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Actions of discrete groups on spheres and real projective spaces

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Abstract. In this paper, we first define discrete, smooth actions on $\mathbb{S}^{2n+1}$, whose limit sets are Cantor sets wildly embedded in $\mathbb{S}^{2n+1}$ (Antoine's necklaces). Secondly, we define Schottky groups on real projective spaces of odd dimensions, $\mathbb{P}^{2n+1}_{\mathbb{R}}$. We lift these actions to (locally) projective actions on the sphere $\mathbb{S}^{2n+1}$ and consider the quotient space of the domain of discontinuity by the group to obtain new examples of manifolds with real projective structures.

Keywords: wild Cantor sets, discrete actions and Kleinian Groups.

Mathematical subject classification: Primary: 57M30; Secondary: 57M45, 57Q45, 30F14.

1 Introduction

M.L. Antoine is one of the great names in the classical study of wild embeddings. A basic example of wild set is *Antoine's necklace*, which may be described as the intersection $X = \cap X_i$ of compact sets $\ldots \subset X_k \subset \ldots \subset X_2 \subset X_1 \subset X_0$. The set X_0 consists of a single unknotted solid torus in $\mathbb{S}^3$; the set X_1 is the union of four unknotted solid tori linked in $\mathrm{Int}(X_0)$; each component A of X_1 contains four solid tori of X_2 linked in A just as the four components of X_1 are linked in X_0; etc. (like the solid tori in Fig. 1). The Cantor set X is homeomorphic to the standard middle-thirds Cantor set $X' \subset [0, 1] \subset \mathbb{S}^1 \subset \mathbb{S}^3$. But no homeomorphism $h\colon \mathbb{S}^3 \to \mathbb{S}^3$ can take the wild Cantor set X into the tame Cantor set X'. This is easily seen from the fact that the simple closed curve

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$J \subset \partial X_0$ represents a nontrivial element of $\Pi_1(\mathbb{S}^3 - X)$ while $\Pi_1(\mathbb{S}^3 - X')$ is trivial. We say that the Cantor set X is *wild* (see Definition 2.4 below).

An example of a topological action on $\mathbb{S}^3$, whose limit set is a wild Cantor set has been constructed by Michael Freedman and Richard Skora ([2]). This action is not quasiconformally conjugate to a *uniformly* quasiconformal action since they have shown that the set of distortions of the quasiconformal homeomorphisms of the action are unbounded (see [16] for the definitions of quasiconformal mapping and quasiconformal distortion).

In this paper, we construct a real analytic action on $\mathbb{S}^3$ in the spirit of Schottky groups, whose limit set is a wild Cantor set. This construction can be generalized to all spheres of odd dimensions.

In the classical case, the Schottky groups are obtained by considering pairwise disjoint $(n-1)$-spheres $S_1, \ldots, S_r$ in $\mathbb{S}^n$ (see [9]). Each sphere S_i plays the role of a *mirror*, i.e. it divides $\mathbb{S}^n$ in two diffeomorphic components, and there exists an *involution* T_i of $\mathbb{S}^n$ interchanging these two components, namely the inversion on S_i.

In our case, roughly speaking, we construct a chain consisting of 4 double links, each link being a closed solid torus, and we replace each link by two disjoint "parallel" links (see Fig. 2). The boundaries of these solid tori are our "mirrors". Our "involutions" consist of conjugates by Möbius transformations of maps $\Psi_\lambda : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$, $\lambda \in \mathbb{R}^+$, defined by $\Psi_\lambda(a, b) = (\lambda b, \lambda^{-1} a)$, for a suitable λ (see Section 2).

In Section 3 we generalize the above construction to obtain wild Cantor sets on $\mathbb{S}^{2n+1}$ as limit sets of discrete groups.

Schottky groups provide us with one of the most interesting families of conformal Kleinian groups. In section 4, we study the analogous construction for groups acting by projective transformations on real projective spaces. These actions can be lifted to the sphere $\mathbb{S}^{2n+1}$. Some of the groups $\widetilde{\Gamma}$ in our examples act freely, properly discontinuously and co-compactly on an invariant open set $\Omega_{\widetilde{\Gamma}} \subset \mathbb{P}^{2n+1}_{\mathbb{R}}$. Moreover, since the action is by restriction of globally defined projective transformations, the compact manifolds $M_{\widetilde{\Gamma}} := \Omega_{\widetilde{\Gamma}}/\widetilde{\Gamma}$ have a projective structure. Manifolds with a projective structure are very interesting and have been studied since the time of Felix Klein's *Earlangen program*. Beautiful examples and more historical references can be obtained in [3], [4], [5] and [15].

2 The Antoine's Necklace as the limit set of a discrete group

Our purpose is to construct an action on $\mathbb{S}^3$ in the spirit of Schottky groups. We will start constructing our "mirrors" and "involutions" on $\mathbb{R}^4$, to obtain a discrete group acting on $\mathbb{S}^3$, whose "limit set" is a Cantor set wildly embedded in $\mathbb{S}^3$.

Let $\tilde{\Psi_\lambda}: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$, $\lambda \in \mathbb{R}^+$ be defined by $\tilde{\Psi_\lambda}(a,b) = (\lambda b, \lambda^{-1}a)$. Then, $\tilde{\Psi_\lambda}$ leaves invariant the set $\hat{E}_\lambda = \{(a,b) \colon ||a||^2 = \lambda^2||b||^2\}$. Clearly, $\hat{E}_\lambda$ separates $\mathbb{R}^4 - \{(0,0)\}$ in two diffeomorphic connected components U and V and these two components are interchanged by $\tilde{\Psi_\lambda}$.

