BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 39(3), 341-354

Holonomy of a foliation by principal curvature lines*

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Abstract. We give a geometrical description of the principal curvature lines on an immersed canal hypersurface, and study the holonomy of the one-dimensional principal foliation that they define.

Keywords: canal hypersurface, principal curvature lines, conformal maps.

Mathematical subject classification: 53A30, 57R30, 53C12.

1 Introduction

In ([Mo]) Monge presents the principal curvature lines of an ellipsoid. They form two orthogonal foliations which have four singular points, the four umbilics of the surface. These umbilic points are of Darbouxian type, [GS1, GS3].

A systematic qualitative study of the behavior of principal curvature lines on surfaces of $\mathbb{R}^3$ was initiated by C. Gutierrez an J. Sotomayor ([GS1, GS2, GS3]). For a survey see [GaS].

They were already known for particular surfaces like regular Dupin cyclides, where principal curvature lines form two foliations by circles. The return map of both curvature foliations is, for Dupin cyclides, the identity.

In this article we will describe the principal curvature lines of a *canal surface* that is the envelope of a one-parameter family of spheres of $\mathbb{R}^3$ or $\mathbb{S}^3$, and more generally the one-dimensional foliation by principal curvature lines of a canal hypersurface, that is the envelope of a one-parameter family of $(n-1)$-dimensional spheres in $\mathbb{R}^n$ or $\mathbb{S}^n$. First results about principal curvature lines of a

Received 18 August 2006.

*This work was done under the project CNPq/473747/2006-5 and Cooperation Brazil-France.

**This author is fellow of CNPq.

***This author used the France-Brazil cooperation agreement.

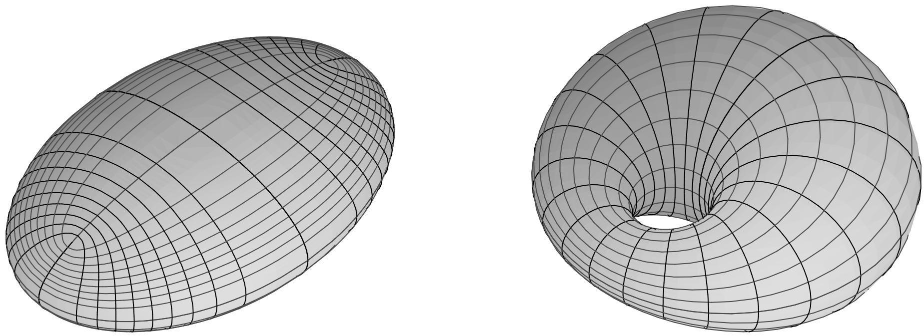

Figure 1: The lines of curvature of an ellipsoid and of a Dupin cyclide.

canal hypersurface in $\mathbb{R}^4$ were already obtained in ([Ga]); examples in dimension two can be found in ([GLS]).

Our main goal in this article is the following global theorem:

Theorem 0.1. *The holonomy of the foliation $\mathcal{F}_1$ defined by the principal curvature lines orthogonal to the characteristic spheres of a regular canal hypersurface is a Möbius map of the chosen characteristic sphere.*

Remark. The holonomy $\varphi_t : CS(t_1) \to CS(t_2), t = t_2 - t_1$ defined by the foliation $\mathcal{F}_1$ is also a flow by conformal maps.

The article will first recall the fact that the holonomy of the lines of curvature of boundary of tubular neighborhood of a curve $C \subset \mathbb{R}^n$ is an isometry.

The proof of our theorem is very similar, and almost computation-free, using the quasi-riemannian structure of the space of spheres.

We will define a canal hypersurface of $\mathbb{R}^{n-2}$; then we will show that the characteristic spheres of a canal are osculating spheres corresponding to the $(n-4)$ equal principal curvatures. We consider the flow given by the last principal curvature of the canal. Extending this flow to the Lorentz space $\mathbb{L}^n$, we prove geometrically that the holonomy of this flow defined on characteristic spheres is a Möbius map (Theorem 0.1).

