BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 39(2), 183-189

A note about the truncation question in percolation of words

Bernardo N.B. de Lima*

Abstract. Consider an independent site percolation model on $\mathbb{Z}^2$, with parameter p, equipped with all horizontal and vertical connections. In this note it is shown that given for any parameter $p \in (0, 1)$, there exists an integer N such that any binary sequence (word) $\xi \in \{0, 1\}^{\mathbb{N}}$ is seen, almost surely, even if all connections whose length is bigger than N are suppressed.

Keywords: percolation of words, truncation question.

Mathematical subject classification: 82B20, 82B41, 82B43.

1 Introduction

Percolation of words on a countable infinite and connected graph G, as proposed in [1], is formulated as follows. Let $G = (\mathbb{V}, \mathbb{E})$ be a graph and assign, independently for each site $v \in \mathbb{V}$, a random variable X_v which takes the value 1 or 0 with probability p or $1 - p$, respectively. This can be done introducing the probability space $(\Omega, \mathcal{F}, P_p)$, where $\Omega = \{0, 1\}^{\mathbb{V}}$, $\mathcal{F}$ is the σ-algebra generated by the cylinder sets in Ω, and $P_p = \prod_{v\in\mathbb{V}} \mu(v)$ is the product of Bernoulli measures with parameter p, such that the random variables (configurations) $\{X_v : v \in \mathbb{V}\}$ take their values. From now on, we denote an element of Ω (configuration) by ω and ω_v denotes the value of $X_v(\omega)$. A path γ on G is a sequence $v_1, v_2, \dots$ of vertices in $\mathbb{V}$, such that $v_i \neq v_j,\ \forall\, i \neq j$ and v_{i+1} is a nearest neighbour of v_i, for all i; that is the edge $\langle v_i, v_{i+1}\rangle$ belongs to $\mathbb{E}$.

A semi-infinite binary sequence $\xi = (\xi_1, \xi_2, \dots) \in \{0, 1\}^{\mathbb{N}}$ will be called a *word*. Given a word $\xi \in \{0, 1\}^{\mathbb{N}}$, a vertex $v \in \mathbb{V}$ and a configuration $\omega \in \Omega$, we say that ξ *is seen in the configuration* ω *from the vertex* v if there exists a self-avoiding path $\gamma = \langle v_0 = v, v_1, v_2, \dots\rangle$ starting from v, such that $\omega_{v_i} = \xi_i,\ \forall i =$

Received 6 August 2007.

*Partially supported by CNPq.

$1, 2, \ldots$. Define the events $Z_\xi(v) = \{\omega \in \Omega \colon \xi$ is seen in ω from v on the graph $G\}$ and $Z_\xi = \cup_{v \in \mathbb{V}} Z_\xi(v)$. When Z_ξ occurs we say that the word ξ percolates. Observe that percolation of $\xi = (1, 1, 1, \ldots)$ (or $(0, 0, 0, \ldots)$) is the ordinary site percolation and percolation of $\xi = (1, 0, 1, 0 \ldots)$ (or $(0, 1, 0, 1, \ldots)$) is the AB-percolation model proposed in [9].

Another variation of the ordinary independent nearest neighbour percolation model are the long range models. The set of vertices of G will be supposed as $\mathbb{Z}^d$, for $d \geq 2$ and the set of edges $\mathbb{E}$ contains edges whose lengths are arbitrarily large. Given a percolation system whose edges have unbounded range and in which percolation occurs almost surely, we ask if the infiniteness of the range is really crucial for percolation to occur. In other words, does this system still percolates if we suppress all edges whose range are bigger than some positive integer N? This is the *truncation question* and there are lot of results in the literature concerning this question, like [10], [11], [2], [4] and [5]. In the present paper, we show a result concerning the truncation question in the context of percolation of words. For this intention will be useful the embedding of graphs developed in [5].