The intersection

$$T_\lambda = \hat{E}_\lambda \cap \mathbb{S}^3 = \left\{(a,b) \colon ||a|| = \frac{|\lambda|}{\sqrt{\lambda^2+1}} \text{ and } ||b|| = \frac{1}{\sqrt{\lambda^2+1}}\right\}$$

is a torus. Let $\bar{T}_\lambda = \{(a,b) \in \mathbb{S}^3 \colon ||a||^2 \leq \lambda^2||b||^2\}$. This is a closed solid torus such that $\partial\bar{T}_\lambda = T_\lambda$. The set $\bar{T}_\lambda$ is a closed tubular neighbourhood in $\mathbb{S}^3$ of the circle $\bar{T}_{S_\lambda} = \{(0,b) \subset \mathbb{S}^3 \colon ||b|| = 1\}$ which we call *the soul* of $\bar{T}_\lambda$. We notice that we can choose λ such that the tubular neighbourhood can be made very thin (i.e. consists of points very close to the soul). For this reason we call λ *the thickness* of $\bar{T}_\lambda$.

Let consider the diffeomorphism of $\mathbb{S}^3$ given by:

$$\Psi_\lambda(a,b) = \left(\frac{\lambda b}{\sqrt{\lambda^2 b^2 + \lambda^{-2}a^2}}, \frac{\lambda^{-1}a}{\sqrt{\lambda^2 b^2 + \lambda^{-2}a^2}}\right).$$

Let $M\ddot{o}b(\mathbb{S}^3)$ denote the group of Möbius transformations of the 3-sphere $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$. Notice that applying Möbius transformations on $\mathbb{S}^3$, each circle (the intersection of a plane in $\mathbb{R}^4$ with $\mathbb{S}^3$) on $\mathbb{S}^3$ can be the image of the soul of a torus $\bar{T}_\lambda$. Thus, there exist $f_1, f_2, f_3, f_4 \in M\ddot{o}b(\mathbb{S}^3)$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}^+$ such that $f_1(\bar{T}_{S_{\lambda_1}})$, $f_2(\bar{T}_{S_{\lambda_2}})$, $f_3(\bar{T}_{S_{\lambda_3}})$, $f_4(\bar{T}_{S_{\lambda_4}})$ form a chain $\tilde{C}_1$ consisting of 4 circles linked in $\mathbb{S}^3$ (see Fig. 1), and the corresponding linked solid tori $f_i(\bar{T}_{\lambda_i})$, $i = 1, \ldots, 4$ are pairwise disjoint.

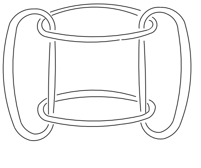

Figure 1: The $\tilde{C}_1$ consisting of 4 solid tori.

This selection of $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}^+$, can be done in such a way that we can add a "parallel" circle to each $f_i(\bar{T}_{S_{\lambda_i}})$ (see Fig. 2), i.e. there exist f_1', f_2', f_3', $f_4' \in M\ddot{o}b(\mathbb{S}^3)$ such that $f_1(\bar{T}_{S_{\lambda_1}})$, $f_2(\bar{T}_{S_{\lambda_2}})$, $f_3(\bar{T}_{S_{\lambda_3}})$, $f_4(\bar{T}_{S_{\lambda_4}})$, $f_1'(\bar{T}_{S_{\lambda_1}})$, $f_2'(\bar{T}_{S_{\lambda_2}})$, $f_3'(\bar{T}_{S_{\lambda_3}})$, $f_4'(\bar{T}_{S_{\lambda_4}})$ form a chain C_1 consisting of 8 components, and the corresponding solid tori $f_i(\bar{T}_{\lambda_i})$, $f_i'(\bar{T}_{\lambda_i})$, $i = 1, \ldots, 4$, are pairwise disjoint.

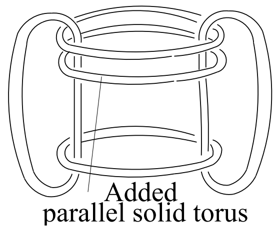

Figure 2: A parallel torus to each original one.

Consider each map $H_i = f_i \circ \Psi_{\lambda_i} \circ f_i^{-1}$ $(H_i' = f_i' \circ \Psi_{\lambda_i} \circ f_i'^{-1})$, $i = 1, \ldots, 4$. By construction, this map sends a copy of the exterior of $f_i(\bar{T}_{\lambda_i})$ $(f_i'(\bar{T}_{\lambda_i}))$ into it. In particular, a copy of the other 7 solid tori is sent into it.

After doing this for each i, we obtain a new chain C_2 consisting of $8 \times 7 = 54$ solid tori. If we apply again the maps H_i, H_i' $i = 1, \ldots, 4$, then we obtain a new chain C_3 consisting of 8×7^2 solid tori. So, in the k^{th}-stage, we obtain the chain C_k consisting of $8 \times 7^{k-1}$. Continuing this process a countable number of times we obtain a sequence $\ldots \subset C_k \subset \ldots \subset C_2 \subset C_1$ of compact sets. Moreover, we may perform this procedure in such a way that the diameters of the components of C_i tend to zero as $i \to \infty$.

We notice that if we compose a large odd number of H_i's or H_i''s we obtain a transformation which is conjugate to a reflection on a torus which becomes of very small diameter.

Let Γ be the group generated by H_i's and H_i''s, $i = 1, \ldots, 4$.

Let $\widetilde{\Gamma}$ be the subgroup of index two of Γ consisting of even words in H_i's and H_i''s, $i = 1, \ldots, 4$.

The "ping-pong" Lemma of F. Klein ([6], Lemma II.24) implies that $\widetilde{\Gamma}$ is a free group.

