_i \geq 1$ for all i. Moreover, $\mathcal{E}^+(E)$ is a semigroup, called the *semigroup of Lipman*. A partial order on $\mathcal{E}^+(E)$ is defined as follows: Given two elements

$$Y_1 = \sum_{i=1}^{n} a_i E_i \quad \text{and} \quad Y_2 = \sum_{i=1}^{n} b_i E_i$$

of $\mathcal{E}^+(E)$, we write $Y_1 \leq Y_2$ if $a_i \leq b_i$ for all i. The smallest element of $\mathcal{E}^+(E)$ can be calculated by Laufer's algorithm (see [5], 4.1). By using a similar process, we can find all other elements in $\mathcal{E}^+(E)$ (see [9, 10]).

Theorem 2.1 (see [2]). *A singularity of a normal analytic surface in $\mathbb{C}^N$ is rational if and only if the arithmetic genus of the smallest element in $\mathcal{E}^+(E)$ associated to a resolution of the singularity vanishes.*

Then we obtain:

Corollary 2.2 (see [2]). *The exceptional fiber of any resolution of a rational singularity is a normal crossing divisor, with each E_i nonsingular and of genus zero, and any two distinct components intersecting transversally in at most one point.*

A proof of this corollary can be also found in [10].

Generally speaking, we are interested in the semigroup of Lipman associated to the exceptional fiber of a resolution of a rational singularity. Nevertheless the same algorithm also holds for nonrational singularities.

3 Generators of the semigroup of Lipman

For any element $Y' = \sum_{i=1}^{n} m_i E_i$ in $\mathcal{G}$ we have

$$M(E) \cdot \big(m_1, m_2, \dots, m_n\big)^t = \big(y_1, y_2, \dots, y_n\big)^t.$$

This says that $(Y' \cdot E_i) = y_i$. Let us denote by δ_i the column matrix with coefficients 0 everywhere except in the i-th row, where the entry is -1, and consider $M(E) \cdot (m_{i1}, m_{i2}, \dots, m_{in})^t = \delta_i$. We have $m_{ij} \in \mathbb{Q}^+$. Set $F_i = \sum_{j=1}^{n} m_{ij} E_j$. Then we can write F_i as $F_i = k_i \cdot F_i{}'$ for some $k_i \in \mathbb{Q}^+$ such that the coordinates of the $F_i{}'$ are positive integers and relatively prime. Hence $F_i{}'$ is an element of $\mathcal{E}^+(E)$.

Definition 3.1. *Let $U^+(E) = \{D_1, \dots, D_n\}$ be a subset of $\mathcal{E}^+(E)$ such that each element in $\mathcal{E}^+(E)$ can be written as a linear combination of D_i's with coefficients in $\mathbb{Z}^+$ (in $\mathbb{Q}^+$). The elements of $U^+(E)$ are called $\mathbb{Z}$-generators (resp. $\mathbb{Q}$-generators) of $\mathcal{E}^+(E)$.*

Remark 3.2. Notice that the F_i' generate $\mathcal{E}^+(E)$ over $\mathbb{Q}^+$.

3.1 $\mathbb{Z}$-generators of the semigroup of Lipman

We present an algorithm to find the minimal generators over $\mathbb{Z}$ of the semigroup of Lipman.

Algorithm. To determine generators over $\mathbb{Z}$ of the semigroup of Lipman, we consider $\mathcal{E}^+(E)$ as $\mathcal{E}^+(E) = \sigma^\vee \cap N$, where $\sigma^\vee$ is the rational polyhedral cone generated by the F_i, and N is the lattice $\mathcal{G}$. The ring $S(E) = \mathbb{C}[\mathcal{E}^+(E)]$ is called the semigroup ring associated to the exceptional fiber E of a resolution of a rational singularity. We can now reduce our problem to one of determining the ring structure of this semigroup ring.