2 The main result

From now on, let $G = (\mathbb{V}, \mathbb{E})$ be the graph in which $\mathbb{V} = \mathbb{Z}^2$ and which contains all vertical and horizontal long range edges

$$\begin{aligned}\mathbb{E} \; = \; \big\{ &\langle (x_1, x_2)(y_1, y_2) \rangle \subset \mathbb{Z}^2 \times \mathbb{Z}^2 \colon (x_1 = y_1 \text{ and } x_2 \neq y_2) \text{ or} \\ &(x_2 = y_2 \text{ and } x_1 \neq y_1) \big\}.\end{aligned}$$

We consider independent site percolation, with parameter p, on the graph G. It is easy to see, using Borel-Cantelli Lemma, that for all $p \in (0, 1)$, we have that $P_p(Z_\xi(0)) = 1, \ \forall\, \xi \in \{0, 1\}^{\mathbb{N}}$. That is, each word ξ is seen from the origin almost surely. In particular $P_p(Z_\xi) = 1$, *i.e.* ξ percolates. The result below shows that it is possible to truncate the graph G, such that each word ξ still percolates with probability one.

Before stating the result, we introduce some notation. Given $x = (x_1, x_2) \in \mathbb{Z}^2$, let $\|(x_1, x_2)\| = |x_1| + |x_2|$. Given a positive integer N, let

$$G_N = (\mathbb{Z}^2, \mathbb{E}_N) \quad \text{where} \quad \mathbb{E}_N = \{\langle xy \rangle \in \mathbb{E} \colon \|x - y\| \leq N\}.$$

That is, G_N can be obtained from G erasing all bonds whose length is larger than N. As before, define $Z_\xi^N(v) = \{\omega \in \Omega \colon \xi$ is seen in ω from v on the graph $G_N\}$ and $Z_\xi^N = \cup_{v \in \mathbb{V}} Z_\xi^N(v)$.

Theorem 1. *For all $p \in (0, 1)$, there exist a positive integer $N = N(p)$ and some constant $c > 0$, such that*

$$P_p(Z_\xi^N(v)) \geq c > 0, \ \forall\, \xi \in \{0, 1\}^{\mathbb{N}}.$$

In particular,

$$P_p(Z_\xi^N) = 1, \ \forall\, \xi \in \{0, 1\}^{\mathbb{N}}.$$

We would like to emphasize that this constant N is a function of p, it can be chosen uniformly in ξ, but not in p.

Proof. Without loss of generality we can suppose that the vertex v is the origin. Given $p \in (0, 1)$, by [7] Theorem 1, there exists some dimension $d = d(p) \geq 3$, such that

$$p_c(\mathbb{Z}^d) < \min\big\{p, 1 - p\big\}$$

where $p_c(\mathbb{Z}^d)$ is the critical threshold of ordinary independent site percolation on the nearest neighbour lattice $\mathbb{Z}^d$. Let $\epsilon = \min\{p, 1 - p\} - p_c(\mathbb{Z}^d)$. By Theorem A of [6], there exists some large integer $L = L(p)$ such that

$$p_c\big(\{0, 1, \ldots, L - 1\}^{d-2} \times \mathbb{Z}^2\big) < p_c(\mathbb{Z}^d) + \frac{\epsilon}{2} < \min\big\{p, 1 - p\big\}.$$

That is, on the slab $\{0, 1, \ldots, L - 1\}^{d-2} \times \mathbb{Z}^2$ both words $\bar{0} = (0, 0, 0, \ldots)$ and $\bar{1} = (1, 1, 1, \ldots)$ are seen from the origin with positive probability. Now the method leading to the result [5] allows to embed the slab $\{0, 1, \ldots, L - 1\}^{d-2} \times \mathbb{Z}^2$ in G. This embedding uses only bonds with $d - 1$ different lengths (see [5]). Then, defining $N = N(p)$ as the largest such length used in the embedding, it follows that

$$P_p\big(Z_{\bar{1}}^N(0)\big) > 0 \qquad \text{and} \qquad P_p\big(Z_{\bar{0}}^N(0)\big) > 0.$$

By Lemma 2 hereafter, this implies

$$P_p\big(Z_\xi^N(0)\big) \geq \min\big\{P_p\big(Z_{\bar{0}}^N(0)\big), P_p\big(Z_{\bar{1}}^N(0)\big)\big\}, \ \forall\, \xi \in \{0, 1\}^{\mathbb{N}}$$

and the proof will be finished.

The last sentence statement of the theorem is true, observing that the event Z_ξ^N is translation invariant, so its probability must be 0 or 1. □

The following lemma is a generalization of the coupling argument given in the proof of Proposition 3.1 in [12] for AB-Percolation. Now, $G = (\mathbb{V}, \mathbb{E})$ is any countably infinite and connected graph; given $v \in \mathbb{V}$ and $\xi \in \{0,1\}^{\mathbb{N}}$ remember that $Z_\xi(v) = \{\omega \in \Omega\colon \xi \text{ is seen in } \omega \text{ from } v \text{ on the graph } G\}$, $\bar{0} = (0,0,0,\dots)$ and $\bar{1} = (1,1,1,\dots)$.

