BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 39(3), 355-370

Parallel and semi-parallel hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$

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Abstract. We give a complete classification of totally umbilical, parallel and semi-parallel hypersurfaces of the Riemannian product space $\mathbb{S}^n \times \mathbb{R}$.

Keywords: hypersurface, totally umbilical, parallel, semi-parallel.

Mathematical subject classification: 53B25.

1 Introduction

Totally umbilical, parallel and semi-parallel submanifolds are natural generalizations of totally geodesic submanifolds. They are important to study because these families of submanifolds provide nice examples and because they give information about the ambient space. In [3] one can find initial work on the geometry of hypersurfaces of the Riemannian product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. In the present paper we give a full classification of totally umbilical, parallel and semi-parallel hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$. By comparing our classifications of totally umbilical hypersurfaces (Theorem 4) and of parallel hypersurfaces (Theorem 6), we remark that, unlike for submanifolds of real space forms, total umbilicity does *not* imply parallelism. In fact, we find a correspondence between totally umbilical hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and solutions of the one-dimensional Sine-Gordon equation from physics. Moreover, our classifications of totally umbilical hypersurfaces (Theorem 4) and of semi-parallel hypersurfaces (Theorem 5) include examples of so-called rotation hypersurfaces, which were introduced in [6].

Received 20 July 2007.

*The author is a postdoctoral researcher supported by the Research Foundation – Flanders (F.W.O.).

2 Preliminaries

Let $F\colon M^n \to \widetilde{M}^{n+1}$ be an isometric immersion of Riemannian manifolds with Levi Civita connections ∇ and $\widetilde{\nabla}$ respectively. Denote by N a unit normal vector field on the hypersurface and let X, Y, Z and W be arbitrary vector fields tangent to M^n. We define the shape operator S by $SX = -\widetilde{\nabla}_X N$, and the second fundamental form h by $h(X, Y) = \langle SX, Y\rangle = \langle X, SY\rangle$. The formula of Gauss states that

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)N. \tag{1}$$

Moreover, the equations of Gauss and Codazzi are given respectively by (cfr. [2])

$$\begin{aligned}\langle \widetilde{R}(X, Y)Z, W\rangle &= \langle R(X, Y)Z, W\rangle + h(X, Z)h(Y, W)\\ &\quad - h(Y, Z)h(X, W),\end{aligned} \tag{2}$$

$$\langle \widetilde{R}(X, Y)Z, N\rangle = (\nabla h)(X, Y, Z) - (\nabla h)(Y, X, Z), \tag{3}$$

where R and $\widetilde{R}$ are the Riemann-Christoffel curvature tensors of M^n and $\widetilde{M}^{n+1}$ respectively. We use the following sign convention: $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. The covariant derivative of h is defined by

$$(\nabla h)(X, Y, Z) = X[h(Y, Z)] - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{4}$$

We say that M^n is *totally geodesic* in $\widetilde{M}^{n+1}$ if $h = 0$, that M^n is *totally umbilical* in $\widetilde{M}^{n+1}$ if h is a scalar multiple of the metric at every point, that M^n is *parallel* in $\widetilde{M}^{n+1}$ if $\nabla h = 0$ and that M^n is *semi-parallel* in $\widetilde{M}^{n+1}$ if $R \cdot h = 0$, where

$$(R \cdot h)(X, Y, Z, W) = -h(R(X, Y)Z, W) - h(Z, R(X, Y)W). \tag{5}$$

Parallel hypersurfaces of real space forms were classified by H.B. Lawson in [7], whereas the classification of semi-parallel hypersurfaces of real space forms was obtained by J. Deprez for Euclidean space in [4] and by F. Dillen for spaces of non-zero constant sectional curvature in [5].

Denote by $\mathbb{E}^{n+2}$ the Euclidean space of dimension $n + 2$. We define the Riemannian product manifold $\mathbb{S}^n \times \mathbb{R}$ as the following subset of $\mathbb{E}^{n+2}$, equipped with the induced metric:

$$\mathbb{S}^n \times \mathbb{R} = \left\{(x_1, \dots, x_{n+2}) \in \mathbb{E}^{n+2} \mid x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\right\}.$$