Let $N = \langle E_1, \ldots, E_n \rangle$ be a lattice generated by the components of E and $M = \langle E_1^*, \ldots, E_n^* \rangle$ be the dual lattice of N with dual pairing denoted by $\langle , \rangle$ such that $\langle E_i, E_j^* \rangle = 1$ if $i = j$, 0 otherwise. We denote $N \otimes_{\mathbb{Z}} \mathbb{R}$ by $N_{\mathbb{R}}$ and the dual space $M \otimes_{\mathbb{Z}} \mathbb{R}$ by $M_{\mathbb{R}}$. Let $\sigma^\vee$ be the rational polyhedral cone defined by $\mathcal{E}^+(E)$ in $N_{\mathbb{R}}$. The semigroup $\sigma^\vee \cap N$ is finitely generated. The associated semigroup ring $S(E)$ is $\mathbb{C}[\sigma^\vee \cap N]$.

We consider $F_1, \ldots, F_n$ as above. Let $N' = \langle F_1, \ldots, F_n \rangle$ be a lattice and M' be the dual lattice of N' generated by $F_1^*, \ldots, F_n^*$ such that $\langle F_i, F_j^* \rangle = 1$ if $i = j$, and 0 otherwise. We obtain:

Proposition 3.3. *The dual lattice M' of N' is generated by the rows of $M(E)$ multiplied by -1.*

Proof. The proof follows from the construction of the F_i. More explicitly, let $F_i = \sum_{j=1}^n a_{ij} E_j$ such that $M(E)(a_{i1}, a_{i2}, \ldots, a_{in})^T = \delta_i$, where δ_i is a $n \times 1$ matrix with i-th row -1 and other entries 0. Since $M(E)$ is invertible we can write

$$(a_{i1}, \ldots, a_{in}) = \frac{1}{\det M(E)} (b_{i1}, \ldots, b_{in})$$

for some b_{ij}. From the following condition $\langle F_i, F_j^* \rangle = 1$ if $i = j$, 0 otherwise we obtain $(c_{ij})(a_{i1}, \ldots, a_{in})^T = -\delta_i$, where

$$F_i^* = \sum_{j=1}^n c_{ij} E_j^*.$$

Thus $(c_{ij}) = -M(E)$.

The lattice M' is a sublattice of finite index of M and $\sigma^\vee \cap N \subset \sigma^\vee \cap N'$. Then we have the following Proposition:

Proposition 3.4. *With the preceding notation,* $\mathbb{C}[\sigma^\vee \cap N] = \mathbb{C}[N']^{\frac{M}{M'}}$.

Proof. See page 34 in Fulton [4].

Remark 3.5. The affine variety $\operatorname{Spec} \mathbb{C}[\sigma^\vee \cap N]$ has only quotient singularities.

To explicitly find the associated semigroup ring $\mathbb{C}[\sigma^\vee \cap N]$, we first determine the ring $\mathbb{C}[N']$ and then the ring of invariants $\mathbb{C}[N']^{\frac{M}{M'}}$ under the $\frac{M}{M'}$-action. As examples, we determine $\mathbb{Z}$-generators of the semigroup of Lipman for simple singularities, a determinantal singularity and a minimally elliptic singularity.

3.2 Examples

Example 3.6. Let E be the exceptional divisor of the minimal resolution of an A_{n-1}-type singularity. We have $E_i \cdot E_j = 1$ if $|j-i| = 1$, -2 if $i = j$ and 0 otherwise. The F_i for $\mathcal{E}^+(E)$ are

$$F_i = \frac{1}{n}\big[(n-i)E_1+2(n-i)E_2+\cdots+i(n-i)E_i+i(n-i-1)E_{i+1}+\cdots+iE_{n-1}\big].$$

Note that the semigroup of Lipman $\mathcal{E}^+(E)$ (respectively the semigroup ring $S(E)$) for a given singularity of type A_{n-1} is sometimes denoted by $\mathcal{E}^+(A_{n-1})$ or $\mathcal{E}^+(\Gamma)$ (respectively $S(A_{n-1})$ or $S(\Gamma)$), where Γ is the dual graph of given singularity. The same notation also applies to other type of singularities.