Definition 2.1. *We define the limit set of* Γ, $\Lambda := \Lambda(\Gamma)$ *to be the set of accumulation points of the* Γ*-orbit of the union* $f_1(\bar{T}_{\lambda_1}) \cup \ldots \cup f_4(\bar{T}_{\lambda_4}) \cup f_1'(\bar{T}_{\lambda_1}) \cup \ldots \cup f_4'(\bar{T}_{\lambda_4})$. *Its complement* $\Omega = \Omega(\Gamma) := \mathbb{S}^3 - \Lambda$ *is the domain of discontinuity.*

Remark 2.2. We note that there is no general definition of a limit set of a discrete group acting on a metric space. For a possible definition, see Kulkarni's definition in [8]. Our definition above is suitable for Schottky groups. We do not know if our definition of limit set coincides with Kulkarni's definition.

By construction, the limit set $\Lambda(\Gamma)$ is given by

$$\Lambda(\Gamma) = \bigcap_{i=1}^{\infty} C_i$$

Lemma 2.3. *The set $\Lambda(\Gamma)$ is a Cantor set.*

Proof. Since each C_k is compact and C_k contains C_{k+1} for each k, these sets satisfy the finite intersection hypothesis and therefore their intersection is non-empty. By construction, it follows that the components of the necklace are single points. In fact, just in the case for standard Schottky groups (see [9]), we can show it is a totally disconnected, compact, perfect metric space. Hence $\Lambda(\Gamma)$ is homeomorphic to the Cantor set. □

Definition 2.4. *A Cantor set $K \subset \mathbb{S}^n$ is tame if the homotopy groups $\Pi_i(\mathbb{S}^n - K) = 0$, for $0 \leq i \leq n-2$ and the group $\Pi_{n-1}(\mathbb{S}^n - K)$ is infinitely generated. Otherwise, K is wild. If K is wild then there is no homeomorphism $h\colon \mathbb{S}^n \to \mathbb{S}^n$ such that $h(K)$ lies in a smoothly embedded arc.*

Lemma 2.5. *The set $\Lambda(\Gamma)$ is wildly embedded in $\mathbb{S}^3$.*

Proof. We will briefly describe the fundamental group of $\mathbb{S}^3 - \Lambda(\Gamma)$. For more details see [10]. Let V be a solid torus such that $C_1 \subset V$. Then the inclusion homomorphism $\Pi_1(\partial V) \to \Pi_1(V - C_1)$ is injective. In particular, for each component $C_{1,j}$ $(j = 1, \ldots, 4)$ of C_1 the inclusion $\partial C_{1,j} \subset C_{1,j} - \mathrm{Int}(C_2)$ induces injective fundamental group homomorphisms. Moreover, we also have that the inclusion homomorphism $\Pi_1(\partial C_{1,j}) \to \Pi_1(V - \mathrm{Int}(C_1))$ is injective.

Consider the diagram of inclusion homomorphisms:

$$\begin{array}{ccc}
\Pi_1(\partial C_{1,1}) & \xrightarrow{i_{1*}} & \Pi_1(C_{1,1} - \mathrm{Int}(C_2)) \\
{\scriptstyle i_{2*}}\big\downarrow & & {\scriptstyle j_{2*}}\big\downarrow \\
\Pi_1(V - \mathrm{Int}(C_1)) & \xrightarrow{j_{1*}} & \Pi_1((V - \mathrm{Int}(C_1)) \cup (C_{1,1} - \mathrm{Int}(C_2))
\end{array}$$

Van Kampen's Theorem implies that if the maps i_{1*} and i_{2*} are injective, then j_{1*} and j_{2*} are also injective. This implies that when we add one component of $C_1 - \mathrm{Int}(C_2)$ to $V - C_1$ the fundamental group is bigger, in other words, the fundamental group of $V - C_1$ is a proper subgroup of the fundamental subgroup of the union of $V - C_1$ with the component (see the argument in pag. [10]).

This argument may be applied again and again to show that all these inclusion homomorphisms are injective:

$$\Pi_1(V - C_1) \to \Pi_1(V - C_2) \to \Pi_1(V - C_3) \to \dots$$

It is clear that these are inclusion of subgroups, i.e. the groups become larger and larger. Similarly with $\mathbb{S}^3$ replacing V.

Thus, the fundamental group $\Pi_1(\mathbb{S}^3 - \Lambda(\Gamma))$ is the direct limit of $\{\Pi_1(\mathbb{S}^3 - C_k),\ \ k = 1, 2, \dots;\quad f_k,\ \ k = 1, 2, \dots)\}$, where $f_k\colon \Pi_1(\mathbb{S}^3 - C_k) \to \Pi_1(\mathbb{S}^3 - C_{k+1})$ is the inclusion map (see Lemma 2.4.1 in [11]).

We have that, $\Pi_1(\mathbb{S}^3 - \Lambda(\Gamma))$ is infinite generated. This implies that $\Lambda(\Gamma)$ is a wild Cantor set in $\mathbb{S}^3$. □

Let $\widetilde{\Gamma}$ be the subgroup of index two of Γ, defined previously, then $\widetilde{\Gamma}$ acts freely and properly discontinuously.

The previous discussion can be summarized in the following theorem.

Theorem 2.6. *There exists a real analytic action of the free group* $\mathbb{F}_8$, $\Phi\colon \mathbb{F}_8 \times \mathbb{S}^3 \to \mathbb{S}^3$, *whose limit set* $\Lambda(\Phi)$ *is a wild Cantor set. The action is proper, free, discontinuous and co-compact on* $\mathbb{S}^3 - \Lambda(\Phi)$.