Then $\mathbb{S}^n \times \mathbb{R}$ is the Riemannian product of the unit sphere $\mathbb{S}^n(1)$ and the real line. Remark that $\xi = (x_1, \dots, x_{n+1}, 0)$ is a unit normal vector field on $\mathbb{S}^n \times \mathbb{R}$

in $\mathbb{E}^{n+2}$. If X and Y are vector fields on $\mathbb{S}^n \times \mathbb{R}$, we denote by $X_{\mathbb{S}^n}$ and $Y_{\mathbb{S}^n}$ the projections of X and Y onto the tangent space to $\mathbb{S}^n(1)$. From (1), we find that the Levi Civita connection $\widetilde{\nabla}$ of $\mathbb{S}^n \times \mathbb{R}$ is given by $\widetilde{\nabla}_X Y = D_X Y + \langle X_{\mathbb{S}^n}, Y_{\mathbb{S}^n}\rangle \xi$, where D is the covariant derivative in $\mathbb{E}^{n+2}$. This expression yields that the curvature tensor $\widetilde{R}$ of $\mathbb{S}^n \times \mathbb{R}$ is determined by

$$\langle \widetilde{R}(X, Y)Z, W\rangle = \langle Y_{\mathbb{S}^n}, Z_{\mathbb{S}^n}\rangle\langle X_{\mathbb{S}^n}, W_{\mathbb{S}^n}\rangle - \langle X_{\mathbb{S}^n}, Z_{\mathbb{S}^n}\rangle\langle Y_{\mathbb{S}^n}, W_{\mathbb{S}^n}\rangle.$$

Now let $F\colon M^n \to \mathbb{S}^n \times \mathbb{R}$ be a hypersurface with unit normal N. Let T denote the projection of the coordinate vector field $\partial_{x_{n+2}}$ onto the tangent space to M^n and denote by θ a function on M^n such that $\cos\theta = \langle N, \partial_{x_{n+2}}\rangle$. This means that $\partial_{x_{n+2}} = T + \cos\theta\, N$. The equations of Gauss and Codazzi, (2) and (3), reduce to

$$\begin{aligned}\langle R(X, Y)Z, W\rangle &= \langle SX, W\rangle\langle SY, Z\rangle - \langle SX, Z\rangle\langle SY, W\rangle \\ &\quad + \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle \\ &\quad + \langle Y, T\rangle\langle W, T\rangle\langle X, Z\rangle + \langle X, T\rangle\langle Z, T\rangle\langle Y, W\rangle \\ &\quad - \langle X, T\rangle\langle W, T\rangle\langle Y, Z\rangle - \langle Y, T\rangle\langle Z, T\rangle\langle X, W\rangle,\end{aligned} \tag{6}$$

$$\nabla_X SY - \nabla_Y SX - S[X, Y] = \cos\theta(\langle Y, T\rangle X - \langle X, T\rangle Y), \tag{7}$$

where X, Y, Z and W are vector fields tangent to M^n. Moreover, by using the fact that $\partial_{x_{n+2}}$ is parallel in $\mathbb{S}^n \times \mathbb{R}$, one obtains

$$\nabla_X T = \cos\theta\, SX, \qquad X[\cos\theta] = -\langle SX, T\rangle. \tag{8}$$

These equations appear in the following existence and uniqueness theorem for immersions of hypersurfaces into $\mathbb{S}^n \times \mathbb{R}$:

Theorem 1 ([3]). *Let M^n be a simply connected Riemannian manifold with Levi Civita connection ∇ and curvature tensor R. Let S be a field of symmetric operators $S_p\colon T_pM^n \to T_pM^n$, and let T and θ be a vector field and a smooth function on M^n such that $\|T\|^2 = \sin^2\theta$. Assume that equations* (6), (7) *and* (8) *are satisfied. Then there exists an isometric immersion $F : M^n \to \mathbb{S}^n \times \mathbb{R}$ with unit normal N, such that the shape operator with respect to this normal is given by S and such that $\partial_{x_{n+2}} = T + \cos\theta\, N$. Moreover, the immersion is unique up to global isometries of $\mathbb{S}^n \times \mathbb{R}$ preserving the orientations of both $\mathbb{S}^n$ and $\mathbb{R}$.*