We denote the ring $\mathbb{C}[N]$ by $\mathbb{C}[x_1, \ldots, x_{n-1}]$. Thus we can write $\mathbb{C}[N'] = \mathbb{C}[u_1, \cdots, u_{n-1}]$, where

$$u_i = x_1^{\frac{(n-i)}{n}} x_2^{\frac{2(n-i)}{n}} \cdots x_i^{\frac{i(n-i)}{n}} x_{i+1}^{\frac{i(n-i-1)}{n}} \cdots x_{n-1}^{\frac{i}{n}}.$$

On the other hand, the finite group $\frac{M}{M'}$ can be described as follows:

Proposition 3.7. *The group $\frac{M}{M'}$ is a cyclic group of order n generated by ϵ for a singularity of type A_{n-1}, where ϵ is a primitive nth root of unity.*

Proof. Let $\overline{F} = F + M' \in M'$ where $F = b_1E_1^* + \cdots + b_{n-1}E_{n-1}^* \in M$. Then $F \in M'$. In other words, there exist integers a_i such that

$$b_1E_1^* + \cdots + b_{n-1}E_{n-1}^* = a_1F_1^* + \cdots + a_{n-1}F_{n-1}^*.$$

Since the F_i^* can be written in terms of the E_i^*, we can solve the a_i in terms of the b_i. More explicitly, we have the following system:

$$a_i = F_i b^t \tag{1}$$

for $i = 1, \ldots, n-1$, where $b = (b_1, b_2, \ldots, b_{n-1})$. If we set $b_i = 0$ for $i \neq 1$, then we can observe that the smallest integer for b_1 satisfying the system (1) should be n. This means that ord $\overline{E_1^*} = n$. Similarly, it can be shown that ord $\overline{E_{n-1}} = n$. It suffices to solve the system (1) for $i = 1$. In other words, the system (1) is equivalent to the following equation

$$a_1 = \frac{1}{n}\big((n-1)b_1 + (n-2)b_2 + \cdots + b_{n-1}\big).$$

From here we can observe that

$$2E_1^* - E_2^*, 3E_1^* - E_3^*, \ldots, (n-2)E_1^* - E_{n-2}, E_1^* + E_{n-1}^* \in M'.$$

Hence M/M' is generated by $\overline{E_1^*}$. Setting $\epsilon = \overline{E_1^*}$, the result follows.

Any element $Y = \sum_{i=1}^{n-1} a_i E_i \in \mathcal{E}^+(A_{n-1})$ should satisfy $M(E)a^t = -b^t$ for some $b_i \geq 0$ where $a = (a_1, a_2 \ldots, a_{n-1})$ and $b = (b_1, b_2 \ldots, b_{n-1})$. This condition is equivalent to $(n-1)b_1 + (n-2)b_2 + \ldots + 2b_{n-2} + b_{n-1} \in n\mathbb{Z}$. Thus we obtain $S(A_{n-1}) = \mathbb{C}[u_1, \ldots, u_{n-1}]^{\langle \epsilon \rangle}$, where the action

$$\epsilon \cdot u_1^{b_1} u_2^{b_2} \cdots u_{n-1}^{b_{n-1}} = \epsilon^{(n-1)b_1 + (n-2)b_2 + \cdots + 2b_{n-2} + b_{n-1}} u_1^{b_1} u_2^{b_2} \cdots u_{n-1}^{b_{n-1}}.$$

For instance, in the case of A_3, the F_i for $\mathcal{E}^+(A_3)$ are

$$F_1 = \frac{1}{4}(3, 2, 1), \; F_2 = \frac{1}{4}(2, 4, 2), \; F_3 = \frac{1}{4}(1, 2, 3).$$

The group $\frac{M}{M'}$ is a cyclic group generated by an element ϵ of order 4. The ring of invariants under the action $\epsilon \cdot u_1^{b_1} u_2^{b_2} u_3^{b_3} = \epsilon^{3b_1 + 2b_2 + b_3} u_1^{b_1} u_2^{b_2} u_3^{b_3}$ should satisfy the following condition $3b_1 + 2b_2 + b_3 \equiv 0 \mod 4$. Therefore the ring of invariants is generated by monomials

$$u_1^4, \; u_2^2, \; u_3^4, \; u_1 u_3, \; u_1^2 u_2, \; u_3^2 u_2.$$

In other words, the corresponding $\mathbb{Z}$-generators of $\mathcal{E}^+(E)$ are

$$(3, 2, 1), \; (1, 2, 1), \; (1, 2, 3), \; (1, 1, 1), \; (2, 2, 1), \; (1, 2, 2).$$

