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On certain curves of genus three in characteristic two

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— Dedicated to the treasured memory of our coauthor, Paulo Henrique Viana

Abstract. We study curves of genus 3 over algebraically closed fields of characteristic 2 with the canonical theta characteristic totally supported in one point. We compute the moduli dimension of such curves and focus on some of them which have two Weierstrass points with Weierstrass directions towards the support of the theta characteristic. We answer questions related to order sequence and Weierstrass weight of Weierstrass points and the existence of other Weierstrass points with similar properties.

Keywords: curves of genus three, characteristic two, theta characteristic.

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1 Introduction

In the present article we study curves of genus 3 over fields of characteristic 2. In the case when the ground field is finite we have a recent wide classification (cf. [NR]). Here we assume the ground field is algebraically closed and we also assume the canonical theta characteristic is totally supported in one point.

So if C is such a curve then we can find a canonical divisor on C of the form $4P_0$ where P_0 is precisely the support of the theta characteristic. These curves must be non-hyperelliptic and hence canonically embedded in $\mathbb{P}^2$ as a plane quartic described by an affine equation in a way that P_0 is the origin. Our first concern here has to do with the moduli dimension of these curves. We have the following result.

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Theorem 1. *The isomorphism classes of irreducible nonsingular projective curves of genus* 3 *over an algebraically closed field of characteristic* 2*, with the canonical theta characteristic totally supported at one point, form an algebraic variety of dimension* 4.

In characteristic 0, a similar problem was studied by A. M. Vermeulen. He proved in 1983 (cf. [V]) that the subspace of $\mathcal{M}_3$ consisting of curves with one hyperflex has dimension 5.

Afterwards we impose an additional condition: that there exists 2 Weierstrass points, say Q_1 and Q_2, with Weierstrass direction towards P_0 which means that $3Q_i + P_0$ is a canonical divisor on C for $i = 1, 2$. These curves are the main subject of this work. We summarize the results we get in the following statement.

Theorem 2. *The isomorphism classes of irreducible nonsingular projective curves of genus* 3 *over an algebraically closed field of characteristic* 2 *with the canonical theta characteristic totally supported at one point with* 2 *Weierstrass directions towards it form an algebraic variety of dimension* 2*. Every such a curve is canonically isomorphic to a plane curve with affine equation*

$$C_{a,b,c} \ : \qquad x + y + ax^3y + bx^2y^2 + cxy^3 = 0$$

where $(a : b : c) \in \mathbb{P}^2$ *does not lie in* $abc = 0$ *nor* $a + b + c = 0$*. We also have:*

(i) *the origin has order sequence* 0, 1, 4 *and its Weierstrass weight is* 5, 8 *or* 20 *according to* $a \neq c$, $a = c \neq b$ *or* $a = b = c$*, respectively, and in any case it is the unique point with this Weierstrass weight; there are other Weierstrass points with order sequence* 0, 1, 4 *only in curves* $C_{a,1,1/a}$ *with* $a \neq 1$*, and these points have Weierstrass weight* 4.

(ii) *all* 4 *infinite points are Weierstrass points with order sequence* 0, 1, 3*, Weierstrass weight* 1*, having Weierstrass directions towards the origin and being the only ones with this latter property.*

(iii) *if* $a = c$ *all Weierstrass points other than the origin are simple Weierstrass points. If* $a = b = c$ *the origin is the unique finite Weierstrass point, otherwise the other 16 Weierstrass points are* 4 *by* 4 *collinear.*

(iv) *there is a Weierstrass point having a Weierstrass direction towards a Weierstrass point at the infinity if and only if* $a^2c + b^3 + b^2a = 0$*. In this case there are* 3 *such Weierstrass points and these points are collinear.*

(v) *there are two Weierstrass points with Weierstrass directions towards two different Weierstrass points at the infinity only in case of the* 3 *curves* $C_{a,1,a}$ *with* $a^3 + a + 1 = 0$.

(vi) *there is a Weierstrass point with Weierstrass direction towards another Weierstrass point having itself a Weierstrass direction towards a Weierstrass point at the infinity only in the case of the* 9 *curves* $C_{\lambda^{-3},1,\lambda^3(1+\lambda^3)}$ *for* λ *a root of* $h_1(\lambda) = \lambda^9 + \lambda^8 + \lambda^7 + \lambda^6 + \lambda^4 + \lambda^2 + 1$.

2 Preliminaries

Let C be an irreducible non-singular algebraic curve of genus 3 over a field of characteristic 2. And let us also assume that there exists a canonical theta characteristic $\theta_0 = |\frac{1}{2}\mathrm{div}(dx)|$ on C (with x a separating variable) which can be represented by a divisor of the form $2P_0$ for a certain $P_0 \in C$. In this situation $4P_0$ is a positive canonical divisor so that the point P_0 has canonical order sequence 0, 1, 4. It follows that the curve is nonhyperelliptic and hence $h^0(\mathcal{O}_C(2P_0)) = 1$ and $2P_0$ is the only positive divisor in the class θ_0.

Since C is nonhyperelliptic let us consider it canonically embedded in $\mathbb{P}^2$ as a smooth plane quartic given by

$$f(x, y) = \sum_{i+j\leq 4} c_{ij}x^i y^j, \qquad c_{ij} \in k. \tag{1}$$

For a general plane curve of genus g over a field of characteristic p we have that the *Cartier operator*

$$\mathcal{C} : \Omega_1(0) \longrightarrow \Omega_1(0)$$

which acts on the space of regular differentials of C can be expressed as

$$\mathcal{C}(hdx) = -\left(\frac{d^{p-1}h}{dx^{p-1}}\right)^{1/p} dx$$

(cf. [SV2]). The *Hasse-Witt invariant* σ is defined as the rank of the matrix

$$\left(h_{ij}\right)\left(h_{ij}^{p}\right)\cdots\left(h_{ij}^{p^{g-1}}\right)$$

for $\left(h_{ij}\right)$ the *Hasse-Witt matrix*, that is, $\left(h_{ij}^{1/p}\right)$ represents the Cartier operator.