Lemma 2. *Consider an independent site percolation model, with parameter* $p \in [0,1]$*, on the graph* G*. Then, given any binary sequence* $\xi \in \{0,1\}^{\mathbb{N}}$ *and* $v_0 \in \mathbb{V}$*, it holds that*

$$P_p\big(Z_\xi(v_0)\big) \geq \min\big\{P_p\big(Z_{\bar{0}}(v_0)\big), P_p\big(Z_{\bar{1}}(v_0)\big)\big\}, \ \forall\, \xi \in \{0,1\}^{\mathbb{N}}.$$

Proof. By symmetry it is enough to consider the case $p \in (0, \frac{1}{2}]$ and to show that $P_p(Z_\xi(v_0)) \geq P_p(Z_{\bar{1}}(v_0))$.

The idea is to make a coupling based on the dynamic construction of the cluster at the origin, in such a way that for each configuration in which the word $(1,1,1,\dots)$ is seen, the given word ξ also is.

Let U_v, $v \in \mathbb{V}$ be independent random variables with the uniform distribution on $[0,1]$, and write $\tilde{\Omega} = [0,1]^{\mathbb{V}}$. With $\tilde{\mathcal{F}}$ the σ-algebra generated by cylinder sets and $\mathbb{P}$ the associated product probability measure on $(\tilde{\Omega}, \tilde{\mathcal{F}})$. Given $p \in [0,1]$ and $v \in \mathbb{V}$ we say that v is p-open if $U_v \leq p$ and p-closed if $U_v > p$.

Now, let $f\colon \mathbb{N} \to \mathbb{V}$ be a fixed ordering of the vertices of the graph G. Construct, dynamically, a sequence $S_n = (A_n, B_n, C_n)$, $\forall n \in \mathbb{N}$ of ordered triple of subsets of $\mathbb{V}$, defined as follows:

$$S_0 = (\emptyset, \emptyset, \emptyset).$$

Given $v_0 \in \mathbb{V}$, let's define v_1 as the earliest vertex in the fixed ordering belonging to $\partial_e(\{v_0\})$. Where, $\partial_e(A) = \{v \in \mathbb{V}\colon v \in A^c \text{ and } \exists u \in A \text{ with } \langle v,u\rangle \in \mathbb{E}\}$. Consider the set $I_1 = \{\xi_1\}$. If $I_1 = \{1\}$, define

$$S_1 = \begin{cases} (A_0 \cup \{v_1\}, B_0, C_0) & \text{if } U_{v_1} \leq p, \\ (A_0, B_0, C_0 \cup \{v_1\}) & \text{if } U_{v_1} > p. \end{cases} \tag{1}$$

If $I_1 = \{0\}$, define

$$S_1 = \begin{cases} (A_0, B_0 \cup \{v_1\}, C_0) & \text{if } U_{v_1} \leq 1-p, \\ (A_0, B_0, C_0 \cup \{v_1\}) & \text{if } U_{v_1} > 1-p. \end{cases} \tag{2}$$

Given $S_n = (A_n, B_n, C_n)$, let v_{n+1} denote the earliest vertex in the fixed ordering belonging to $\partial_e(\{v_0\} \cup A_n \cup B_n \cup C_n)$. If that vertex does not exist, define $S_m = S_n$, $\forall m > n$; otherwise, consider the set

$$I_{n+1} = \{x \in \{0,1\}\colon \exists\, k \in \mathbb{N}, \exists \text{ path } \langle v_0, v_{i_1}, \dots, v_{i_k}, v_{i_{k+1}} = v_{n+1}\rangle$$

with

$$v_{i_j} \in A_n \cup B_n, \xi_j = 1 \text{ if } v_{i_j} \in A_n, \xi_j = 0 \text{ if } v_{i_j} \in B_n, \forall j \leq k \text{ and } \xi_{k+1} = x\}.$$