Example 3.8. Let E be the exceptional divisor of the minimal resolution of a D_n-type singularity for $n \geq 4$. The intersection matrix $M(D_n)$ is defined by $(E_i \cdot E_j) = 1$ if $j = i+1$ for $i = 1, \ldots, n-2$, $(E_{n-2} \cdot E_n) = 1$ and $(E_i \cdot E_j) = 0$ otherwise. We have $\det M(D_n) = 4$ when n is even; $\det M(D_n) = -4$ when n is odd. Then the F_i for $\mathcal{E}^+(E)$ are the following:

$$\begin{aligned} F_i &= E_1 + 2E_2 + \cdots + (i-1)E_{i-1} + i(E_i + E_{i+1} + \cdots + E_{n-2}) \\ &\quad + \frac{i}{2}\big(E_{n-1} + E_n\big) \quad \text{for} \quad i \leq n-2, \\ F_{n-1} &= \frac{1}{2}\big(E_1 + 2E_2 + \cdots + (n-2)E_{n-2}\big) + \frac{n}{4}E_{n-1} + \frac{(n-2)}{4}E_n, \\ F_n &= \frac{1}{2}\big(E_1 + 2E_2 + \cdots + (n-2)E_{n-2}\big) + \frac{(n-2)}{4}E_{n-1} + \frac{n}{4}E_n. \end{aligned}$$

We let $\mathbb{C}[N] = \mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[N'] = \mathbb{C}[u_1, \ldots, u_n]$, where

$$\begin{aligned}
u_i &= x_1 x_2^2 \cdots x_{i-1}^{i-1} x_i^i x_{i+1}^i \cdots x_{n-2}^i x_{n-1}^{\frac{i}{2}} x_n^{\frac{i}{2}} \quad \text{for} \quad i \leq n-2,\\
u_{n-1} &= x_1^{\frac{1}{2}} x_2^{\frac{2}{2}} \cdots x_{n-2}^{\frac{(n-2)}{2}} x_{n-1}^{\frac{n}{4}} x_n^{\frac{(n-2)}{4}},\\
u_n &= x_1^{\frac{1}{2}} x_2^{\frac{2}{2}} \cdots x_{n-2}^{\frac{(n-2)}{2}} x_{n-1}^{\frac{(n-2)}{4}} x_n^{\frac{n}{4}}.
\end{aligned}$$

The finite group $\frac{M}{M'}$ is more complicated to describe explicitly in the case of D_n than in the case of A_n.

Let $\overline{F} = F + M' \in M'$ where $F = b_1 E_1^* + \cdots + b_n E_n^* \in M$. Then $F \in M'$. In other words, there exist integers a_i such that

$$b_1 E_1^* + \cdots + b_n E_n^* = a_1 F_1^* + \cdots + a_n F_n^*.$$

Since the F_i^* can be written in terms of the E_i^*, we can solve the a_i in terms of the b_i as in the case of the proof of Proposition 3.7. More explicitly, we have the following system:

$$a_i = F_i b^t \tag{2}$$

for $i = 1, \ldots, n$, where $b = (b_1, b_2, \ldots, b_n)$.

Proposition 3.9.

(a) *If n is odd, then the ring of invariants is generated by the following:*

$$\begin{gathered}
u_1^2, u_2, u_3^2, u_4, \ldots, u_{n-4}^2, u_{n-3}, u_{n-2}^2, u_{n-1}^4, u_n^4,\\
u_i u_j \ \text{ for } \ i, j = 1, 3, \ldots, n-2 \ \text{ and } \ i \neq j, u_{n-1} u_n,\\
u_i u_{n-1}^2 \ \text{ for } \ i = 1, 3, \ldots, n-2, u_i u_n^2 \ \text{ for } \ i = 1, 3, \ldots, n-2.
\end{gathered}$$

(b) *If n is even, then the ring of invariants is generated by the following:*

$$\begin{gathered}
u_1^2, u_2, u_3^2, u_4, \ldots, u_{n-4}, u_{n-3}^2, u_{n-2}, u_{n-1}^2, u_n^2,\\
u_i u_j \ \text{ for } \ i, j = 1, 3, \ldots, n-3 \ \text{ and } \ i \neq j,\\
u_i u_{n-1} u_n \ \text{ for } \ i = 1, 3, \ldots, n-3.
\end{gathered}$$

Proof. Assume that (a) holds. From the system of the equations (2), if we set $b_i = 0$ for all $i \neq 1$, then the smallest integer for b_1 will be 2. Similarly, we can prove that the smallest integers for other b_i are 2 for $i = 3, \ldots, n-2$,