In general a curve C of genus 3 admits a number of 7, 4, 2 or 1 bitangents, depending on the values $\sigma = 3, 2, 1$ or 0 of the Hasse-Witt invariant (cf. [SV2] pg. 60). In our case, as the canonical theta characteristic θ_0 does have a section, the Cartier operator necessarily has a non-trivial kernel, and so $\sigma = 2, 1$ or 0.

In the case of a non-singular plane curve (such as our canonical plane quartic) given by $f(x, y) = 0$ as in (1), the differential $\omega = \frac{dx}{f_y}$ is regular and

$$H^0(\mathcal{O}_C(\operatorname{div}(\omega)) = \left\{h \in k[x, y] \mid \deg(h) \leq \deg(f) - 3\right\}.$$

We have that ([SV2] Theorem 1.1) in the case of characteristic 2 yields the formula for the Cartier operator

$$\mathcal{C}(h\omega) = \left(\frac{\partial^2}{\partial x \partial y} hf\right)^{1/2} \omega,$$

giving the Hasse-Witt matrix

$$H = \begin{pmatrix} c_{11} & c_{01} & c_{10} \\ c_{31} & c_{21} & c_{30} \\ c_{13} & c_{03} & c_{12} \end{pmatrix}.$$

We will use the theory of Weierstrass points, for which we refer to ([SV1] Section 1). In the case of a non-singular plane curve the results we need are collected below.

To compute the (canonical) Weierstrass points in curves of genus 3 over a field of characteristic 2 we use the classical *Wronskian* since from [K] there are no non-classical curves in this situation. Using the separating variable x we obtain

$$W_x^{0,1,2} = \det \begin{pmatrix} 1 & x & y \\ 0 & 1 & D_x^{(1)}(y) \\ 0 & 0 & D_x^{(2)}(y) \end{pmatrix} = D_x^{(2)}(y),$$

where $D_x^{(i)}(y)$ stands for the i^{th} *Hasse-Schmidt derivative*. To compute the Wronskian we take *generic Taylor expansions*

$$\begin{aligned} \mathcal{T}(x) &= x + t \\ \mathcal{T}(y) &= y + D_x^{(1)}(y)t + D_x^{(2)}(y)t^2 + \cdots \end{aligned}$$

and use that

$$f(\mathcal{T}(x), \mathcal{T}(y)) = \sum_{i,j} c_{ij} \mathcal{T}(x)^i \mathcal{T}(y)^j = 0.$$

This yields

$$D_x^{(2)}(y) = \frac{w_K}{f_y^3},$$

where the numerator w_K is given by

$$\begin{aligned} w_K &= f_x f_y \sum_{i,j\equiv 1 \mod 2} c_{ij} x^{i-1} y^{j-1} \\ &+ f_y^2 \sum_{i\equiv 2,3 \mod 4} c_{ij} x^{i-2} y^j \\ &+ f_x^2 \sum_{j\equiv 2,3 \mod 4} c_{ij} x^i y^{j-2}, \end{aligned}$$

for

$$f_x = \sum_{i\equiv 1 \mod 2} c_{ij} x^{i-1} y^j \qquad \text{and} \qquad f_y = \sum_{j\equiv 1 \mod 2} c_{ij} x^i y^{j-1}.$$

The *ramification divisor* (cf. [SV1] pg. 3) is given by

$$\begin{aligned} \mathcal{R}_K &= \text{div } (D_x^{(2)}(y)) + 3 \text{ div } (dx) + 3E \\ &= \text{div } (w_K) + 3 \text{ div } \left(\frac{dx}{f_y}\right) + 3E \\ &= \text{div } (w_K) + 6E \end{aligned}$$

where $E = \text{div } (\omega) = \text{div } \left(\frac{dx}{f_y}\right)$ is the intersection divisor of the curve with the infinite line. The finite Weierstrass points are thus the zeros (counted with multiplicities) of the numerator w_K of $D_x^{(2)}(y)$. In our case of genus 3 and characteristic 2 there are altogether 24 Weierstrass points.

If P is a Weierstrass point then its order sequence may be only $0, 1, 3$ or $0, 1, 4$ and as a consequence of ([SV1] Theorem 1.5) its Weiertrass weight is 1 in the first case and greater than 2 in the second. The intersection divisor of the tangent of the curve at P is then $3P + Q$ (with $P = Q$ if the order sequence at P is $0, 1, 4$). If Q itself is a Weierstrass point then we will say that P has a *Weierstrass direction towards* Q and

$$P \xrightarrow{\mathcal{W}} Q$$

will denote this situation.

3 The moduli problem

We start this section with a result used to rule out some (in principle) possible degenerate situations. Recall that there are curves in prime characteristics having just one Weierstrass point.

Proposition 3.1. *The Weierstrass points of a smooth plane quartic in characteristic* 2 *with the canonical theta characteristic supported at one point are non collinear.*

Proof. If all Weierstrass points were collinear then we can suppose that the line containing them is the infinite line. In this case the numerator w_K of the Wronskian would be a non-zero constant, as there will be no finite Weierstrass points.