We will now recall the definition of a special class of hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$, proposed in [6], namely *rotation hypersurfaces*. Consider a three-dimensional subspace P^3 of $\mathbb{E}^{n+2}$ containing the x_{n+2}-axis. Then $(\mathbb{S}^n \times \mathbb{R}) \cap P^3$ is a cylinder $\mathbb{S}^1 \times \mathbb{R}$. Let P^2 be a two-dimensional subspace of P^3, also through the x_{n+2}-axis. Denote by $\mathcal{J}$ the group of isometries of $\mathbb{E}^{n+2}$ which leave $\mathbb{S}^n \times \mathbb{R}$ globally invariant and which leave P^2 pointwise fixed. Finally, let α be a curve in $\mathbb{S}^1 \times \mathbb{R}$ which does not intersect P^2. Then the rotation hypersurface M^n of $\mathbb{S}^n \times \mathbb{R}$ with profile curve α and axis P^2 is defined as the $\mathcal{J}$-orbit of α. It is clear from the definition that the velocity vector of α is proportional to T, unless α lies in a plane orthogonal to $\partial_{x_{n+2}}$, in which case $T = 0$. In the following, we will always assume that P^3 is spanned by ∂_{x_1}, $\partial_{x_{n+1}}$ and $\partial_{x_{n+2}}$ and that P^2 is spanned by ∂_{x_1} and $\partial_{x_{n+2}}$. In [6] it was proved that there exists a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n, with $T = \|T\| e_1$, such that the shape operator S takes the form

$$S = \begin{pmatrix} \lambda & & & \\ & \mu & & \\ & & \ddots & \\ & & & \mu \end{pmatrix}.$$

Moreover, if α is not a vertical line in $\mathbb{S}^1 \times \mathbb{R}$, it can be locally parametrized as $\alpha(s) = \big(\cos s, 0, \ldots, 0, \sin s, a(s)\big)$, and we have

$$\lambda = -\frac{a''(s)}{\big(1 + a'(s)^2\big)^{3/2}}, \qquad \mu = -\frac{a'(s)\cot s}{\big(1 + a'(s)^2\big)^{1/2}}. \tag{9}$$

If α is a vertical line $\alpha(s) = \big(\cos c, 0, \ldots, 0, \sin c, s\big)$ for some real constant c, we have

$$\lambda = 0, \qquad \mu = -\cot c. \tag{10}$$

Finally, we mention the following characterization:

Theorem 2 ([6]). *Take $n \geq 3$ and let $F : M^n \to \mathbb{S}^n \times \mathbb{R}$ be a hypersurface with shape operator*

$$S = \begin{pmatrix} \lambda & & & \\ & \mu & & \\ & & \ddots & \\ & & & \mu \end{pmatrix},$$

with $\lambda \neq \mu$. Suppose that $ST = \lambda T$ and assume that there is a functional relation $\lambda(\mu)$. Then M^n is an open part of a rotation hypersurface.

3 Totally umbilical hypersurfaces

In this section, we classify totally umbilical hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$. Let us first note that there are only few totally geodesic hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$.

Theorem 3. *Let M^n be a totally geodesic hypersurface of $\mathbb{S}^n \times \mathbb{R}$. Then M^n is an open part of a hypersurface $\mathbb{S}^n(1) \times \{t_0\}$ for $t_0 \in \mathbb{R}$, or of a hypersurface $\mathbb{S}^{n-1} \times \mathbb{R}$.*

Proof. Let M^n be totally geodesic in $\mathbb{S}^n \times \mathbb{R}$. It follows from the equation of Codazzi (7) that there are two cases to consider, namely $T = 0$ and $\cos\theta = 0$.

In the first case, M^n is everywhere orthogonal to $\partial_{x_{n+2}}$. This gives the first family of hypersurfaces mentioned in the theorem.

In the second case, M^n is everywhere tangent to $\partial_{x_{n+2}}$ and we have a hypersurface of type $\bar{M}^{n-1} \times \mathbb{R}$, where $\bar{M}^{n-1}$ is a hypersurface of $\mathbb{S}^n(1)$. It is easy to see that $\bar{M}^{n-1} \times \mathbb{R}$ is totally geodesic in $\mathbb{S}^n \times \mathbb{R}$ if and only if $\bar{M}^{n-1}$ is totally geodesic in $\mathbb{S}^n(1)$. Hence, we obtain the second family of hypersurfaces of the theorem. □

In the following proposition we remark a correspondence between totally umbilical hypersurfaces and solutions of the one-dimensional Sine-Gordon equation from physics.