Remark 2.7. Michael Freedman and Richard Skora [2] have constructed strange discrete *topological* actions analogous to our construction, however our construction is by *real analytic* diffeomorphisms of the sphere.

A *fundamental domain* for Γ is $D = \mathbb{S}^3 - \cup_{i=1}^{4} \operatorname{Int}(f_i(\bar{T}_{\lambda_i}) \cup f_i'(\bar{T}_{\lambda_i}))$. The quotient space $\Omega/\,\Gamma$ is homeomorphic to D.

A fundamental domain for $\widetilde{\Gamma}$ is $D \cup H_1(D)$. Since $\widetilde{\Gamma}$ acts freely, properly and discontinuously on its domain of discontinuity and its fundamental domain is compact, the quotient space $\Omega(\widetilde{\Gamma})/\widetilde{\Gamma}$ is a smooth compact manifold without boundary $M_{\widetilde{\Gamma}}$. Since the action is by orientation-preserving diffeomorphisms, $M_{\widetilde{\Gamma}}$ is orientable.

3 Discrete actions on higher dimensional spheres

The above construction can be generalized to odd dimensional spheres $\mathbb{S}^{2n+1}$, to obtain a discrete, real analytic action on $\mathbb{S}^{2n+1}$, whose "limit set" is a Cantor set wildly embedded in $\mathbb{S}^{2n+1}$.

We define $\Psi_\lambda : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, $\lambda \in \mathbb{R}^+$, as above, i.e. $\Psi_\lambda(a, b) = (\lambda b, \lambda^{-1} a)$. So, Ψ_λ leaves invariant the set $\hat{E}_\lambda = \{(a, b) : ||a||^2 = \lambda^2 ||b||^2\}$. Again, $\hat{E}_\lambda$ separates $\mathbb{R}^{2n+2} - \{(0, 0)\}$ in two diffeomorphic components U and V and these two components are interchanged by Ψ_λ.

The intersection $T_\lambda = \hat{E}_\lambda \cap \mathbb{S}^{2n+1}$ is homeomorphic to $\mathbb{S}^n \times \mathbb{S}^n$. Let $\bar{T}_\lambda \cong \mathbb{S}^n \times \mathbb{D}^{n+1} \subset \mathbb{S}^{2n+1}$ be a closed tubular neighborhood in $\mathbb{S}^{2n+1}$ of $\bar{T}_{S_\lambda} = \{(0, b) \subset \mathbb{S}^{2n+1} : ||b|| = 1\}$ such that $\partial \bar{T}_\lambda = T_\lambda$. The n-sphere $\bar{T}_{S_\lambda}$ is called *the soul* of $\bar{T}_\lambda$. We notice that we can choose λ such that the tubular neighbourhood can be made very thin (i.e. consists of points very close to the soul). For this reason, as before, we call λ *the thickness* of $\bar{T}_\lambda$.

Let PC be the group of homeomorphisms of $\mathbb{S}^n$ generated by projective transformations (action of $GL(n+1, \mathbb{R})$ on rays starting at the origin) and conformal transformations.

As in the previous section, we have that applying Möbius transformations $f_1, \dots f_4$ on $\mathbb{S}^{2n+1}$, we can form a chain consisting of 4 linked components $f_i(\bar{T}_{\lambda_i})$, $i = 1, \dots, 4$ which are pairwise disjoint. The selection of the scalar numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}^+$, can be done in such a way that we can set a parallel component to each $f_i(\bar{T}_{\lambda_i})$ (see Fig. 2), obtaining a new chain C_1 consisting of 8 components, $f_i(\bar{T}_{\lambda_i})$, $f_i'(\bar{T}_{\lambda_i})$ $i = 1, \dots, 4$, which are pairwise disjoint.

As above, there exists maps H_i, (H_i') $i = 1, \dots, 4$ that send a copy of the exterior of the corresponding $f_i(\bar{T}_{\lambda_i})$ $(f_i'(\bar{T}_{\lambda_i}))$ into it. Let Γ be the group generated by $H_i, H_i', i = 1, \dots, 4$. Then Γ is a discrete subgroup of PC. Let $\Lambda(\Gamma)$ be the limit set (see definition 2.1).

Lemma 3.1. *The set $\Lambda(\Gamma)$ is a Cantor set.*

Proof. The proof is straightforward from Lemma 2.3. □

Lemma 3.2. *The set $\Lambda(\Gamma)$ is wildly embedded on $\mathbb{S}^{2n+1}$.*

Proof. (Compare proof of Lemma 2.5). The construction is essentially the same as in dimension 3, therefore we will use the same notation for different stages, e.g. C_1, for the first stage, etc. We will briefly describe the n^{th}-singular homology group of $\Lambda(\Gamma)$. Let V be a solid torus such that $C_1 \subset V$. Then the inclusion homomorphism $H_n(\partial V) \to H_n(V - C_1)$ is injective. In particular, for each component $C_{1,j}$ $(j = 1, \dots, 4)$ of C_1 the inclusion $\partial C_{1,j} \subset C_{1,j} - \mathrm{Int}(C_2)$ induces injective n^{th}-singular homology group homomorphisms. Moreover, we also have that the inclusion homomorphism $H_n(\partial C_{1,j}) \to H_n(V - \mathrm{Int}(C_1))$ is injective.