We can further assume that the canonical theta characteristic θ_0 contains the divisor $2P_0$, for $P_0 = (1\colon 0\colon 0)$, and that the tangent line of the curve at P_0 is given by $y = 0$. This implies that $c_{10} = c_{20} = c_{30} = c_{40} = 0$ in (1). Moreover, as $2P_0$ is a divisor in the class θ_0 it follows that $4P_0 = \operatorname{div}(y)\frac{dx}{f_y}$, the intersection divisor of the line $y = 0$ with the curve, is a canonical divisor in the kernel of the Cartier operator. Given the above expression of the Hasse-Witt matrix this implies $c_{12} = 0$. With these normalizations the equation (1) for the curve simplifies to

$$\begin{aligned} f(x, y) = 1 &+ c_{01}y + c_{11}xy + c_{02}y^2 + c_{21}x^2y + c_{03}y^3 + c_{31}x^3y \\ &+ c_{22}x^2y^2 + c_{13}xy^3 + c_{04}y^4. \end{aligned}$$

The expression w_K is then given by

$$\begin{aligned} w_K = &\left[\left(c_{21}^2c_{31} + c_{11}c_{31}^2\right)y\right]x^5 + \left[\left(c_{21}^2c_{22} + c_{31}^2c_{02}\right)y^2 + \left(c_{21}^3 + c_{31}^2c_{01}\right)y\right]x^4 \\ &+ \left[\left(c_{11}c_{13}^2 + c_{03}^2c_{31}\right)y^5 + \left(c_{01}^2c_{31} + c_{11}^3\right)y\right]x + \left(c_{13}^2c_{02} + c_{03}^2c_{22}\right)y^6 \\ &+ \left(c_{13}^2c_{01} + c_{03}^2c_{21}\right)y^5 + \left(c_{01}^2c_{22} + c_{11}^2c_{02}\right)y^2 + \left(c_{01}^2c_{21} + c_{11}^2c_{01}\right)y. \end{aligned}$$

If we successively subtract multiples of $f(x, y)$ from the expression w_K so as to cancel in the resulting expressions the initial terms with respect to the lexicographic order with $x > y$, we obtain a remainder of the form

$$r = c_{31}^5x^3 + \cdots .$$

This remainder must be a constant, and so $c_{31} = 0$, but this condition will result in a singularity at P_0, and so the result is proved. □

For the remainder we use other normalizations for such a curve. Now we bring the support P_0 of the positive divisor in the canonical theta characteristic θ_0 to the origin $(0\colon 0\colon 1)$ and force the tangent of the curve at P_0 to be the line given by $x = y$. This tangent line intersects the curve at P_0 with contact multiplicity 4, and thus never meets the curve again.

Theorem 3.2. *The isomorphism classes of curves of genus three having the canonical theta characteristic represented by a positive divisor supported at one point form an algebraic variety of dimension* 4.

Proof. In terms of the coefficients of the equation (1) defining the curve the normalizations stated above imply $c_{00} = c_{10} + c_{01} = c_{20} + c_{11} + c_{02} = c_{30} + c_{21} + c_{12} + c_{03} = 0$.

As $2P_0$ is a divisor in the class θ_0 it follows that $4P_0 = \operatorname{div}(x+y)\frac{dx}{f_y}$, the intersection divisor of the line $x = y$ with the curve, is a canonical divisor in the kernel of the Cartier operator. Given the above expression of the Hasse-Witt matrix this implies $c_{21} + c_{30} = c_{03} + c_{12} = 0$.

From the preceding proposition we can use a projective plane transformation in order to take one Weierstrass point to the location $Q_1 = (1\colon 0\colon 0)$ and another to $Q_2 = (0\colon 1\colon 0)$. The projective automorphisms that fix these normalizations of the origin $P_0 = (0\colon 0\colon 1)$, of the tangent at the origin $y = x$ and of the two infinite Weierstrass points $Q_1 = (1\colon 0\colon 0)$ and $Q_2 = (0\colon 1\colon 0)$ form a subgroup G of $PGl_2(k)$ consisting of (classes of) matrices

$$G := \left\{ \begin{pmatrix} s_{00} & 0 & 0 \\ 0 & s_{00} & 0 \\ 0 & 0 & s_{22} \end{pmatrix} ;\ s_{00}, s_{22} \neq 0 \right\} \Big/ k^*. \tag{2}$$

As a consequence of the choice of points Q_1 e Q_2 we have $c_{04} = c_{40} = 0$. The tangent lines at these points (after the necessary homogenizations and dehomogeneizations) are, respectively,

$$c_{30}z + c_{31}y = 0 \qquad \text{and} \qquad c_{03}z + c_{13}x = 0.$$

These are Weierstrass points, and so these equations divide the quadratic parts appearing in the local expression of f. These divisibility conditions yield

$$c_{31}^2 c_{20} + c_{30}^2 c_{22} + c_{30}c_{31}c_{21} = 0 \qquad \text{and} \qquad c_{13}^2 c_{02} + c_{03}^2 c_{22} + c_{03}c_{13}c_{12} = 0,$$

and these may be rewritten as

$$c_{31}^2 c_{20} + c_{30}^2 c_{22} + c_{30}^2 c_{31} = 0 \qquad \text{and} \tag{3}$$

$$c_{13}^2 c_{02} + c_{03}^2 c_{22} + c_{03}^2 c_{13} = 0. \tag{4}$$

The result is now just parameter counting, once we observe that the two conditions above are algebraically independent. The curve is given by the equation $f = f_1 + f_2 + f_3 + f_4 = 0$, where f_i is the homogeneous part of degree i so that, with the chosen normalizations,