1 Canals and the space of spheres

1.1 The simplest case

Embedded tubular neighborhoods of closed curves tubes are special canals. Lines of curvature on the boundary of a tubular neighborhood of a curve $C \subset \mathbb{R}^3$ form two families: a foliation $\mathcal{F}_2$ by circles, geodesic on the boundary of the tube, and a family of curves orthogonal to the previous circles. If the boundary of the tube is not a torus of revolution, the other family of lines of curvature is not a family of circles. The holonomy of the foliation $\mathcal{F}_1$ given by this other family of lines of curvature is a flow by isometries, therefore the return map on one characteristic circle is a rotation of angle α.

This fact is a corollary of Gauss lemma (see for example [doCa]) that we can restate as

Lemma 1.1 (Gauss lemma). *The holonomy of the foliation formed by the curves orthogonal to a geodesic foliation is a local isometry from one geodesic to another geodesic.*

It is proven in [GS3] that the rotation angle α is the torsion integral $\left(-\frac{1}{2\pi}\int_C \tau ds\right)$ along the curve C, and that any value of α can be achieved. Let us give here the "plumber's proof" of this result.

First construct a polygon matching two incomplete rectangles contained in different planes P and Q as on Figure 2a. Then replace each vertex by a small arc of circle in the plane defined by the two incident sides, see Figure 2b. We can now replace each arc formed by one of these arcs of circle and two small segment by an arc of plane convex C^∞ curve keeping just a half of the two segment parts. Our curve, Figure 2c, is a tubular neighborhood of the closed curve obtained smoothing the curve of Figure 2b.

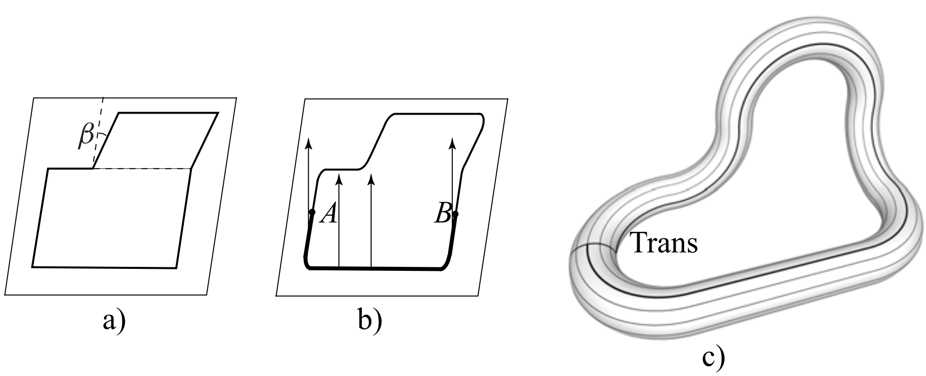

Figure 2: How to obtain a given rotation R_α as Poincaré map.

Let us, when the planes make an angle α, follow the holonomy of the foliation $\mathcal{F}_2$ along a loop coming back to a given geodesic circle of the tube. It is a rotation of angle α. Along the arc AB of the curve of Figure 2b, the upward vertical direction provides the curve (thicker on Figure 2c) that we will follow, using the holonomy, up to our starting point.

Construction. First, notice that by pasting quarters of tori, it is possible to build a closed tube where the holonomy is a rotation of angle $\pi/2$ (take $\beta = \pi/2$ in Figure 2a).

Slanting the vertical part of the Figure 2a (the angle with the horizontal is now $0 \leq \alpha < \pi$), and modifying subsequently the after next quarter of torus to get a portion of torus of "angle" α, it is possible to achieve any rotation of angle α.

We can extend the result on two-dimensional tubes at the beginning of Subsection 1.1 to tubes in $\mathbb{R}^n$. The first foliation $\mathcal{F}_2$ is now a foliation by spheres $\Sigma^{(n-1)}$ of constant radius r, while the other $\mathcal{F}_1$ is still a one dimensional foliation, by curves orthogonal to the spheres of $\mathcal{F}_2$.