$b_{n-1} = b_n = 4, b_i = 1$ otherwise. In other words, ord $\overline{E_i^*} = 2$ for $i = 1, 3, \ldots, n-2$, ord $\overline{E_{n-1}^*} =$ ord $\overline{E_n^*} = 4$, ord $\overline{E_i^*} = 1$ otherwise. By a simple observation the system of the equations is equivalent to the following system of equations:

$$\begin{aligned} b_1 + b_3 + \cdots + b_{n-2} + b_{n-1} + \frac{b_{n-1} + b_n}{2} \in 2\mathbb{Z}, \\ b_1 + b_3 + \cdots + b_{n-2} + b_n + \frac{b_{n-1} + b_n}{2} \in 2\mathbb{Z}. \end{aligned} \tag{3}$$

where b_i can be at most 2 for $i = 1, 3, \ldots, n-2 \mod 2$; at most 4 mod 4 for $i = n-1, n$; 1 otherwise. Then by solving the system of equations (3) (a) follows.

Now we assume that n is even. In a similarly way, we can prove that ord $\overline{E_i^*} = 2$ for $i = 1, 3, \ldots, n-3, n-1, n$; 1 otherwise. Finding generators for the ring of invariants is equivalent to solving the following system of equations:

$$b_{n-1} + b_n \in 2\mathbb{Z}, \quad b_1 + b_3 \ldots + b_{n-3} + b_n \in 2\mathbb{Z},$$
$$b_1 + b_3 \ldots + b_{n-3} + b_{n-1} \in 2\mathbb{Z}.$$

We can observe that the b_i can be at most 2 for $i = 1, 3, \ldots, n-3, n-1, n \mod 2$. By solving this system we can deduce that (b) follows.

This proposition explicitly gives $\mathbb{Z}$-generators for $\mathcal{E}^+(D_n)$. For instance, in the case of D_4, the F_i for the semigroup are

$$F_1 = E_1 + E_2 + \frac{1}{2}\big(E_3 + E_4\big), \qquad F_2 = E_1 + 2E_2 + E_3 + E_4,$$
$$F_3 = \frac{1}{2}\big(E_1 + 2E_2\big) + E_3 + \frac{1}{2}E_4, \quad F_4 = \frac{1}{2}\big(E_1 + 2E_2\big) + \frac{1}{2}E_3 + E_4.$$

In this case we can explicitly determine the group $M/M' \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Any element $Y = \sum_{i=1}^{n-1} a_i E_i \in \mathcal{E}^+(D_n)$ should satisfy $M(E)a^t = -b^t$ for some $b_i \geq 0$ where $a = (a_1, a_2 \ldots, a_n)$ and $b = (b_1, b_2 \ldots, b_n)$. This condition is equivalent to the following the system of equations:

$$b_3 + b_4 \in 2\mathbb{Z}, \quad b_1 + b_4 \in 2\mathbb{Z}, \quad b_1 + b_3 \in 2\mathbb{Z},$$

where b_1, b_3, b_4 can be at most 2 mod 2 and $b_2 = 1$. This gives the generators for the ring of invariants: $u_1^2,\ u_3^2,\ u_4^2,\ u_2,\ u_1u_3u_4$.

Example 3.10. Let E be the exceptional divisor of the minimal resolution of an E_n-type singularity for $n = 6, 7, 8$. The intersection matrix $M(E_n)$ is given by $(E_i \cdot E_i) = -2$ for $i = 1, \ldots, n$, $(E_3 \cdot E_n) = 1$, $(E_i \cdot E_j) = 1$ for $i = 1, \ldots, n-2$ if $j = i + 1$ and 0 otherwise.