$$\begin{aligned} f_1(x, y) &= x + y \\ f_2(x, y) &= c_{20}x^2 + (c_{20} + c_{02})xy + c_{02}y^2 \\ f_3(x, y) &= c_{30}x^3 + c_{30}x^2 y + c_{03}xy^2 + c_{03}y^3 \\ f_4(x, y) &= c_{31}x^3 y + c_{22}x^2 y^2 + c_{13}xy^3, \end{aligned}$$

is isomorphic, through a projective plane transformation given by an element of the group G described above, to the curve given by

$$f_1(x, y) + \alpha^{-1} f_2(x, y) + \alpha^{-2} f_3(x, y) + \alpha^{-3} f_4(x, y) = 0,$$

for $\alpha = \frac{s_{00}}{s_{22}}$. In these equations the conditions (3) and (4) have not yet been taken into consideration. □

4 Curves with two Weierstrass directions towards the support of the canonical theta characteristic

We can ask the question of when the curve has two Weierstrass points Q_1 and Q_2 with Weierstrass directions towards P_0:

$$Q_1 \xrightarrow{\mathcal{W}} P_0 \qquad \text{and} \qquad Q_2 \xrightarrow{\mathcal{W}} P_0.$$

We can certainly use a projective plane transformation to bring these Weierstrass points to the chosen infinite locations $(1\colon 0\colon 0)$ and $(0\colon 1\colon 0)$. This implies $c_{30} = c_{03} = 0$, and from (3) and (4) we deduce

$$c_{31}^2 c_{20} = 0 \qquad \text{and} \qquad c_{13}^2 c_{02} = 0.$$

On the other hand, if $c_{31} = 0$ or $c_{13} = 0$ then the infinite points Q_1 or Q_2 are singular, respectively, and so both c_{31} and c_{13} are non-zero, so that if $c_{30} = c_{03} = 0$ then also $c_{20} = c_{02} = 0$.

If $c_{30} = c_{03} = 0$ and $c_{20} = c_{02} = 0$ then the equation of the curve simplifies to

$$C_{a,b,c} \; : \qquad x + y + ax^3y + bx^2y^2 + cxy^3 = 0. \tag{5}$$

This family of curves were used in [RV] for displayed examples of minimal curves and eventually minimal curves.

Remark 4.1. We left the details to the reader to check that a curve $C_{a,b,c}$ as above is irreducible if and only if $a + b + c \neq 0$.

The proof of the following result is straightforward:

Proposition 4.2. *The curve $C_{a,b,c}$ is singular if and only if $abc = 0$.*

The intersection divisor with the infinite line is given by

$$Q_1 + Q_2 + Q_\delta + Q_{\delta+1},$$

where

$$Q_\delta = \left(\frac{b}{a}\delta : 1 : 0\right) \qquad \text{and} \qquad Q_{\delta+1} := \left(\frac{b}{a}\delta + \frac{b}{a} : 1 : 0\right),$$

for δ is any root of the Artin-Schreier equation

$$t^2 + t + \frac{ca}{b^2} = (t + \delta)(t + \delta + 1) = 0.$$

Under the hypothesis $a, b, c \neq 0$ these points are all distinct. Moreover, the next result shows that these 4 infinite points are Weierstrass points with order sequence $0, 1, 3$ and have Weierstrass directions towards the origin and there is no finite Weierstrass point with Weierstrass direction towards the origin.

Theorem 4.3. *In the family of curves $C_{a,b,c}$ the Hasse-Witt invariant σ is 0 or 2 according to $a = c$ or not. The rank of the Cartier operator is always 2.*

In the curve $C_{a,b,c}$ the origin is a Weierstrass point having order sequence $0, 1, 4$ and its Weierstrass weight is 5, 8 or 20 according to $a \neq c, a = c \neq b$ or $a = b = c$, respectively, and in any case it is the unique point with this Weierstrass weight. All 4 infinite points are Weierstrass points with order sequence $0, 1, 3$, Weierstrass weight 1, having Weierstrass directions towards the origin and being the only ones with this latter property. There are other Weierstrass points with order sequence $0, 1, 4$ only in curves $C_{a,1,1/a}$ with $a \neq 1$, and these points have Weierstrass weight 4.

Proof. The Hasse-Witt matrix of the curve $C_{a,b,c}$ is given by

$$H = \begin{pmatrix} 0 & 1 & 1 \\ a & 0 & 0 \\ c & 0 & 0 \end{pmatrix}$$

and thus the Hasse-Witt invariant σ, which is the rank of the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ a & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ a^2 & 0 & 0 \\ c^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ a^4 & 0 & 0 \\ c^4 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^2+c^2 & a^2+c^2 \\ a(a^4+c^4) & 0 & 0 \\ c(a^4+c^4) & 0 & 0 \end{pmatrix}$$

is given by

$$\sigma = \begin{cases} 2 & \text{if } a \neq c \\ 0 & \text{if } a = c \end{cases}$$

If $a = c$ then the origin P is the unique point having 0, 1, 4 as order-sequence. In fact, if Q is another point with this order-sequence then we have $4P \sim 4Q$ where $\sim$ means linearly equivalent to. Since there are no 2-torsion points (the Hasse invariant is zero) we get $2P \sim 2Q$, but the curve is not hyperelliptic.