The proof is identical replacing Gauss lemma by Reinhart's theorem ([Re], [To] chapter 6, Corollary 6.6):

Proposition 1.2. *Let M be a submanifold of the euclidean space $\mathbb{R}^n$ with a totally geodesic foliation $\mathcal{F}_2$ of codimension* 1*. Then a holonomy map of the orthogonal foliation $\mathcal{F}_1$ along a path $\gamma \subset L$, where L is a leaf $\mathcal{F}_1$, seen as a map from a leaf of $\mathcal{F}_2$ to another (or the same) leaf of $\mathcal{F}_2$, is a local isometry.*

Remark. Reinhart's theorem implies that the orthogonal trajectories to a one parameter family of planes in $\mathbb{R}^3$ have, where they define a foliation, orthogonal trajectories the holonomy of which are isometries from domains of one of the planes where they are defined to their image in another plane. Considering the family of planes normal to the core C of a tube, we extend simultaneously the foliation $\mathcal{F}_2$ to a domain of $\mathbb{R}^3$ and the holonomy maps from circles of given radii in the normal plane which belong to a given tube around a curve C, to at least discs of the same radius in the normal plane. The remark is also valid for tubes of higher dimension.

Our goal is to show that Theorem 0.1 concerning general regular canals can be obtained by a similar argument using the hyperplanes of the Lorentz space $\mathbb{L}^5$ orthogonal to the curve defining the canal as an envelope of spheres instead of the euclidean planes orthogonal to the core C of the euclidean tube.

1.2 The space of spheres

In ([Dar1], Livre II chap VI: Les coordonnées pentasphériques, pages 265–284) or ([Dar2] livre V, chap II, Les coordonnées pentasphériques pages 379–404) Darboux locates a sphere (or plane) of $\mathbb{R}^3$ by five coordinates, defining their "power" with respect to five mutually orthogonal spheres (one of them is imaginary, that is the square of its radius is a negative number). In modern language, this amounts to visualize the set of oriented spheres-or-planes of $\mathbb{R}^3$, or better of oriented spheres of $\mathbb{S}^3$, as a quadric Λ contained in the Lorentz space $\mathbb{L}^5$ of dimension five. Basic references are Berger ([Be] chapter 20), Cecil ([Ce]) and Hertrich-Jeromin [H-J].

Let $\mathcal{L}$ be the Lorentz quadratic form defined on the n-dimensional space $\mathbb{L}^n$ by:

$$\mathcal{L}\big(x_1, x_2, \ldots, x_{n-1}, x_n\big) = \big(x_1\big)^2 + \big(x_2\big)^2 + \cdots + \big(x_{n-1}\big)^2 - \big(x_n\big)^2$$

We will call *light cone* the isotropic cone of $\mathcal{L}$. We note also $\mathcal{L}$ the associated bilinear form, and call $\mathcal{L}$-orthogonal vectors a, b such that $\mathcal{L}(a, b) = 0$. The set of oriented $(n-3)$-spheres of the sphere $\mathbb{S}^{n-2}$ admits a bijection with the set of points of the quadric Λ^{n-1} of equation $\mathcal{L} = 1$; for a proof, see [H-J].

The points at infinity of the light cone form two $(n-2)$-dimensional spheres. We retain the "positive" one $\mathbb{S}^{n-2}_{\infty}$, that is the points at infinity of the light cone in the upper half space $x_n > 0$.

Definition 1.3. *A vector v of $\mathbb{L}^n$ is called* space-like *if $\mathcal{L}(v) > 0$. It is called* time-like *if $\mathcal{L}(v) < 0$. A line is called space-like (resp time-like) if it contains a space-like (resp time-like) vector.*

A subspace h is space-like if all its non-zero vectors are space-like. It is of mixed type *if it contains space-like and time-like vectors.*

Proposition 1.4. *Let c be a path in Λ^{n-1}. If at each point $c(t)$ of the path, the tangent vector $v(t)$ satisfies:*

*$\mathcal{L}(v(t)) > 0$, (*space-like *curve), the corresponding family of spheres Σ_t admits an envelope which is the union of spheres of a one-parameter family: the* characteristic spheres *$CS(t)$;*

*$\mathcal{L}(v(t)) < 0$, (*time-like *curve), at any point of the path, the spheres Σ_t are nested.*

Proof. See [H-J], chapter 1. □

Let $\mathcal{G}$ be the group of linear isomorphisms of $\mathbb{L}^n$ leaving $\mathcal{L}$ invariant.

Theorem 1.2 that we used in the n-dimensional Euclidean space $\mathbb{R}^n$ for the foliation defined by the curves orthogonal to a one parameter family of hyperplanes is also valid in $\mathbb{L}^n$ with the orthogonality defined using the quadratic form $\mathcal{L}$, or in submanifolds of $\mathbb{L}^n$ like the quadric Λ^{n-1}.