By Mayer-Vietoris Theorem we have that the maps j_{1*} and j_{2*} are injective.

$$\begin{array}{ccc} H_n(\partial C_{1,1}) & \xrightarrow{i_{1*}} & H_n(C_{1,1} - \mathrm{Int}(C_2)) \\ {\scriptstyle i_{2*}}\big\downarrow & & \big\downarrow{\scriptstyle j_{2*}} \\ H_n(V - \mathrm{Int}(C_1)) & \xrightarrow{j_{1*}} & H_n((V - \mathrm{Int}(C_1)) \cup (C_{1,1} - \mathrm{Int}(C_2)) \end{array}$$

In other words, adding one component of $C_1 - \mathrm{Int}(C_2)$ to $V - C_1$ has simply enlarged the n^{th} homology group.

This argument may be applied again and again to show that all these inclusion homomorphisms are injective:

$$H_n(V - C_1) \to H_n(V - C_2) \to H_n(V - C_3) \to \dots$$

It is clear that these are inclusions of subgroups, i.e. the groups become larger and larger. Similarly with $\mathbb{S}^{2n+1}$ replacing V.

Thus, the n^{th}-singular homology group $H_n(\mathbb{S}^{2n+1} - \Lambda(\Gamma))$ is the direct limit of $H_n(\mathbb{S}^{2n+1} - C_k),\ k = 1, 2, \dots;\ f_k,\ k = 1, 2, \dots)\}$ where $f_k\colon H_n(\mathbb{S}^{2n+1} - C_k) \to H_n(\mathbb{S}^{2n+1} - C_{k+1})$ is the inclusion map (see [1]).

We have that $H_n(\mathbb{S}^{2n+1} - \Lambda(\Gamma))$ is infinitely generated. By Hurewicz homomorphism, this implies that $\Pi_n(\mathbb{S}^{2n+1} - \Lambda(\Gamma))$ is infinitely generated. Hence, $\Lambda(\Gamma)$ is a wild Cantor set in $\mathbb{S}^{2n+1}$. □

The previous discussion can be summarized in the following theorem.

Theorem 3.3. *For any $2m \geq 8$ there exists a real analytic action of the free group $\mathbb{F}_{2m} \subset PC$, $\Phi\colon \mathbb{F}_{2m} \times \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$, $n \geq 1$, whose limit set $\Lambda(\Phi)$ is a wild Cantor set. The action is proper, free, discontinuous and co-compact on $\mathbb{S}^{2n+1} - \Lambda(\Phi)$.*

The *fundamental domain* for Γ is $D = \mathbb{S}^{2n+1} - \cup_{i=1}^{4} \mathrm{Int}(f_i(\bar{T}_{\lambda_i}) \cup f_i'(\bar{T}_{\lambda_i}))$. The quotient space Ω/Γ is homeomorphic to D.

Let $\widetilde{\Gamma}$ be the subgroup of index two of Γ consisting of even words. Its fundamental domain is $D \cup H_1(D)$. Since $\widetilde{\Gamma}$ acts freely, properly and discontinuously on its domain of discontinuity and its fundamental domain is compact, then the quotient space $\Omega(\widetilde{\Gamma})/\widetilde{\Gamma}$ is a smooth compact manifold without boundary $M_{\widetilde{\Gamma}}$. Since the action is by orientation-preserving diffeomorphisms, $M_{\widetilde{\Gamma}}$ is orientable.

4 Real projective Schottky groups on projective spaces and spheres

The purpose of this section is to construct Schottky groups on the $(2n+1)$-real projective space, $\mathbb{P}^{2n+1}_{\mathbb{R}}$ and $(2n+1)$-sphere, $\mathbb{S}^{2n+1}$. In the complex case, complex Schottky groups on $\mathbb{P}^{2n+1}$ were constructed by Seade-Verjovsky (see [14]). It is clear that their construction works perfectly well in the real case but for the sake of completeness we present it explicitly in this paper. *We remark that the proofs of* [14] *still apply for the present case.* However we believe it is important to write explicitly the real case, since it is not presented in [14].

4.1 Real projective Schottky groups on projective spaces

As in the previous section, we will start by constructing our “mirrors” and “involutions” in $\mathbb{P}^{2n+1}_{\mathbb{R}}$.

Consider the subspaces of $\mathbb{R}^{2n+2} = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ defined by $\hat{L}_0 := \{(a,0) \in \mathbb{R}^{2n+2}\}$ and $\hat{M}_0 := \{(0,b) \in \mathbb{R}^{2n+2}\}$. Let $\hat{S}$ be the involution of $\mathbb{R}^{2n+2}$ defined by $\hat{S}(a,b) = (b,a)$. This interchanges $\hat{L}_0$ and $\hat{M}_0$.

Let $\Phi\colon \mathbb{R}^{2n+2} \to \mathbb{R}$ be given by $\Phi(a,b) = ||a||^2 - ||b||^2$. Then, $\hat{E}_{\hat{S}} := \Phi^{-1}(0) = \{(a,b)|\ ||a|| = ||b||\}$ is invariant under multiplication by real numbers. Hence, it is an embedded cone in $\mathbb{R}^{2n+2}$ over $\mathbb{S}^{n+1} \times \mathbb{S}^{n+1}$, with vertex at $0 \in \mathbb{R}^{2n+2}$. Clearly, $\hat{E}_{\hat{S}}$ separates $\mathbb{R}^{2n+2} - \{(0,0)\}$ in two diffeomorphic connected components U and V, which contain respectively $\hat{L}_0 - \{(0,0)\}$ and $\hat{M}_0 - \{(0,0)\}$. These two components are interchanged by the involution $\hat{S}$ and $\hat{E}_{\hat{S}}$ stays invariant. Notice that every linear subspace $\hat{K} \subset \mathbb{R}^{2n+2}$ of dimension $n+2$ containing $\hat{L}_0$ meets transversely $\hat{E}_{\hat{S}}$ and $\hat{M}_0$, since through every point in $\hat{E}_{\hat{S}}$ there exists and affine line in $\hat{K}$ which is transverse to $\hat{E}_{\hat{S}}$. Therefore a tubular neighborhood V of $\hat{M}_0 - \{(0,0)\}$ in $\mathbb{P}^{2n+1}_{\mathbb{R}}$ is obtained, whose normal disc fibers are of the form $\hat{K} \cap V$, with $\hat{K}$ as above.