The numerator of the Wronskian for the curves $C_{a,b,c}$ is given by

$$\begin{aligned} w_K &= f_x f_y\big(ax^2+cy^2\big) + f_y^2\big(axy+by^2\big) + f_x^2\big(bx^2+cxy\big) \\ &= bx^2+by^2+(a+c)xy+\big(ax^2+cy^2\big)+\big(ax^2+cy^2\big)^2(x+y). \end{aligned}$$

The following hold:

$$\begin{aligned} \operatorname{div}(x) &= 2Q_2 + P_0 - \big[Q_1 + Q_\delta + Q_{\delta+1}\big] \\ \operatorname{div}(y) &= 2Q_1 + P_0 - \big[Q_2 + Q_\delta + Q_{\delta+1}\big]. \end{aligned}$$

Because of ([SV1] Theorem 1.5) the origin is a Weierstrass point with weight greater than 2 having x as a local parameter, and the following expansions hold

$$\begin{aligned} y &= x + (a+b+c)x^4 + (a+c)(a+b+c)x^7 + \\ &\quad + ((a+c)^2(a+b+c) + (b+c)(a+b+c)^2)x^{10} + \cdots \\ w_K &= (a+c)(a+b+c)x^5 + (a+b+c)^2(b+c)x^8 + \cdots \end{aligned}$$

If $a = c$ then the order of w_K at the origin is greater than 5, and the expansions are

$$\begin{aligned} y &= x + bx^4 + (b+a)b^2x^{10} + ab^3x^{13} + ab^4(a+b)x^{19} + \cdots \\ w_K &= b^2(b+a)x^8 + b^4(b^3+a^3+b^2a)x^{20} + \cdots \end{aligned}$$

If also $b + c = 0$ (so that $a = b = c$) then the order of w_K at the origin is 20, and the origin is the unique finite Weierstrass point. Summarizing we have:

$$v_{P_0}(w_K) = v_{P_0}(\mathcal{R}_K) = \begin{cases} 5 & \text{if } a \neq c \\ 8 & \text{if } a = c \neq b \\ 20 & \text{if } a = b = c \end{cases}$$

At the points Q_1 and Q_2 the following holds:

$$v_{Q_2}(w_K) = v_{Q_1}(w_K) = -5 \qquad \text{and} \qquad v_{Q_2}(\mathcal{R}_K) = v_{Q_1}(\mathcal{R}_K) = 1,$$

and so these points are Weiertrass points with orders 0, 1, 3 and weight 1.

The other infinite points Q_δ and $Q_{\delta+1}$ are Weierstrass points with orders 0, 1, 3 and weight 1, since taking the local parameter $t = 1/y$ the local expansion of x begins as

$$x = \frac{b}{a}\delta t^{-1} + \cdots,$$

and thus

$$ax^2 + cy^2 = \left(\frac{b^2}{a}\delta^2 + c\right)t^{-2} + \cdots; \qquad \text{as} \qquad \delta^2 + \delta = \frac{ca}{b^2}$$

this first coefficient

$$\frac{b^2}{a}\delta^2 + c = \frac{b^2}{a}\left(\delta^2 + \frac{ca}{b^2}\right) = \frac{b^2}{a}\delta$$

is nonzero. At Q_δ, for instance, the tangent is given by $x + \frac{b}{a}\delta y = 0$, and the intersection divisor of this tangent with the curve is given by $3Q_\delta + P_0$, so that Q_δ and $Q_{\delta+1}$ have Weierstrass directions towards the origin.

A line in the pencil through the origin has the equation $y = \alpha x$, which taken into the equation for $C_{a,b,c}$ gives

$$x\big(1 + \alpha + x^3\big(a\alpha + b\alpha^2 + c\alpha^3\big)\big) = 0.$$

If $\alpha \neq 1$ this equation does not have a multiple root, showing that no finite point of the curve other than the origin has its tangent passing through the origin. As a consequence, the infinite points are the only ones in the curve with Weierstrass direction towards the origin.

Second coordinates of other Weierstrass points are given by the Sylvester resultant $R(f, w_K)$ between f and w_K:

$$\begin{aligned} R(f, w_K) = {} & y^{20}(a^2(c^4ab^4 + c^4b^5 + c^5b^4)) + y^{17}(a^2(ac^4b^3 + b^5c^3 + b^4c^4 + a^2c^3b^3)) \\ & + y^{14}(a^2(b^7 + c^3a^4 + a^2c^5 + a^2b^4c + a^3c^2b^2 + a^2c^2b^3 + b^6a + b^4c^3)) \\ & + y^{11}(a^2(b^5c + a^3c^2b + b^4a^2 + b^5a + c^2a^4 + a^2c^4 + b^4c^2 + ba^2c^3)) \\ & + y^8(a^2(b^3a^2 + a^4c + b^2a^3 + a^2c^3)) + y^5(a^2(a^4 + a^2c^2 + ba^3 + ba^2c)). \end{aligned}$$

The roots of $R(f, w_K)$ are the second coordinates of finite Weierstrass points, but the counting of multiplicities needs some care: $R(f, w_K)$ has a multiple root in a higher Weierstrass point (that is, one having a higher weight in the ramification divisor or, equivalently (cf. [SV1] Theorem 1.5), having orders $0, 1, 4$), but a multiple root of $R(f, w_K)$ would also happen if 2 distinct Weierstrass points had the same second coordinate. Note that the origin is counted with multiplicity 5 if $a \neq c$, 8 if $a = c \neq b$ and 20 if $a = b = c$.