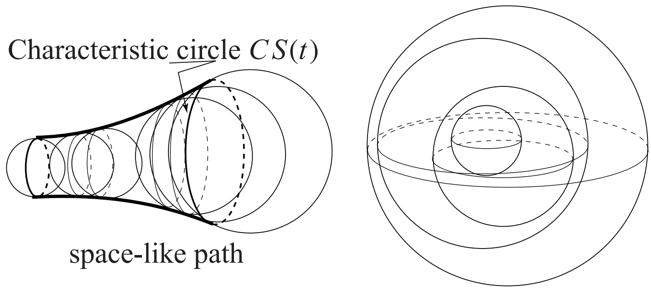

Figure 3: Spheres corresponding to space-like and time-like paths.

The group $\mathcal{G}$ leaves the light-cone globally invariant, sending rays to rays. The group $\mathcal{G}$ leaves invariant the connected components of the light-cone minus the origin. This defines an action on the sphere $\mathbb{S}^{n-2}$. Let us choose a "round" metric on $\mathbb{S}^n$ choosing an affine section of the light-cone by a space-like hyperplane. Then the group $\mathcal{G}$ acts on $\mathbb{S}^{n-2}$ by conformal transformations. Sometimes, as in dimension two, we will use the notation $\mathcal{M}$ (*Möbius group*) instead of repeating: $\mathcal{G}$ acting on the sphere, section of the light-cone.

In fact any local conformal C^∞ diffeomorphism of $\mathbb{S}^{n-2}$, $(n-2) \geq 3$ is the restriction of an element of the Möbius group $\mathcal{M}$. This is a famous theorem by Liouville (see [Li] and [doCa] chapter 9). The result is true in any dimension greater or equal to 3 (see for example [Bl], where the smoothness hypothesis is C^4). We know that the result is false in dimension 2: there are much more holomorphic maps that homographies. In dimension one the conformality condition is just meaningless.

1.3 Envelopes of spheres

1.3.1 Canal hypersurfaces

Thinking of the case of envelopes of spheres in $\mathbb{R}^3$ the name *canal hypersurface* is given to the envelope of one-parameter families of spheres in $\mathbb{R}^{n-2}$ or $\mathbb{S}^{n-2}$.

Definition 1.5. *A canal hypersurface is a hypersurface envelope of the spheres of a space-like curve* $c \subset \Lambda^{n-1}$.

Such a hypersurface may have singular points. However a second order condition guarantees it is immersed.

Proposition 1.6. *Let c be a space-like curve parametrized by the arc-length s. Let w^T be the orthogonal projection on $T_{c(s)}\Lambda$ of a vector $w \in \mathbb{L}^n$. Let in particular*

$$\overrightarrow{k_g}(s) = \left(\ddot{c}\,(s)^T\right)$$

be the geodesic curvature vector of c at $c(s)$. If the geodesic curvature vector satisfies the inequality

$$\mathcal{L}\left(\overrightarrow{k_g}\right) < 0,$$

the canal hypersurface is immersed and we will call it regular.

Proof. See [H-J], chapter 1. □

1.3.2 The Dupin cyclides

Let P be an affine space-like plane in $\mathbb{L}^n$. The intersection $P \cap \Lambda^{n-1}$ can be empty, a point or a circle (for the euclidean metric on P induced from the Lorentz form).

Definition 1.7. *The envelope of the spheres which are, in Λ^{n-1}, the points of a circle $P \cap \Lambda$, P a space-like plane, is a surface which is called a* Dupin cyclide.

If the plane P is space-like and satisfies $\inf_{x\in P} \mathcal{L}(x) < 0$, then the Dupin cyclide is regular, that is the envelope is the image of an embedding of $\mathbb{S}^{n-2} \times \mathbb{S}^1$.

In the 3-dimensional case it is an embedded torus (see Figure 1). We will see in next section that the lines of principal curvature are particularly simple on 3 dimensional Dupin cyclides (see Figure 1).