There are four possibilities for I_{n+1}: If $I_{n+1} = \emptyset$, define $S_m = S_n,\ \forall m > n$. If $I_{n+1} = \{1\}$, define

$$S_{n+1} = \begin{cases} (A_n \cup \{v_{n+1}\}, B_n, C_n) & \text{if } U_{v_{n+1}} \leq p, \\ (A_n, B_n, C_n \cup \{v_{n+1}\}) & \text{if } U_{v_{n+1}} > p. \end{cases} \tag{3}$$

If $I_{n+1} = \{0\}$ define

$$S_{n+1} = \begin{cases} (A_n, B_n \cup \{v_{n+1}\}, C_n) & \text{if } U_{v_{n+1}} \leq 1-p, \\ (A_n, B_n, C_n \cup \{v_{n+1}\}) & \text{if } U_{v_{n+1}} > 1-p. \end{cases} \tag{4}$$

and if $I_{n+1} = \{0, 1\}$, define

$$S_{n+1} = \begin{cases} (A_n \cup \{v_{n+1}\}, B_n, C_n) & \text{if } U_{v_{n+1}} \leq p, \\ (A_n, B_n \cup \{v_{n+1}\}, C_n) & \text{if } U_{v_{n+1}} > p. \end{cases} \tag{5}$$

Observe that (A_n), (B_n) and (C_n) are non decreasing sequences and define $A = \cup_{n\in\mathbb{N}} A_n$, $B = \cup_{n\in\mathbb{N}} B_n$ and $C = \cup_{n\in\mathbb{N}} C_n$. Given the word ξ, A and B are the set of vertices that form the cluster of the origin, each such vertex being 1 and 0 depending if it belongs to A or B, respectively. C is the set of vertices that belong to the external boundary of the cluster of the origin; and I_{n+1} is the set of possible values that allows v_{n+1} to belong to the cluster of the origin. By dynamic construction

$$\{\tilde{\omega} \in \tilde{\Omega}\colon |(A \cup B)(\tilde{\omega})| = \infty\} \subset \{\tilde{\omega} \in \tilde{\Omega}\colon \forall\, k \in \mathbb{N}, \exists \text{ path } \langle v_0, v_{i_1}, \ldots, v_{i_k}\rangle$$

with

$$v_{i_j} \in A \cup B, \xi_j = 1 \text{ if } v_{i_j} \in A, \xi_j = 0 \text{ if } v_{i_j} \in B, \forall j \leq k\},$$

then

$$P_p(Z_\xi(0)) \geq \mathbb{P}\{\tilde{\omega} \in \tilde{\Omega}\colon |A \cup B(\tilde{\omega})| = \infty\}. \tag{6}$$

By definition of C, each vertex $v \in C$ is p-closed (remember that $p \leq 1/2$). Thus, if $C(p)$ is the p-open cluster of the origin, we have that $A \cup B \supset C(p)$, end so

$$\begin{aligned} \mathbb{P}\{\tilde{\omega} \in \tilde{\Omega}\colon |A \cup B(\tilde{\omega})| = \infty\} &\geq \mathbb{P}\{\tilde{\omega} \in \tilde{\Omega}\colon |C(p)(\tilde{\omega})| = \infty\} \\ &= P_p(Z_{\bar{1}}(0)). \end{aligned} \tag{7}$$

Combining inequalities 6 and 7, the lemma is proved. □

3 An open question

Let us suppose that the sequence of digits in the word ξ is a sequence of independent Bernoulli random variables with parameter α, therefore each word ξ take its values in the probability space $(\{0,1\}^{\mathbb{N}}, \mathcal{A}, Q_\alpha)$, where $\mathcal{A}$ is the σ-algebra generated by the cylinder sets in $\{0,1\}^{\mathbb{N}}$, and $Q_\alpha = \prod_{n\in\mathbb{N}} \mu(n)$ is the product of Bernoulli measures with parameter α.

The theorem above shows that for all $p \in (0,1)$ there exists some integer $N(p) = N$ such that

$$Q_\alpha\{\xi \in \{0,1\}^{\mathbb{N}} \colon P_p(Z_\xi^N) = 1\} = Q_\alpha(\{0,1\}^{\mathbb{N}}) = 1.$$

Given $\omega \in \Omega$, define

$$S(\omega) = \big\{\xi \in \{0,1\}^{\mathbb{N}} : \xi \text{ is seen on } G_N \text{ from some vertex in the configuration } \omega\big\},$$

and

$$\Lambda = \big\{(\xi,\omega) \in \{0,1\}^{\mathbb{N}} \times \Omega \colon \xi \text{ is seen on } G_N \text{ from some vertex in the configuration } \omega\big\}.$$

The $\mathcal{A} \times \mathcal{F}$ measurability of Λ is shown in Proposition 1 of [1]. Then, using Fubini's Theorem, we obtain

$$P_p\big\{\omega \in \Omega \colon Q_\alpha(S(\omega)) = 1\big\} = 1.$$

When this occur we say that *the random word percolates*.