Let S be the linear projective involution of $\mathbb{P}^{2n+1}_{\mathbb{R}}$ defined by $\hat{S}$. Then $\hat{E}_{\hat{S}}$ projects to a codimension one submanifold of $\mathbb{P}^{2n+1}_{\mathbb{R},}$ that we denote by E_S. Thus the submanifold E_S is an invariant set of S. Moreover, it is a $\mathbb{S}^{2n+1}$-bundle over $\mathbb{P}^n_{\mathbb{R}}$ and it separates $\mathbb{P}^{2n+1}_{\mathbb{R}}$ in two connected components which are interchanged by S and each one is diffeomorphic to a tubular neighborhood of the canonical $\mathbb{P}^n_{\mathbb{R}}$ in $\mathbb{P}^{2n+1}_{\mathbb{R}.}$

Definition 4.1. *We call E_S the canonical mirror and S the canonical involution.*

The previous discussion still applies to the following more general case.

Lemma 4.2. *Let λ be a positive real number and consider the involution*

$$\hat{S}_\lambda \colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1},$$

given by $\hat{S}_\lambda(a, b) = (\lambda b, \lambda^{-1} a)$. Then $\hat{S}_\lambda$ also interchanges $\hat{L}_0$ and $\hat{M}_0$, and the set

$$\hat{E}_\lambda = \left\{(a, b) \colon ||a||^2 = \lambda^2 ||b||^2\right\}$$

satisfies, with respect to $\hat{S}_\lambda$, the analogous properties of $\hat{E}_{\hat{S}}$ and $\hat{S}$ described in the above discussion.

Again $\hat{S}_\lambda$ projects to a linear involution S_λ on $\mathbb{P}^{2n+1}_{\mathbb{R}}$ and $\hat{E}_\lambda$ projects to a codimension one submanifold E_λ of $\mathbb{P}^{2n+1}_{\mathbb{R}}$. Thus S_λ and E_λ satisfy the analogous properties of S and E_S.

Observe that the manifold E_λ gets thinner as λ tends to ∞, and it approaches the L_0-axes. Consider now two arbitrary disjoint projective subspaces L and M of dimension n in $\mathbb{P}^{2n+1}_{\mathbb{R},}$ and the corresponding linear subspaces $\hat{L}$, $\hat{M}$ of $\mathbb{R}^{2n+2}$. So $\mathbb{R}^{2n+2} = \hat{L} \bigoplus \hat{M}$ and there is a linear automorphism $\hat{H}$ that sends $\hat{L}$ to $\hat{L}_0$ and $\hat{M}$ to $\hat{M}_0$. The automorphism $\hat{T} = \hat{H}^{-1} \circ \hat{S}_\lambda \circ \hat{H}$, $\lambda \in \mathbb{R}_+$, is an involution that defines an involution $T = H^{-1} \circ S_\lambda \circ H$ of $\mathbb{P}^{2n+1}_{\mathbb{R}}$ that interchanges L and M. Then we have that T has a mirror, i.e. an invariant set $E = E_T \subset \mathbb{P}^{2n+1}_{\mathbb{R},}$ which separates $\mathbb{P}^{2n+1}_{\mathbb{R}}$ in two connected components which are interchanged by T. Each component is diffeomorphic to a tubular neighborhood of the canonical $\mathbb{P}^n_{\mathbb{R}} \subset \mathbb{P}^{2n+1}_{\mathbb{R}.}$ Moreover, given an arbitrary tubular neighborhood U of L, we can choose T so that the corresponding mirror E_T is contained in the interior of U.

We have that every linear projective involution T of $\mathbb{P}^{2n+1}_{\mathbb{R}}$ that interchanges L and M is conjugate in $PSL(2n+2, \mathbb{R})$ to the canonical involution S. In fact, let $\hat{L}$ and $\hat{M}$ be linear subspaces of $\mathbb{R}^{2n+2}$ as above. Let $\{l_1, \ldots, l_{n+1}\}$ be a basis of $\hat{L}$. Then $\{l_1, \ldots, l_{n+1}, \hat{T}(l_1), \ldots, \hat{T}(l_{n+1})\}$ is a basis of $\mathbb{R}^{2n+2}$. The linear transformation that sends the canonical basis of $\mathbb{R}^{2n+2} = \mathbb{R}^{n+1} \bigoplus \mathbb{R}^{n+1}$ to this basis induces a projective transformation which realizes the required conjugation.

Definition 4.3. *We call mirrors in $\mathbb{P}^{2n+1}_{\mathbb{R}}$ to the images of E_S under the action of $PSL(2n+2, \mathbb{R})$. A mirror is the boundary of a tubular neighborhood of a $\mathbb{P}^n_{\mathbb{R}}$ in $\mathbb{P}^{2n+1}_{\mathbb{R},}$ i.e. it is an $\mathbb{S}^{2n+1}$-bundle over $\mathbb{P}^n_{\mathbb{R}}$.*

The above discussion gives us the following result.