Remark 3.11. By the formula $F_i = M(E)^{-1}\delta_i$ we can obtain:

(i) if $n = 6$ then $\det M(E_6) = 3$ and

$$F_1 = \frac{1}{3}(4, 5, 6, 4, 2, 3), \quad F_2 = \frac{1}{3}(5, 10, 12, 8, 4, 6),$$
$$F_3 = (2, 4, 6, 4, 2, 3), \quad F_4 = \frac{1}{3}(4, 8, 12, 10, 5, 6),$$
$$F_5 = \frac{1}{3}(2, 4, 6, 5, 4, 3), \quad F_6 = (1, 2, 3, 2, 1, 2);$$

(ii) if $n = 7$ then $\det M(E_7) = -2$ and

$$F_1 = (2, 3, 4, 3, 2, 1, 2), \quad F_2 = (3, 6, 8, 6, 4, 2, 4),$$
$$F_3 = (4, 8, 12, 9, 6, 3, 6), \quad F_4 = \frac{1}{2}(6, 12, 18, 15, 10, 5, 9),$$
$$F_5 = (2, 4, 6, 5, 4, 2, 3), \quad F_6 = \frac{1}{2}(2, 4, 6, 5, 4, 3, 3),$$
$$F_7 = \frac{1}{2}(4, 8, 12, 9, 6, 3, 7);$$

(iii) if $n = 8$ then $\det M(E_8) = 1$ and

$$F_1 = (4, 7, 10, 8, 6, 4, 2, 5), \quad F_2 = (7, 14, 20, 16, 12, 8, 4, 10),$$
$$F_3 = (10, 20, 30, 24, 18, 12, 6, 15), \quad F_4 = (8, 16, 24, 20, 15, 10, 5, 12),$$
$$F_5 = (6, 12, 18, 15, 12, 8, 4, 9), \quad F_6 = (4, 8, 12, 10, 8, 6, 3, 6),$$
$$F_7 = (2, 4, 6, 5, 4, 3, 2, 3), \quad F_8 = (5, 10, 15, 12, 9, 6, 3, 8);$$

We give the following proposition for E_6, E_7, E_8 without proof. It can be proved by using the same technique as in the other examples above.

Proposition 3.12.

(i) *In the case of an E_6-type singularity, the generators for the ring of invariants are*

$$u_1^3, u_2^3, u_3, u_4^3, u_5^3, u_6, u_1u_2, u_1u_5, u_2u_4, u_4u_5, u_2^2u_5, u_2u_5^2, u_1^2u_4, u_1u_4^2,$$

where

$$u_1 = x_1^{\frac{4}{3}}x_2^{\frac{5}{3}}x_3^{\frac{6}{3}}x_4^{\frac{4}{3}}x_5^{\frac{2}{3}}x_6, \quad u_2 = x_1^{\frac{5}{3}}x_2^{\frac{10}{3}}x_3^{\frac{12}{3}}x_4^{\frac{8}{3}}x_5^{\frac{4}{3}}x_6^{\frac{6}{3}},$$
$$u_3 = x_1^2x_2^4x_3^6x_4^4x_5^2x_6^3, \quad u_4 = x_1^{\frac{4}{3}}x_2^{\frac{8}{3}}x_3^{\frac{12}{3}}x_4^{\frac{10}{3}}x_5^{\frac{5}{3}}x_6^{\frac{6}{3}},$$
$$u_5 = x_1^{\frac{2}{3}}x_2^{\frac{4}{3}}x_3^{\frac{6}{3}}x_4^{\frac{5}{3}}x_5^{\frac{4}{3}}x_6, \quad u_6 = x_1x_2^2x_3^3x_4^2x_5x_6^2.$$

(ii) *In the case of an E_7-type singularity, the generators for the ring of invariants are*

$$u_1, u_2, u_3, u_4^2, u_5, u_6^2, u_7^2, u_4u_6, u_4u_7, u_6u_7,$$

where

$$u_1 = x_1^2x_2^3x_3^4x_4^3x_5^2x_6x_7^2, \qquad u_2 = x_1^3x_2^6x_3^8x_4^6x_5^4x_6^2x_7^4,$$

$$u_3 = x_1^4x_2^8x_3^{12}x_4^9x_5^6x_6^3x_7^6, \qquad u_4 = x_1^{\frac{6}{2}}x_2^{\frac{12}{2}}x_3^{\frac{18}{2}}x_4^{\frac{15}{2}}x_5^{\frac{10}{2}}x_6^{\frac{5}{2}}x_7^{\frac{9}{2}},$$

$$u_5 = x_1^2x_2^4x_3^6x_4^5x_5^4x_6^2x_7^3, \qquad u_6 = x_1^{\frac{2}{2}}x_2^{\frac{4}{2}}x_3^{\frac{6}{2}}x_4^{\frac{5}{2}}x_5^{\frac{4}{2}}x_6^{\frac{3}{2}}x_7^{\frac{3}{2}},$$

$$u_7 = x_1^{\frac{4}{2}}x_2^{\frac{8}{2}}x_3^{\frac{12}{2}}x_4^{\frac{9}{2}}x_5^{\frac{6}{2}}x_6^{\frac{3}{2}}x_7^{\frac{7}{2}}.$$