Proposition 1. *Let M^n be a totally umbilical hypersurface of $\mathbb{S}^n \times \mathbb{R}$ with angle function θ and let p be a point of M^n where $\sin\theta \neq 0$. Then there exist local coordinates $(u, v_1, \ldots, v_{n-1})$ on an open neighbourhood U of p in M^n such that θ only depends on u and such that $\phi := 2\theta$ satisfies the one-dimensional Sine-Gordon equation*

$$\phi'' + \sin\phi = 0. \tag{11}$$

Conversely, starting with an open subset $U \subseteq \mathbb{R}^n$ with coordinates $(u, v_1, \ldots, v_{n-1})$ and a solution $\phi(u)$ of (11), which is nowhere zero on U, we can put $\theta = \frac{\phi}{2}$ and we can define a function λ and a Riemannian metric on U such that there exists an isometric immersion $F\colon U \to \mathbb{S}^n \times \mathbb{R}$ with shape operator $S = \lambda\,\mathrm{id}$ and angle function θ.

In the proof of this proposition, we will use the following result.

Proposition 2 ([8]). *Let $M = N_1 \times_f N_2$ be a warped product of semi-Riemannian manifolds with Levi Civita connection ∇ and curvature tensor R. Denote* *by R^{N_1} and R^{N_2} the lifts of the curvature tensors of N_1 and N_2 respectively. If X, Y and Z are lifts of vector fields on N_1 and U, V and W are lifts of vector fields on N_2, then*

(i) $R(X,Y)Z$ *is the lift of* $R^{N_1}(X,Y)Z$,

(ii) $R(X,U)Y = \frac{H^f(X,Y)}{f}U$, *where* H^f *is the Hessian of* f,

(iii) $R(X,Y)U = R(V,W)X = 0$,

(iv) $R(U,X)V = \frac{\langle U,V\rangle}{f}\nabla_X(\text{grad } f)$,

(v) $R(U,V)W = R^{N_2}(U,V)W - \frac{\langle \text{grad } f, \text{grad } f\rangle}{f^2}(\langle V,W\rangle U - \langle U,W\rangle V)$.

Proof of Proposition 1. Assume that M^n is totally umbilical in $\mathbb{S}^n \times \mathbb{R}$ with shape operator $S = \lambda\,\text{id}$. Since $\sin\theta \neq 0$ at p, there exists an open neighbourhood U of p in M^n on which $\sin\theta$ is nowhere zero. Let X be a vector field tangent to U. Then the equation of Codazzi (7) and the second equation of (8) yield

$$\begin{cases} X[\lambda] = -\cos\theta\,\langle X,T\rangle\,, \\ X[\cos\theta] = -\lambda\,\langle X,T\rangle\,. \end{cases} \tag{12}$$

Moreover, the first equation of (8) yields that the orthogonal complement of span$\{T\}$ is integrable. Indeed, if $X, Y \perp T$, then we obtain

$$\langle [X,Y],T\rangle = \langle \nabla_X Y - \nabla_Y X, T\rangle = -\langle Y, \nabla_X T\rangle + \langle X, \nabla_Y T\rangle = 0.$$

This means that we can choose coordinates $(u, v_1, \ldots, v_{n-1})$ on U such that $\partial_u = \frac{T}{\sin\theta}$ and $\partial_u \perp \partial_{v_i}$. The system (12) yields that $\partial_{v_i}\lambda = \partial_{v_i}\theta = 0$, such that λ and θ are functions of u only, and that

$$\begin{cases} \lambda' = -\cos\theta\sin\theta, \\ \theta' = \lambda. \end{cases} \tag{13}$$

Remark that the function θ satisfies the equation $\theta'' = -\cos\theta\sin\theta$. After the substitution $\phi = 2\theta$, we obtain (11).

Conversely, let us start with an open part U of $\mathbb{R}^n$, with coordinates $(u, v_1, \ldots, v_{n-1})$ and with a non-vanishing solution $\phi(u)$ of the Sine-Gordon equation (11). On U, we define a Riemannian metric

$$g = du^2 + \sum_{i,j=1}^{n-1} g_{ij}\,dv_i dv_j,$$

the function $\theta(u) = \frac{\phi(u)}{2}$, the vector field $T = \sin\theta\, \partial_u$ and the field of operators $S = \theta'$ id. These data satisfy the equation of Codazzi (7) and the second equation of (8). From Theorem 1, we know that there exists an isometric immersion of (U, g) into $\mathbb{S}^n \times \mathbb{R}$ with shape operator S, structure vector field T and angle function θ if and only if the equation of Gauss (6) and the first equation of (8) are satisfied. These equations are equivalent to