A Dupin cyclide of dimension $n - 3$ contained in $\mathbb{S}^{n-2}$ is the envelope of a the one-parameter family of spheres $P \cap \Lambda^{n-2}$. It is also the envelope of a $(n-4)$-family of spheres: the spheres of the intersection $Q \cap \Lambda^{n-2}$, where Q is an affine subspace of $\mathbb{L}^n$ orthogonal to the 3-dimensional subspace $\operatorname{span}(O, P)$ of $\mathbb{L}^n$. Let p be the vectorial plane parallel to P and q the vectorial plane parallel to Q. The subspaces P and Q intersect the line $(p \oplus q)^\perp$ in two points x_P and x_Q which satisfy, as in the 3-dimensional case (see [LW]) the equality $\mathcal{L}(x_P, x_Q) = 1$. We will see in next section that the characteristic circles of the last $(n - 4)$-family of spheres are lines of principal curvature of the Dupin cyclide. They are closed and therefore the corresponding holonomy map is the identity of a $(n - 4)$-dimensional sphere, as in the 3-dimensional case Dupin $\subset \mathbb{S}^3 \subset \mathbb{L}^5$, where the return map of a family of lines of principal curvature is seen on one orthogonal circle belonging to the other family of lines of principal curvature.

2 Configuration of principal curvature lines

At every point m of a hypersurface M^{n-3} we can define an unitary frame $(e_1, e_2, \ldots, e_{n-3})$ of eigen vectors of the second fundamental form. Let e_{n-2} be a unit vector normal to the hypersurface such that the frame $(e_1, e_2, \ldots, e_{n-3}, e_{n-2})$ is positive. Let $k_1, \ldots, k_{n-3}$ be the principal curvatures of M^{n-3} in the directions of $e_1 \ldots, e_{n-3}$. When M^{n-3} is a regular canal hypersurface, one has

$$k_2 = k_3 = \cdots = k_{n-3} = k;$$

we suppose that $k_1 \neq k$. The $(n-4)$-dimensional plane field generated by $(e_2, \ldots e_{n-3})$ is integrable and its integral foliation is formed by the characteristic spheres of the canal. The term foliation is used even when the canal hypersurface is only immersed. It makes sense at least locally (see also the construction of the Subsection 3.1).

Our previous definition of principal directions seems to be a metric one. In fact the values of the principal curvatures at a point depend on the ambient (euclidean or "round") metric, but the *directions* do not. They depend only on the conformal structure of $\mathbb{S}^{n-2}$, that is they are invariant by the action of the Möbius group on the ambient space.

Principal directions are also preserved by a stereographic projection $\mathbb{S}^{n-2} \to \mathbb{R}^{n-2}$.

To see that, consider the set of tangent spheres at a point $m \in M^{n-3}$. In the quadric $\Lambda^{n-1} \subset \mathbb{L}^n$ this set of spheres is a light-ray. Very small spheres correspond to the extremities of the ray, see also [H-J].

Suppose now that M^{n-3} is a regular canal hypersurface. A sphere Σ^{n-3} of the family generating the canal is tangent to M^{n-3} along a $(n-4)$-dimensional sphere C. This sphere is therefore an osculating sphere, and $n-4$ principal curvatures, say $k_2 = k_3, \cdots, k_{n-3} = k$, of M^{n-3} coincide; they are equal to $1/R$ where R is the radius of the sphere Σ^{n-2} with a proper sign. The first principal direction of curvature is therefore orthogonal to the characteristic spheres. We will use this definition to study the holonomy of the foliation $\mathcal{F}_1$ that this first principal direction of curvature defines.

When $M^{(n-3)}$ is a Dupin cyclide, it is also the envelope of an $(n-4)$-dimensional family of spheres. Each sphere of this second family is tangent to $M^{(n-3)}$ along a circle, which is a line of principal curvature of $M^{(n-3)}$; the sphere is an osculating sphere of radius $R = (1/k_1)$.

Therefore the Dupin cyclide has one family of lines of principal curvature which is a family of circles, orthogonal to the previous one-parameter family of $(n-4)$-dimensional characteristic spheres.

3 Global results

The determination of the holonomy of the lines of principal curvature on Dupin cyclide is immediate, as there are not only closed curves but round circles: it is the identity.

Let us now prove Theorem 0.1. As announced at the end of Section 1.1, we will give a geometric proof of the Theorem 0.1, inducing the curvature lines on the canal of $\mathbb{S}^{n-2}$ from a foliation defined on the Lorentz space $\mathbb{L}^n$.