If the curve has Weierstrass points with orders $0, 1, 4$ other than the origin then the polynomial $R(f, w_K)$ has other multiple roots. This situation is given by the discriminant of $R(f, w_K)/(y^5)$. To simplify the computation of this discriminant we set $b = 1$, which is allowed because of the action of G; this discriminant is then given by

$$\operatorname{disc}\left(\frac{R(f, w_K)}{y^5}\right) = (a + c)^2(a + c + 1)^2(1 + ac)^{12}.$$

The factor $a + c$ is expected: the origin in this case has multiplicity greater than 5. The second factor is $a + c + 1 = a + b + c$, which is never zero. If the third factor is zero then $a = \dfrac{1}{c}$, and then

$$a^3\frac{R(f, w_K)}{y^5} = \big(a^2 + a + 1\big)\big(y^3 + a(a + 1)\big)\big(y^{12} + a^4(a + 1)\big).$$

If $a^2 + a + 1 = 0$ then $a \in \mathbb{F}_4 \setminus \mathbb{F}_2$, but then $c = \dfrac{1}{a} = a + 1 = a^2$ and hence $a^2 + a + 1 = a + b + c = 0$, which is against our moduli hypothesis. If $a \notin \mathbb{F}_4$ then the three distinct roots of

$$y^{12} = a^4(a + 1)$$

are second coordinates of Weierstrass points with order sequence $0, 1, 4$, all of then having weight four, and the three distinct roots of

$$y^3 = a(a + 1)$$

are second coordinates of Weierstrass points with order sequence $0, 1, 3$, all of then having weight 1, as follows from direct computations. $\square$

Theorem 4.4. *The isomorphism classes of curves of genus* 3 *having the canonical theta characteristic represented by a positive divisor supported at one point having* 2 *Weierstrass directions towards it form an algebraic variety of dimension two.*

Proof. There are exactly 4 Weierstrass points having Weierstrass directions towards the origin, and we may choose 2 of them to the normalized positions $(1\colon 0\colon 0)$ and $(0\colon 1\colon 0)$ in 12 ways. Having chosen these 2 infinite Weiertrass points, the group that fixes these normalizations is the group G described in (2) above. This group fixes the point $P_0 = (0\colon 0\colon 1)$ and each point of the infinite line $Z = 0$, acting by homothety on finite points: $(a\colon b\colon 1) \mapsto (\alpha a\colon \alpha b\colon 1)$, where $\alpha := \frac{s_{00}}{s_{22}}$. Thus the curve $C_{a,b,c}$ is isomorphic to the curve $C_{\lambda a,\lambda b,\lambda c}$ for $\lambda \neq 0$. □

Theorem 4.5. *In the curve $C_{a,b,a}$ all Weierstrass points other than the origin are simple Weierstrass points. If $b = a$ the origin is the unique finite Weierstrass point, otherwise the other 16 Weierstrass points are* 4 *by* 4 *collinear. These* 4 *lines containing them are lines in the pencil of lines through* $(1\colon 1\colon 0)$*, and so is the canonical bitangent.*

Proof. If $a = c$ then there are no other bitangents, and thus all Weierstrass points except the origin have orders 0, 1, 3 and weight one. Besides the infinite points, these Weierstrass points are given by the zeros of

$$\begin{aligned} w_K(x, y) &= a^2(x+y)^5 + (a+b)(x+y)^2 \\ &= (x+y)^2\big[a^2(x+y)^3 + a + b\big] \end{aligned}$$

As $x \neq y$ away from the origin, these Weierstrass points occur in pairs $(x\colon y\colon 1)$ and $(y\colon x\colon 1)$, the above equation being symmetric in x and y, as expected: if $a = c$ then $x \leftrightarrow y$ is an automorphism of the curve. The above equation gives a relation among the coordinates of Weierstrass points

$$x + y = \left(\frac{a+b}{a^2}\right)^{1/3}. \tag{6}$$

If $a = b = c$ only the origin is a finite Weierstrass point, as we have seen. If $a = c \neq b$ then $\frac{a+b}{a^2}$ never vanishes and has 3 distinct cubic roots, and thus the other 12 Weierstrass points are on the 3 lines given by (6), occurring in 3 sets of 4 collinear points. These 3 lines are concurrent at the infinite point $(1: 1: 0)$, which is also a point in the canonical bitangent $x = y$. $\square$

If $a \neq c$ there are 3 non-canonical bitangents, corresponding to $\alpha_0 + \alpha_1 x + \alpha_2 y = 0$ where

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} \quad \text{is a solution of} \quad H\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ a & 0 & 0 \\ c & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_0^2 \\ \alpha_1^2 \\ \alpha_2^2 \end{pmatrix}$$

(cf. [SV2] Proposition 3.3). Therefore, for each one of the 3 distinct roots of $\alpha^3 = a + c \neq 0$, there is one non-canonical bitangent given by

$$\alpha^{1/2} + a^{1/2}x + c^{1/2}y = 0.$$

These non-canonical bitangents are concurrent at the infinite point $(c^{1/2}\colon a^{1/2}\colon 0)$.

5 Other Weierstrass directions

Having seen that the infinite points have Weierstrass directions towards the origin, and also that these are the only Weierstrass points with this property, one can ask when a finite Weierstrass point has a Weierstrass direction towards an infinite Weierstrass point Q_i. This is the content of the following result.

Proposition 5.1. *The curve $C_{a,b,c}$ has a Weierstrass point P having a Weierstrass direction towards a Weierstrass point Q_1, as in the situation*

$$P \xrightarrow{\mathcal{W}} Q_1 \xrightarrow{\mathcal{W}} P_0 \tag{7}$$

if and only if $a^2c + b^3 + b^2a = 0$. In this case there are 3 such Weierstrass points P, and these points are collinear.