(iii) *In the case of an E_8-type singularity, the generators for ring of invariants are*

$$u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8,$$

where

$$u_1 = x_1^4x_2^7x_3^{10}x_4^8x_5^6x_6^4x_7^2x_8^5, \qquad u_2 = x_1^7x_2^{14}x_3^{20}x_4^{16}x_5^{12}x_6^8x_7^4x_8^{10},$$

$$u_3 = x_1^{10}x_2^{20}x_3^{30}x_4^{24}x_5^{18}x_6^{12}x_7^6x_8^{15}, \qquad u_4 = x_1^8x_2^{16}x_3^{24}x_4^{20}x_5^{15}x_6^{10}x_7^5x_8^{12},$$

$$u_5 = x_1^6x_2^{12}x_3^{18}x_4^{15}x_5^{12}x_6^8x_7^4x_8^9, \qquad u_6 = x_1^4x_2^8x_3^{12}x_4^{10}x_5^8x_6^6x_7^3x_8^6,$$

$$u_7 = x_1^2x_2^4x_3^6x_4^5x_5^4x_6^3x_7^2x_8^3, \qquad u_8 = x_1^5x_2^{10}x_3^{15}x_4^{12}x_5^9x_6^6x_7^3x_8^8.$$

Example 3.13. Now we consider a determinantal rational surface singularity. Let E be the exceptional fiber of the minimal resolution of an A_2^2-type singularity such that $E_4 \cdot E_i = 0$ for $i = 1, 3$, $E_4 \cdot E_2 = 1$, $E_4^2 = -3$ and $E_i \cdot E_j = 1$ if $|j - i| = 1$, -2 if $i = j$ and 0 otherwise for $i = 1, 2, 3$.

From the formula $F_i = M(E)^{-1}\delta_i$ for each i we obtain

$$F_1 = \frac{1}{8}(7, 6, 3, 2), \quad F_2 = \frac{1}{8}(6, 12, 6, 4),$$

$$F_3 = \frac{1}{8}(3, 6, 7, 2), \quad F_4 = \frac{1}{8}(2, 4, 2, 4)$$

Now we consider the ring $\mathbb{C}[N] = \mathbb{C}[x_1, x_2, x_3, x_4]$. Then we obtain $\mathbb{C}[N'] = \mathbb{C}[u_1, u_2, u_3, u_4]$, where

$$u_1 = x_1^{\frac{7}{8}}x_2^{\frac{6}{8}}x_3^{\frac{3}{8}}x_4^{\frac{2}{8}}, \quad u_2 = x_1^{\frac{6}{8}}x_2^{\frac{12}{8}}x_3^{\frac{6}{8}}x_4^{\frac{4}{8}}, \quad u_3 = x_1^{\frac{3}{8}}x_2^{\frac{6}{8}}x_3^{\frac{7}{8}}x_4^{\frac{2}{8}}, \quad u_4 = x_1^{\frac{2}{8}}x_2^{\frac{4}{8}}x_3^{\frac{2}{8}}x_4^{\frac{4}{8}}.$$

In order to understand the group structure of M/M', we consider an element $\overline{F} = F + M' \in M'$ where $F = b_1E_1^* + b_2E_2^* + b_3E_3^* + b_4E_4^* \in M$. Hence $F \in M'$. Therefore there exist elements a_i such that

$$b_1E_1^* + b_2E_2^* + b_3E_3^* + b_4E_4 = a_1F_1^* + a_2F_2^* + a_3F_3^* + a_4F_4^*.$$