Consider in $\mathbb{L}^n$ the one-parameter family of hyperplanes H_t normal to $c \subset \mathbb{L}^n$ at each point $c(t)$. As this is a family of totally geodesic subspaces of $\mathbb{L}^n$, the foliation $\tilde{\mathcal{F}}_1$ of $\mathbb{L}^n$ defined by the curves orthogonal to the hyperplanes is Riemannian. It is a true foliation out of the envelope of the family of hyperplanes.

Notice that the hyperplane $H(t)$ is of mixed type, and therefore is never tangent to the light cone. Notice also that all the affine hyperplanes $H(t)$ are in fact vectorial ones, as the line $\operatorname{span}(c(t))$ is orthogonal to the curve c at $c(t)$, and therefore is contained in $H(t)$.

To avoid any difficulty coming from the fact the family $\{H(t)\}$ does not foliate the ambient Lorentz space, we will deal with a slightly more general situation. It will apply in particular to the foliation $\tilde{\mathcal{F}}_1$ normal to the family H_t where it is defined.

3.1 Parametrically normal flow of a family of hypersurfaces

First of all, we will define the flow normal to a one-parameter family of hypersurfaces $H(t)$.

Definition 3.1. *Let $H(t)$, t in an interval I of $\mathbb{R}$, be a differentiable family of hyperplanes of the n-dimensional ambient space E. Let $c\colon I \to E$ be a differentiable curve. The curve c is said* parametrically normal to H if for all t, $c(t)$ is on $H(t)$ and $\dot{c}(t)$ is orthogonal to $H(t)$ ($\dot{c}(t)$ maybe zero).

Definition 3.2. *A flow $F : I \times \mathbb{R}^{n-1} \to E$, F differentiable, is* parametrically adapted *to the family H if for every $t \in I$ and $x \in E$ the point $F(x, t)$ is in $H(t)$ and if the restriction of F to the subspaces $t \times E \subset I \times E$ are diffeomorphisms F_t on their images $H(t)$. The flow is* parametrically normal *to the family H if it is parametrically adapted, and if moreover the vector $dF[(\partial/\partial t)(F(x, t))]$ is everywhere normal to $H(t)$ (it may be zero).*

Proposition 3.3. *For a given t_0 in I and x_0 in $H(t_0)$, there exists one, and only one curve c parametrically normal to H, and c depends continuously on x_0.*

Proof. Let $F : I \times \mathbb{R}^{n-1} \to E$ be a parametrically adapted flow for the family H. Let $c(t) = F(t, d(t))$ be the image by F of a curve in the product $I \times \mathbb{R}^{n-1}$. It satisfies $c(t) \in H(t)\ \forall t$. The advantage of dealing with the curves $d(t)$ is that they will be solutions of an ordinary differential equation in $\mathbb{R}^{n-1}$.

Let $\{e_i\}$ be the canonical basis of $\mathbb{R}^{n-1}$ and $\{e_i(t)\}$ be the base of $H(t)$ formed by the images $dF_t(e_i)$. For any t_1, the condition that $\overset{\bullet}{c}(t_1)$ is normal to $H(t_1)$ is equivalent to the condition that $\overset{\bullet}{c}(t_1)$ is orthogonal to each $e_i(t_1)$. We can express $\overset{\bullet}{c}(t_1)$ in terms of d:

$$\overset{\bullet}{c}(t_1) = dFt_1(d(t_1))(\overset{\bullet}{d}(t_1)) + dF(t_1, d(t_1))\left(\frac{\partial}{\partial t}\right).$$

We have two terms. The first term is tangent to $H(t_1)$ and is due to the motion $d(t)$ of the point in the model $\mathbb{R}^{n-1}$; it is a linear and non-degenerate function of $\overset{\bullet}{d}(t_1)$; let us call it $D(t_1)$. The second term is transverse to $H(t_1)$ (except in degenerate cases). It depends on $d(t_1)$ but not on its derivative; let us call it $T(t_1)$.