Proof. Such curves $C_{a,b,c}$ have a finite Weierstrass point with a "horizontal" tangent $y + \mu = 0$ passing through Q_1. Substitution into the equation (5) of the curve $C_{a,b,c}$ yields the polynomial

$$a\mu x^3 + b\mu^2 x^2 + (c\mu^3 + 1)x + \mu = \sum_i c_i x^i.$$

The line $y + \mu = 0$ is the tangent of a Weierstrass point with Weiertrass direction towards Q_1 if and only if this polynomial is a cube $c_3(x + \lambda)^3$, and this will happen if and only if $\lambda = \dfrac{c_2}{c_3} = \dfrac{c_1}{c_2} = \dfrac{c_0}{c_1}$, which is equivalent to

$$c_2^2 + c_1c_3 = \mu\big[(b^2 + ac)\mu^3 + a\big] \;=\; 0 \quad \text{and}$$
$$c_1^2 + c_0c_2 = c^2\mu^6 + b\mu^3 + 1 \;=\; 0.$$

This will happen only if $ca^2 + b^3 + b^2a = 0$. In this case, normalizing to $b = 1$, there are 3 such Weierstrass points given by $P_\lambda := (\lambda\colon a\lambda\colon 1)$, for $\lambda = \frac{1}{a^{1/3}}$ a cubic root of $\frac{1}{a}$. The tangent $y = a\lambda$ meets the curve again at the infinite point $Q_1 = (1\colon 0\colon 0)$ so that P_λ has a Weierstrass direction towards Q_1 which has a Weierstrass direction towards P_0:

$$P_\lambda \xrightarrow{\mathcal{W}} Q_1 \xrightarrow{\mathcal{W}} P_0.$$

These points are collinear, the line $y = ax$ passing through all of them. □

Corollary 5.2. *The isomorphism classes of curves of genus* 3 *having the canonical theta characteristic represented by a positive divisor supported at one point P_0 having* 2 *Weierstrass points Q_i with Weierstrass directions towards P_0, and having a Weierstrass point P with a Weierstrass direction towards one of the points Q_i as in*

$$P \xrightarrow{\mathcal{W}} Q_i \xrightarrow{\mathcal{W}} P_0 \tag{8}$$

form an algebraic variety of dimension 1.

Proof. The infinite Weierstrass point Q_i, towards which a finite Weierstrass point has a Weierstrass direction, may be taken, up to isomorphism, to be $Q_1 = (1\colon 0\colon 0)$, and then one can use the above proposition. The equation $ca^2 + b^3 + b^2a = 0$ clearly defines a subvariety of dimension 1. □

Corollary 5.3. *The three curves $C_{a,1,a}$ with $a^3 + a + 1 = 0$ are the only curves of genus* 3 *having the canonical theta characteristic represented by a positive divisor supported at one point P_0 having* 2 *Weierstrass points Q_i and Q_j with Weierstrass directions towards P_0 and having two Weierstrass points P_i and P_j with Weierstrass directions towards different points Q_i and Q_j as in*

$$P_i \xrightarrow{\mathcal{W}} Q_i \xrightarrow{\mathcal{W}} P_0 \qquad \text{and} \qquad P_j \xrightarrow{\mathcal{W}} Q_j \xrightarrow{\mathcal{W}} P_0.$$

Proof. Again without loss of generality one may assume $Q_i = Q_1 = (1: 0: 0)$ and $Q_j = Q_2 = (0: 1: 0)$. The analysis for the first infinite point Q_1 having Weierstrass directions towards it carries over to Q_2, *mutatis mutandis*, to yield the condition $ac^2 + b^3 + b^2c = 0$. Together with the normalization $b = 1$, the two resulting equations yield

$$(a + c)(ac + 1) = 0,$$

but $ac = 1$ implies $ca^2 + b^3 + b^2a = 1 \neq 0$. The only possibility of having a common solution of the two equations is $a = c$, and this yields $a^3 + a + 1 = 0$. □

One can ask if a longer chain than (8) is possible. This is answered in the following result.

Proposition 5.4. *There are exactly* 9 *curves* $C_{a,b,c}$ *having a configuration like*

$$P \xrightarrow{w} P_\lambda \xrightarrow{w} Q_1 \xrightarrow{w} P_0.$$

These curves are $C_{\lambda^{-3},1,\lambda^3(1+\lambda^3)}$ *for* λ *a root of*

$$h_1(\lambda) = \lambda^9 + \lambda^8 + \lambda^7 + \lambda^6 + \lambda^4 + \lambda^2 + 1.$$

Proof. In the proof of Proposition 5 it was seen that the Weierstrass points having Weierstrass directions towards Q_1 are given by $P_\lambda = (\lambda : a\lambda : 1)$. The pencil of lines through P_λ consists of the line $y = ax$ and the lines with equations $y = (a + \frac{\gamma}{\lambda})x + \gamma$. Substitution of this last equation in the equation (5) of the curve yields the polynomial

$$\begin{aligned}\sum_i c_i x^i &= \left[\lambda^6(1 + \lambda^3)\gamma^3 + \lambda^7\gamma^2 + \lambda^5\gamma + \lambda^3 + 1\right]x^3 + \left[\lambda + \lambda^4 + \lambda^8\gamma^2\right]x^2 \\ &\quad + \left[\lambda^8(1 + \lambda^3)\gamma^3 + \lambda^2(1 + \lambda^3)\right]x + \lambda^5\gamma.\end{aligned}$$