An interesting and difficult question is to find out whether it is true or not that

$$P_p\big\{\omega \in \Omega \colon S(\omega) = \{0,1\}^{\mathbb{N}}\big\} = 1.$$

In words, is it possible to find some truncation length $\tilde{N}(p)$ (maybe different from the previous $N(p)$) such that we can see all words instead of Q_α*-almost all* words P_p-almost surely? Observe that we can see all words on the graph G with infinite range bonds, P_p-almost surely.

Knowing whether it is possible to see all words on some graph where the random word percolates is a difficult question in general. For example, using the upper bound given in [3] and the same modification of the coupling argument in [12] used above, it is possible to see that the random word percolates on $\mathbb{Z}^3$ for $p \in (p_c(\mathbb{Z}^3), 1 - p_c(\mathbb{Z}^3))$, but is still a open question if it is possible or not to see all words, like mentioned in the Open Question 2 in [1].

In this direction, two interesting results are: the Theorem 5 in [1], an example of tree where the random word percolates but not all words, and the paper [8], where it is proved that on the closed packed graph of $\mathbb{Z}^2$, for $p \in (1 - p_c(\mathbb{Z}^2), p_c(\mathbb{Z}^2))$, all words are seen P_p-almost surely.

References

[1] I. Benjamini and H. Kesten. *Percolation of arbitrary words in* $\{0, 1\}^{\mathbb{N}}$. The Annals of Probability, **23** (3) (1995), 1024–1060.

[2] N. Berger. *Transience, Recurrence and Critical Behavior for Long-Range Percolation*. Commun. Math. Phys., **226** (2002), 531–558.

[3] M. Campanino and L. Russo. *An upper bound for the critical percolation for the three dimensional cubic lattice*. The Annals of Probability, **13** (1985), 478–491.

[4] S. Friedli and B.N.B. de Lima. *On the truncation systems with non-Summable Interactions*. Journal of Statistical Physics, **122**(6) (2006), 1215–1236.

[5] S. Friedli, B.N.B. de Lima and V. Sidoravicius. *On Long range Percolation with Heavy Tails*. Eletronic Communications in Probability, **9** (2004), 175–177.

[6] G. Grimmett and J.M. Marstrand. *The Supercritical Phase of Percolation is Well Behaved*. Proc. Roy. Soc. London Ser A, **430** (1990), 439–457.

[7] H. Kesten. *Asymptotics in high dimensions for percolation*. Disorder in physical systems, 219–240, Oxford Sci. Publ., Oxford Univ. Press, New York, (1990).

[8] H. Kesten, V. Sidoravicius and Y. Zhang. *Percolation of arbitrary words on the closed-packed graph of* $\mathbb{Z}^2$. Eletronic Journal of Probability, **6** (2001), 1–27.

[9] T. Mai and J.W. Halley. *AB Percolation on a triangular lattice*. In ordering two dimensions (S.K. Sinha, ed), 369–371. North Holland, Amsterdam (1980).

[10] R. Meester and J.E. Steif. *On the Continuity of the Critical Value for Long Range Percolation in the Exponential Case*. Commun. Math. Phys., **180** (1996), 483–504.

[11] V. Sidoravicius, D. Surgailis and M.E. Vares. *On the Truncated Anisotropic Long-Range Percolation on* $\mathbf{Z}^2$. Stoch. Proc. and Appl., **81** (1999), 337–349.

[12] J.C. Wierman. *AB Percolation: a brief survey*. Combinatorics and Graph Theory, **25** (1989), 241–251, Banach Center Publications.

Bernardo N.B. de Lima
Universidade Federal de Minas Gerais
Av. Antônio Carlos, 6627
Cx. Postal 702, CEP 30123-970
Belo Horizonte, MG
BRAZIL

E-mail: bnblima@mat.ufmg.br