We have, for all i, the conditions

$$\left\langle D(t_1)(\overset{\bullet}{d}(t_1)) + T(t_1), e_i(t_1)\right\rangle = 0,$$

which is equivalent to

$$\begin{aligned} \left\langle D(t_1)(\overset{\bullet}{d}(t_1)), e_i(t_1)\right\rangle &= -\left\langle T(t_1), e_i(t_1)\right\rangle, \qquad \text{or} \\ \left\langle \overset{\bullet}{d}(t_1), D(t_1)^T(e_i(t_1))\right\rangle &= -\left\langle T(t_1), e_i(t_1)\right\rangle. \end{aligned}$$

Since the vectors $e_i(t_1)$ are in the image of $D(t_1)$ and independent, the vectors $D(t_1)^T(e_i(t_1))$ are also independent; thus, as the restriction of Q to $H(t)$ is definite, we have $n-1$ independent linear conditions on $\overset{\bullet}{d}(t_1)$.

Then $\overset{\bullet}{d}(t_1)$ is uniquely defined by $d(t_1)$ and the condition that $\overset{\bullet}{c}(t_1)$ must be normal to $H(t_1)$. In $\mathbb{R}^{n-1}$, the curve d is the solution of an *ordinary differential equation*. Therefore, all the classical results of existence and uniqueness of solutions are valid. □

The family of curves parametrically normal to H gives a family of diffeomorphisms F_t, $t = t_2 - t_1$ from $H(t_1)$ to $H(t_2)$. This family shall be called the parametrically normal flow of H.

The speed vector field along $H(t)$ shall be called the parametrically normal vector field. Vectors of that vector field shall be called parametrically normal to $H(t)$.

Corollary 3.4. *If c is a curve with $c(t)$ on $H(t)$ for all t and if for a given t_0, $\dot{c}(t_0)$ is normal to $H(t_0)$, then $\dot{c}(t_0)$ is the parametrically normal vector to $H(t_0)$ at $c(t_0)$.*

Proposition 3.5. *If H is a family of vectorial hyperplanes in $\mathbb{R}^n$, with any non-degenerate bilinear form Q, such that the restriction of Q to any hyperplane $H(t)$ is also non-degenerate, then the parametrically normal flow F_t, $t = t_2 - t_1$ between $H(t_1)$ and $H(t_2)$ is a flow by isometries.*

With no loss of generality we can suppose that the map F is such that all the maps F_t are isometries, for Q, that is linear maps preserving the quadratic form Q.

Lemma 3.6. *If H is a family of vectorial hyperplanes, then the solution of the differential equation in $\mathbb{R}^{n-1}$ satisfy $e_i(\varphi_t) = A(t)(e_i)$ where $A(t)$ is linear and Q-skewsymmetric.*

Proof. Let $\nu(t)$ be a normal unit vector to $H(t)$ with a consistent orientation. Let $r(t)$ be the Q-isometry that fixes $\mathrm{Vect}(\nu(t_0), \nu(t))$, sends $\nu(t_0)$ to $\nu(t)$ and is the identity on the space $\mathrm{Vect}(\nu(t_0), \nu(t))^{\perp}$, and let $\dot{r}(t)$ be the derivative of r with regard to t in the space of linear transformations of $\mathbb{R}^n$. For all x, $\dot{r}(t_0)(x)$ is orthogonal to $H(t_0)$, then $\dot{r}(t_0)(x)$ is the parametrically normal vector field on $H(t_0)$ at x. $\square$

Remark. If $E = \mathbb{L}^n$ is endowed with the Lorentz form, as $\mathcal{L}$ is not a positive definite bilinear form, we should impose that $H(t)$ is never tangent to the isotropic cone; this is the case for the family of hyperplanes normal to a space-like curve $c \subset \Lambda^4$.

Remark. This flow restricts to an isometric flow on the space of spheres. This shows directly that it induces a map from $(n-5)$-dimensional spheres of a characteristic sphere $\dot{c}(t)^{\perp} \cap c(t)^{\perp}$ to $(n-5)$-dimensional spheres of the same or another characteristic sphere, as the hyperplane F_t intersects Λ^{n-2} in the set of spheres orthogonal to the derivated sphere $\dot{c}(t)^{\perp}$, which in turn intersect $\dot{c}(t)^{\perp} \cap c(t)^{\perp}$ in a set of spheres orthogonal to the characteristic spheres. Each of them intersect the characteristic sphere in $(n-5)$-dimensional sphere.

Proof. Let us now prove the Proposition 3.5.