The point P_λ has a Weierstrass direction towards it if and only if this polynomial is a cube; the conditions $\dfrac{c_2}{c_3} = \dfrac{c_1}{c_2} = \dfrac{c_0}{c_1}$ now yield

$$\begin{aligned}c_2^2 + c_1c_3 &= \lambda^7(1 + \lambda^6)\gamma^6 + \lambda^8(1 + \lambda^3)\gamma^5 + \lambda^6\gamma^4 \\ &\quad + \lambda^2(1 + \lambda^3)\gamma^2 + (1 + \lambda^3)\gamma = 0 \\ c_1^2 + c_0c_2 &= \lambda^{12}(1 + \lambda^6)\gamma^6 + \lambda^9\gamma^3 + \lambda^2(1 + \lambda^3)\gamma + (1 + \lambda^6) = 0.\end{aligned}$$

Considered as polynomials in γ these two conditions have a common root if and only if their Sylvester resultant

$$\begin{aligned}\mathcal{R}_1 &= (1+\lambda)^7(\lambda^2+\lambda+1)^7(\lambda^6+\lambda^3+1)(\lambda^9+\lambda^8+\lambda^7+\lambda^6+\lambda^4+\lambda^2+1)\\ &\quad (\lambda^{18}+\lambda^{17}+\lambda^{15}+\lambda^{11}+\lambda^{10}+\lambda^9+\lambda^8+\lambda^7+\lambda^6+\lambda^2+1)\end{aligned}$$

vanishes. The value of λ determines the curve $C_{a,1,\frac{1+a}{a^2}}$ using $a=\lambda^{-3}$, while γ determines the tangent at P_λ in the above curve. It turns out that $\lambda^3 \neq 1$, for otherwise $c=0$, which is a condition outside our moduli hypothesis. This observation leaves the first two factors of $\mathcal{R}_1$ out of consideration.

With the same procedure x can be eliminated using the form of the tangent $x=\beta y+\lambda+\frac{\beta}{\lambda^2}$, which taken into the equation (5) yields an equation in y that will be a cube exactly when two conditions on β and λ are met. The Sylvester resultant of this two conditions is

$$\begin{aligned}\mathcal{R}_2 \;=\; & (1+\lambda)^6(\lambda^2+\lambda+1)^6(\lambda^9+\lambda^8+\lambda^7+\lambda^6+\lambda^4+\lambda^2+1)\\ & (\lambda^{18}+\lambda^{17}+\lambda^{15}+\lambda^{11}+\lambda^{10}+\lambda^9+\lambda^8+\lambda^7+\lambda^6+\lambda^2+1)\end{aligned}$$

As a consequence a Weierstrass direction towards P_λ occurs only in roots of

$$\begin{aligned}h_1(\lambda) &= \lambda^9+\lambda^8+\lambda^7+\lambda^6+\lambda^4+\lambda^2+1 \qquad \text{or}\\ h_2(\lambda) &= \lambda^{18}+\lambda^{17}+\lambda^{15}+\lambda^{11}+\lambda^{10}+\lambda^9+\lambda^8+\lambda^7+\lambda^6+\lambda^2+1.\end{aligned}$$

Both $h_1(\lambda)$ and $h_2(\lambda)$ are irreducible over $\mathbb{F}_2$, but while $h_1(\lambda)$ is irreducible also over $\mathbb{F}_4=\mathbb{F}_2[\alpha]$, with $\alpha^2+\alpha+1=0$. The following factorization of $h_2(\lambda)$ holds

$$h_2(\lambda)=h_1(\alpha\lambda)h_1(\alpha^2\lambda).$$

Thus if $h_1(\lambda)=0$ then all three points $P_\lambda=(\lambda\colon a\lambda\colon 1)$, $P_{\alpha\lambda}=(\alpha\lambda\colon a\alpha\lambda\colon 1)$ and $P_{\alpha^2\lambda}=(\alpha^2\lambda\colon a\alpha^2\lambda\colon 1)$ have in $C_{\lambda^{-3},1,\lambda^3(1+\lambda^3)}$ Weierstrass directions towards them.

While each root of $h_1(\lambda)=0$ accounts for 3 different Weierstrass points P_λ, $P_{\alpha\lambda}$ and $P_{\alpha^2\lambda}$, each having a Weierstrass direction towards it, different such triples occur at different curves, as no 2 roots of $h_1(\lambda)$ have the same cube: this is seen because if λ_1 and λ_2 were distinct roots of $h_1(\lambda)$ with $\lambda_1^3=\lambda_2^3$ then $\lambda_1=\alpha\lambda_2$, and $\lambda=\lambda_2$ would be a common root of $h_1(\lambda)$ and of $h_1(\alpha\lambda)$, and the following would hold:

$$\begin{aligned}0 &= h_1(\lambda)+h_1(\alpha\lambda)\\ &= \alpha\lambda^8+\alpha^2\lambda^7+\alpha^2\lambda^4+\alpha\lambda^2\\ &= \alpha\lambda^2(\lambda+1)(\lambda+\alpha)(\lambda+\alpha^2)(\lambda^3+\alpha\lambda^2+1).\end{aligned}$$

As $\lambda \neq 0$ and $\lambda^3 \neq 1$ the first four factors in this equation are not genuine values of λ. For the last one has just to observe that

$$h_1(\lambda) = (\lambda^3 + \alpha\lambda^2 + 1)(\alpha^2\lambda^2 + \alpha^2\lambda^5 + \lambda^6) + 1 + \alpha\lambda^2,$$

to see that λ cannot simultaneously be a root of $h_1(\lambda)$ and of $\lambda^3 + \alpha\lambda^2 + 1$ without being one of $\lambda^3 + 1$. □

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