Theorem 4.4. *Let $\mathcal{L} := \{(L_1, M_1), \ldots, (L_r, M_r)\}$, $r > 1$, be a set of r pairs of projective subspaces of dimension n of $\mathbb{P}^{2n+1}_{\mathbb{R}}$, all of them pairwise disjoint. Then:*

1. *There exist involutions* $T_1, \ldots, T_r$ *of* $\mathbb{P}_{\mathbb{R}}^{2n+1}$, *such that each* T_i, $i = 1, \ldots, r$, *interchanges* L_i *and* M_i *and the corresponding mirrors* E_{T_i} *are all pairwise disjoint.*

2. *If we choose the* T_i'*s in this way, then the subgroup of* $PSL(2n+2, \mathbb{R})$ *that they generate is Kleinian.*

3. *Moreover, given a constant* $C > 0$, *we can choose the* T_i*'s so that if* $T := T_{j_1} \cdots T_{j_k}$ *is a reduced word of length* $k > 0$ *(i.e.,* $j_1 \neq j_2 \neq \cdots \neq j_{k-1} \neq j_k$*), then* $T(N_i)$ *is a tubular neighborhood of the projective subspace* $T(L_i)$ *which becomes very thin as k increases:* $d(x, T(L_i)) < C\lambda^k$ *for all* $x \in T(N_i)$, *where* N_i *is the connected component of* $\mathbb{P}^{2n+1} - E_{T_i}$ *that contains* L_i, *for all* $i = 1, \ldots, r$.

Definition 4.5. *A Kleinian group constructed as above will be called a* projective Schottky group.

Definition 4.6. *Given a projective Schottky group* Γ, *we define its limit set* $\Lambda := \Lambda(\Gamma)$ *to be the set of accumulation points of the* Γ*-orbit of the union* $L_1 \cup \ldots \cup L_r$. *Its complement* $\Omega = \Omega(\Gamma) := \mathbb{P}_{\mathbb{R}}^{2n+1} - \Lambda$ *is the region of discontinuity.*

The next results describe the domain of discontinuity and the limit set of real projective Schottky groups.

Lemma 4.7. *Let* Γ *be a projective Schottky group in* $\mathbb{P}_{\mathbb{R}}^{2n+1}$, *generated by involutions* $\{T_1, \ldots, T_r\}$, $n \geq 1$, $r > 1$. *Let* $W = \mathbb{P}_{\mathbb{R}}^{2n+1} - \bigcup_{i=1}^{r} \mathrm{Int}(N_i)$, *where* $\mathrm{Int}(N_i)$ *is the interior of the tubular neighborhood* N_i. *Then W is a compact fundamental domain for the action of* Γ *on* $\Omega(\Gamma)$. *The action on* $\Omega(\Gamma)$ *is properly discontinuous and* $\Omega(\Gamma) = \bigcup_{\gamma \in \Gamma} \gamma(W)$.

Proof. The proof is straightforward from Theorem 2.2 in [14]. □

Theorem 4.8. *Let* Γ *be a projective Schottky group in* $\mathbb{P}_{\mathbb{R}}^{2n+1}$, *generated by involutions* $\{T_1, \ldots, T_r\}$, $n \geq 1$, $r > 1$, *as in Theorem* 4.4. *Let* $\Omega(\Gamma)$ *be the region of discontinuity of* Γ *and let* $\Lambda(\Gamma) = \mathbb{P}_{\mathbb{R}}^{2n+1} - \Omega(\Gamma)$ *be the limit set. Then,*

1. *If* $r > 2$, *then* $\Lambda(\Gamma)$ *is a solenoid (lamination), homeomorphic to* $\mathbb{P}_{\mathbb{R}}^{n} \times C$, *where C is a Cantor set,* Γ *acts minimally on the set of projective subspaces in* $\Lambda(\Gamma)$ *considered as a closed subset of the Grassmannian* $G_{2n+1,n}$.

2. *If $r > 2$, let $\tilde{\Gamma} \subset \Gamma$ be the index two subgroup consisting of the elements which are reduced words of even length. Then $\tilde{\Gamma}$ acts freely on $\Omega(\Gamma)$. The compact manifold with boundary $\widetilde{W} = W \cup T_1(W)$ is a fundamental domain for the action of $\tilde{\Gamma}$ on $\Omega(\Gamma)$. We also call $\tilde{\Gamma}$ a projective Schottky group.*

3. *Each element $\gamma \in \tilde{\Gamma}$ leaves invariant two copies, P_1 and P_2, of $\mathbb{P}^n_{\mathbb{R}}$ in $\Lambda(\Gamma)$. For every $L \subset \Lambda(\Gamma)$, $\gamma^i(L)$ converges to P_1 (or to P_2) as $i \to \infty$ (or $i \to -\infty$).*

Proof. The proof is straightforward from Theorem 2.2 in [14]. $\square$

Remarks 4.9.

1. The limit set $\Lambda(\Gamma)$ is the intersection of nested sets. In fact, $\Lambda(\Gamma) = \cap_{i=1}^{\infty} \gamma_i(N_{j(i)})$, where $\{\gamma_i\}_{i=1}^{\infty}$ is a sequence of distinct elements of Γ and $j : \mathbb{N} \to \{1, \ldots, r\}$ is a function such that $\gamma_{i+1}(N_{j(i+1)}) \subset \gamma_i(N_{j(i)})$.

2. If $r = 2$, then $\Gamma \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, the infinite dihedral group, and $\Lambda(\Gamma)$ is the union of two disjoint projective subspaces L and M of dimension n.

Next, we will describe the quotients $\Omega(\Gamma)/\Gamma$ and $\Omega(\Gamma)/\tilde{\Gamma}$, where Γ and $\tilde{\Gamma}$ are the above groups.