In the space of affine transformations of $\mathbb{R}^n$, the equation $\dot{T}(t) = A(t) + C(t)$ has an unique solution T_{t_0} such that $T_{t_0}(t_0) = Id$. Since $\dot{T}_{t_0}(t)$ coincides with the parametrically normal vector field, the restriction of $T_{t_0}(t)$ to $F(t_0)$ is the parametrically normal flow.

Furthermore, since for all t $A(t)$ is skewsymmetric, $T_{t_0}(t)$ is an isometry for all t. □

3.2 Parametrically normal flow to derivated spheres

Let c be the curve in Λ^{n-1} representing a canal hypersurface. Let $H = \{H(t) \subset \mathbb{L}^n\}$ be the family of hyperplanes normal to the curve c. The trace of $H(t)$ on the sphere at infinity is the derivated sphere $D(t) = [\dot{c}(t)]^{\perp}$ orthogonal to the canal hypersurface (it contains the characteristic circle $CC(t)$). We will now show that the parametrically normal vector field to these derivated spheres is the flow induced by the parametrically normal vector field to H.

First notice, that, given a point $x_0 \in \mathcal{L}ight \setminus \{O\} \cap H$, where H is a vectorial hyperplane of mixed type, a vector orthogonal to H at x_0 is tangent to $\mathcal{L}ight$ (see Figure 4).

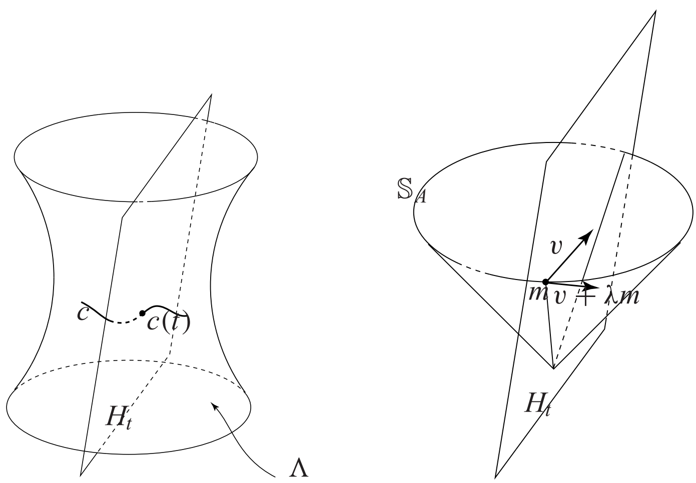

Figure 4: Hyperplane H_t normal to a space-like curve $c \subset \Lambda$ and intersection of the light-cone with the hyperplane and normal vector to H_t at $m \in \mathbb{S}_A = \mathcal{L}ight \cap A$.

We have seen that the flow T_t is given by Lorentz isometries. Each Lorentz isometry induces a linear transformation of the light-cone into itself, therefore a transformation of the sphere $\mathbb{S}_\infty$. Such a transformation is called a Möbius map. We need to choose a metric on $\mathbb{S}_\infty$ to give a meaning to the sentence: a Möbius map is conformal. For that, let us choose a space-like affine subspace A. The intersection of A with the homogenous cone $\mathcal{L}ight$ is a model $\mathbb{S}_A$ of the sphere at infinity; it inherits from A an euclidean metric. Identifying $\mathcal{L}ight \cap A$ and $\mathbb{S}_\infty$, we see that, for this metric, maps induced from Lorentz isometries are conformal. We need now only to check that the flow T_t^A induced on $\mathbb{S}_A$ from T_t, which is a flow by conformal maps, has orbits which are orthogonal to the derivated spheres $[\dot{c}(t)]^\perp$.

It is indeed the case as the tangent vector to the orbits of T_t^A at $m \in \mathbb{S}_A$ is of the form $v + \lambda m$, as it is obtained from the tangent vector v to the orbits of T_t at m by adding a multiple of m. As the light-ray $\mathbb{R} \cdot m$ is orthogonal to $T_m\mathcal{L}ight$, $v + \lambda m$ is orthogonal to the derivated sphere $[\dot{c}(t)]^\perp \cap \mathbb{S}_A$ (see Figure 4).

This ends the geometrical proof of Theorem (0.1).

Notice that in our proof we never use the fact that the canal is regular. It means that the line of curvature flow goes across the singular locus of a non-regular canal.

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