$$\langle R(\partial_u, X)\partial_u, Y\rangle = -\big(\cos^2\theta + (\theta')^2\big)\,\langle X, Y\rangle\,, \tag{14}$$

$$\langle R(X, Y)\partial_u, Z\rangle = 0, \tag{15}$$

$$\langle R(X, Y)Z, W\rangle = \big(1 + (\theta')^2\big)(\langle X, W\rangle\,\langle Y, Z\rangle - \langle X, Z\rangle\,\langle Y, W\rangle), \tag{16}$$

$$\nabla_{\partial_u}\partial_u = 0, \tag{17}$$

$$\sin\theta\,\nabla_X\partial_u = (\cos\theta)\theta' X, \tag{18}$$

where X, Y, Z and W are vector fields on U orthogonal to T, R is the Riemann-Christoffel curvature tensor of (U, g) and ∇ is the Levi Civita connection of (U, g). From (18) we find

$$\begin{aligned} \partial_u g_{ij} &= \partial_u\left\langle\partial_{v_i}, \partial_{v_j}\right\rangle = \left\langle\nabla_{\partial_u}\partial_{v_i}, \partial_{v_j}\right\rangle + \left\langle\partial_{v_i}, \nabla_{\partial_u}\partial_{v_j}\right\rangle \\ &= 2(\cot\theta)\theta'\left\langle\partial_{v_i}, \partial_{v_j}\right\rangle = 2(\cot\theta)\theta' g_{ij}. \end{aligned}$$

Hence, $g_{ij} = \sin^2\theta\, c_{ij}(v_1, \ldots, v_{n-1})$ and the metric g takes the form of a warped product metric

$$g = du^2 + \sin^2\theta(u)\sum_{i,j=1}^{n-1} c_{ij}\big(v_1, \ldots, v_{n-1}\big)dv_i dv_j = du^2 + \sin^2\theta g_c. \tag{19}$$

Equations (17) and (18) are now satisfied and the question is whether we can choose the functions c_{ij} such that the curvature tensor R of (U, g) satisfies equations (14), (15) and (16).

From Proposition 2, we obtain for $X \perp T$

$$R(\partial_u, X)\partial_u = \frac{H^{\sin\theta}(\partial_u, \partial_u)}{\sin\theta}X = \big(-(\theta')^2 + \cot\theta(\theta'')\big)X = -\big((\theta')^2 + \cos^2\theta\big)X.$$

Hence, equation (14) is satisfied for any choice of c_{ij}. On the other hand, we have for $X, Y, Z \perp T$

$$\begin{aligned} R(X, Y)Z &= R_c(X, Y)Z - \frac{\|\mathrm{grad}(\sin\theta)\|^2}{\sin^2\theta}\big(\langle Y, Z\rangle\,X - \langle X, Z\rangle\,Y\big) \\ &= R_c(X, Y)Z - (\cot^2\theta)(\theta')^2\big(\langle Y, Z\rangle\,X - \langle X, Z\rangle\,Y\big), \end{aligned}$$

where R_c is the curvature tensor associated to the metric g_c. Hence, equation (15) is satisfied and equation (16) is equivalent to

$$R_c(X,Y)Z = \left(1 + \frac{(\theta')^2}{\sin^2\theta}\right)\big(\langle Y,Z\rangle X - \langle X,Z\rangle Y\big)$$

or, equivalently,

$$R_c(X,Y)Z = \big((\theta')^2 + \sin^2\theta\big)\big(g_c(Y,Z)X - g_c(X,Z)Y\big).$$

Remark that $(\theta')^2 + \sin^2\theta$ is constant, since $\phi = 2\theta$ satisfies the equation (11). Hence, the last relation is satisfied if and only if the metric g_c has constant curvature $c = (\theta')^2 + \sin^2\theta$. $\square$

In [9], the first named author classified totally umbilical surfaces in $\mathbb{S}^2 \times \mathbb{R}$ by means of an explicit parametrization. It turns out that the totally umbilical surfaces in $\mathbb{S}^2 \times \mathbb{R}$, which are not totally geodesic, are rotation (hyper)surfaces. The following theorem is a straightforward generalization of that result.