Proposition 4.10. *Let L be a copy of the projective space $\mathbb{P}^n_{\mathbb{R}}$ in $\mathbb{P}^{2n+1}_{\mathbb{R}}$ and let x be a point in $\mathbb{P}^{2n+1}_{\mathbb{R}} - L$. Let $K_x \subset \mathbb{P}^{2n+1}_{\mathbb{R}}$ be the unique copy of the projective space $\mathbb{P}^{n+1}_{\mathbb{R}}$ in $\mathbb{P}^{2n+1}_{\mathbb{R}}$ that contains L and x. Then K_x intersects transversely every other copy of $\mathbb{P}^n_{\mathbb{R}}$ embedded in $\mathbb{P}^{2n+1}_{\mathbb{R}} - L$, and this intersection consists of one single point. Thus, given two disjoint copies L and M of $\mathbb{P}^{2n+1}_{\mathbb{R}}$ in $\mathbb{P}^{2n+1}_{\mathbb{R}}$, there is a canonical projection map*

$$\pi := \pi_L : \mathbb{P}^{2n+1}_{\mathbb{R}} - L \to M,$$

which is a submersion. Each fiber $\phi^{-1}(x)$ is diffeomorphic to $\mathbb{R}^{n+1}$.

Proof. It is straightforward. $\square$

Theorem 4.11. *Let Γ be a projective Schottky group as in Theorem* 4.8*, with $r > 2$. Let $\tilde{\Gamma} \subset \Gamma$ be the index two subgroup. Let W be the fundamental domain of Γ. Then,*

1. *The map $\psi : W \to \mathbb{P}^n_{\mathbb{R}}$ is a locally trivial differentiable fiber bundle with fiber $\mathbb{S}^{n+1} - \mathrm{Int}(D_1) \cup \cdots \cup \mathrm{Int}(D_r)$, where $\mathrm{Int}(D_i)$ is the interior of a smooth closed $n+1$-disc D_i in $\mathbb{S}^{n+1}$ and the D_i's are pairwise disjoint.*

2. *The domain of discontinuity $\Omega(\Gamma)$ fibers differentiably over $\mathbb{P}^n_{\mathbb{R}}$ with fiber $\mathbb{S}^{n+1}$ minus a Cantor set.*

3. *The space $\Omega(\Gamma)/\tilde{\Gamma}$ is a compact manifold that fibers over $\mathbb{P}^n_{\mathbb{R}}$, with fiber $(\mathbb{S}^n \times \mathbb{S}^1)\# \cdots \#(\mathbb{S}^n \times \mathbb{S}^1)$, the connected sum of $r - 1$ copies of $(\mathbb{S}^n \times \mathbb{S}^1)$.*

Proof. The proof is straightforward from Theorem 2.2 in [14]. □

Remark 4.12. The compact manifolds $M_{\tilde{\Gamma}} := \Omega_{\tilde{\Gamma}}/\widetilde{\Gamma}$ have a projective structure, since the action is by restriction of globally defined projective transformations, see [3], [4], [5] and [15].

4.2 Real Projective Schottky Groups on Spheres

Let $p\colon \mathbb{S}^{2n+1} \to \mathbb{P}^{2n+1}$ be a two-fold covering map. Note that the group $PSL(2n+2, \mathbb{R})$ can be lifted to the group $\pm SL(2n+2, \mathbb{R})$. Then each n-projective space L_i is lifted to a n-sphere S_i in $\mathbb{S},^{2n+1}$ $i = 1, \ldots, r$. Each involution T_i can be lifted to an involution $\hat{T}_i$ in $\mathbb{S},^{2n+1}$ $i = 1, \ldots, r$. Let $\hat{\Gamma}$ be the group generated by $\hat{T}_i$, $i = 1, \ldots, r$. Then $\hat{\Gamma}$ is a discrete subgroup of $SL(2n+1, \mathbb{R})$.

From the above and Theorem 4.8, we have the following result.

Corollary 4.13. *Let $S_1, \ldots S_r$ be n-spheres as above and let $\hat{\Gamma}$ be the corresponding discrete subgroup. Then the linking number $l(S_i, S_j) = 1$ for $i \neq j$ and the limit set $\Lambda(\hat{\Gamma})$ is a solenoid, homeomorphic to $\mathbb{S}^n \times C$, where C is a Cantor set.*

Remark 4.14. In these examples, the Cantor set C is tame.

Next, we will describe the quotients $\Omega(\hat{\Gamma})/\hat{\Gamma}$ and $\Omega(\hat{\Gamma})/\tilde{\hat{\Gamma}}$, where $\hat{\Gamma}$ is the above group and $\tilde{\hat{\Gamma}} \subset \hat{\Gamma}$ is a subgroup consisting of even words, i.e. $\tilde{\hat{\Gamma}}$ is the orientation-preserving index two subgroup of $\hat{\Gamma}$. From Theorem 4.11, we have the following result.

Corollary 4.15. *Let $\hat{\Gamma}$ be the lifting of a projective Schottky group Γ via the covering map $p\colon \mathbb{S}^{2n+1} \to \mathbb{P}^{2n+1}$, with $r > 2$. Let $\tilde{\hat{\Gamma}} \subset \hat{\Gamma}$ be the index two subgroup. Then,*

1. *The domain of discontinuity $\Omega(\hat{\Gamma})$ fibers differentiably over $\mathbb{S}^n$ with fiber $\mathbb{S}^{n+1}$ minus a Cantor set.*
2. *The space $\Omega(\hat{\Gamma})/\tilde{\hat{\Gamma}}$ is a compact manifold that fibers over $\mathbb{S}^n$, with fiber $(\mathbb{S}^n \times \mathbb{S}^1)\# \cdots \#(\mathbb{S}^n \times \mathbb{S}^1)$, the connected sum of $r-1$ copies of $(\mathbb{S}^n \times \mathbb{S}^1)$.*

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