Since the F_i^* can be written in terms of the E_i^* we can solve the a_i in terms of the b_i. In other words,

$$a_i = F_i b^t, \tag{4}$$

where $b = (b_1, b_2, b_3, b_4)$. Since the $a_i \in \mathbb{Z}$ we can deduce that $\operatorname{ord}(\overline{E_1^*}) = \operatorname{ord}(\overline{E_3^*}) = 8, \operatorname{ord}(\overline{E_2^*}) = \operatorname{ord}(\overline{E_4^*}) = 4$. Under the M/M'-action the ring of invariants $u_1^{b_1}u_2^{b_2}u_3^{b_3}u_4^{b_4}$ come from the solutions (b_1, b_2, b_3, b_4) of the system of equations (4). This system is equivalent to the following system of equations:

$$6b_1 + 7b_2 + 3b_3 + 2b_4 \in 8\mathbb{Z}, \quad 2b_1 + b_2 + b_3 + 2b_4 \in 4\mathbb{Z}.$$

Therefore by solving the latter system of equations we can obtain the generators of the ring of invariants:

$$u_1^8, u_2^4, u_3^8, u_4^4, u_1u_3^3, u_1^3u_3, u_1^5u_3^7, u_1^7u_3^5, u_2u_4, u_3^2u_4, u_1^2u_4, u_2^3u_3^2, u_2^2u_3^4,$$
$$u_2u_3^6, u_1^2u_2^3, u_1^4u_2^2, u_1^6u_2, u_1^4u_3^6u_4, u_1^6u_3^4u_4, u_1^3u_3^5u_4^2, u_1^2u_3^4u_4^3,$$
$$u_1^4u_3^2u_4^3, u_1u_3u_4^3, u_1u_2u_3.$$

Example 3.14. We examine a Tr-type singularity, which is a minimally elliptic singularity. In the minimal resolution of a Tr-type singularity there are three nonsingular rational curves E_i meeting transversely at the same point. We are interested in such a Tr-type singularity for which the self intersections of the exceptional curves E_i are $E_1^2 = E_2^2 = -2$ and $E_3^2 = -3$.

By a similar argument as above we can determine the F_i:

$$F_1 = \frac{1}{3}(5, 4, 1), \ F_2 = \frac{1}{3}(4, 5, 1), \ F_3 = \frac{1}{3}(3, 3, 3).$$

Now we consider the ring $\mathbb{C}[N] = \mathbb{C}[x_1, x_2, x_3]$. Then we obtain $\mathbb{C}[N'] = \mathbb{C}[u_1, u_2, u_3]$ where

$$u_1 = x_1^{\frac{5}{3}}x_2^{\frac{4}{3}}x_3^{\frac{1}{3}}, \ u_2 = x_1^{\frac{4}{3}}x_2^{\frac{5}{3}}x_3^{\frac{1}{3}}, \ u_3 = x_1x_2x_3.$$

By examining the structure of M/M' we obtain that M/M' is a cyclic group of order 3 generated by $\overline{E_1^*}$ and $\operatorname{ord}(\overline{E_1^*}) = \operatorname{ord}(\overline{E_2^*}) = 3, \operatorname{ord}(\overline{E_3^*}) = 1$. We set $\overline{E_i^*} = \epsilon$. Therefore the ring of invariants under the action

$$\epsilon \cdot u_1^{b_1}u_2^{b_2}u_3^{b_3} = \epsilon^{5b_1+4b_2+3b_3}u_1^{b_1}u_2^{b_2}u_3^{b_3}$$

should satisfy the following condition $2b_1 + b_2 \equiv 0 \mod 3$. Hence we can obtain the generators of the ring of invariants as follows:

$$u_1^3, u_2^3, u_3, u_1u_2.$$

Remark 3.15. With the preceding notation, the ring of invariants $\mathbb{C}[N']^{\frac{M}{M'}}$ is given by the monomials lying on the compact part of the Newton polygon of

$$u_1^{d_1^{(Y)}} u_2^{d_2^{(Y)}} \cdots u_n^{d_n^{(Y)}}$$

for each $Y \in \mathcal{E}^+(E)$, where $d_i^{(Y)} = -(Y \cdot E_i)$.

Let

$$S := \left\{ \left(d_1^{(Y)}, d_2^{(Y)}, \ldots, d_n^{(Y)}\right) \mid d_i^{(Y)} = -(Y \cdot E_i) \text{ for all } Y \in \mathcal{E}^+(E) \right\}.$$

Consider the convex hull of the points $(d_1^{(Y)}, d_2^{(Y)}, \ldots, d_n^{(Y)})$ in S. Then the elements of S which are on the compact part of the boundary of the convex hull (called the Newton polygon) generate S.

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