Theorem 4. *Let M^n be a totally umbilical hypersurface of $\mathbb{S}^n \times \mathbb{R}$, with angle function θ and let p be a point of M^n where $\sin\theta \neq 0$. Then there exist coordinates $(u, v_1, \ldots, v_{n-1})$ on an open neighbourhood U of p in M^n such that θ only depends on u, the shape operator is $S = \theta' \,\mathrm{id}$, and*

$$(\theta')^2 + \sin^2\theta = c, \tag{20}$$

where c is a strictly positive real constant. Moreover, M^n is locally isometric to a rotation hypersurface with profile curve

$$\alpha(u) = \frac{1}{\sqrt{c}}\left(\sin\theta(u), 0, \ldots, 0, \theta'(u), \sqrt{c}\int \sin\theta\, du\right). \tag{21}$$

Conversely, all rotation hypersurfaces with profile curve (21)*, where θ and c satisfy* (20)*, are totally umbilical in $\mathbb{S}^n \times \mathbb{R}$.*

Proof. Let M^n be totally umbilical in $\mathbb{S}^n \times \mathbb{R}$ with shape operator $S = \lambda\,\mathrm{id}$ and angle function θ. Since $\sin\theta \neq 0$ at p, there exists an open neighbourhood U of p in M^n on which $\sin\theta$ is nowhere zero. In the proof of Proposition 1, we obtained that there exist local coordinates $(u, v_1, \ldots, v_{n-1})$ on U such that λ and θ only depend on u and satisfy (13). This implies that $\lambda^2 + \sin^2\theta = (\theta')^2 + \sin^2\theta = c$ is constant. Remark that $c > 0$ since $\sin\theta$ is nowhere zero on U.

As the function θ satisfies (20), the proof of Proposition 1 shows that in the local coordinates $(u, v_1, \ldots, v_{n-1})$, the induced metric on U is $g = du^2 + \sin^2\theta\, g_c(v_1, \ldots, v_{n-1})$, where g_c is a Riemannian metric of constant sectional curvature c. It follows from Theorem 1 that there exists, up to isometries of $\mathbb{S}^n \times \mathbb{R}$, a unique immersion $F\colon U \to \mathbb{S}^n \times \mathbb{R}$ such that F is isometric; the projection of $\partial_{x_{n+2}}$ on $F(U)$ is $F_*(\sin\theta\,\partial_u)$; the angle between the unit normal N and $\partial_{x_{n+2}}$ is θ and the shape operator is $S = \theta'\,\mathrm{id}$. A straightforward computation yields that the immersion

$$F(u, v_1, \ldots, v_{n-1}) = \frac{1}{\sqrt{c}}\left(\varphi_1 \sin\theta(u), \ldots, \varphi_n \sin\theta(u), \theta'(u), \sqrt{c}\int \sin\theta(u)\, du\right),$$

where $\big(\varphi_1(v_1, \ldots, v_{n-1}), \ldots, \varphi_n(v_1, \ldots, v_{n-1})\big)$ is a parametrization of the unit sphere $\mathbb{S}^{n-1}(1) \subset \mathbb{E}^n$, satisfies these four conditions. □

Remark. Changing the coordinate u to $\bar{u}$, with $\partial_{\bar{u}} = \sin\theta\,\partial_u$, equation (20) becomes

$$(\theta')^2 + \sin^4\theta = c\sin^2\theta.$$

After putting $\theta = \arctan(f)$, we obtain

$$\left(\frac{f'}{1+f^2}\right)^2 + \left(\frac{f}{\sqrt{1+f^2}}\right)^4 = c\left(\frac{f}{\sqrt{1+f^2}}\right)^2,$$

or, equivalently,

$$\frac{f'}{f\sqrt{c+(c-1)f^2}} = \pm 1.$$

Direct integration yields

$$f = \frac{4ce^{\pm\sqrt{c}\bar{u}+d}}{4c(1-c) + e^{2(\pm\sqrt{c}\bar{u}+d)}}.$$

A similar approach to the original equation $(\theta')^2 + \sin^2\theta = c$ with respect to the u-coordinate, would lead to an elliptic integral.

4 Semi-parallel hypersurfaces

In this section, we give a classification of semi-parallel hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$. First, we characterize them in terms of their shape operator.

Lemma 1. *Let M^n be a semi-parallel hypersurface of $\mathbb{S}^n \times \mathbb{R}$ and define T and θ as above. Then there exists a local orthonormal frame field $\{e_1, \dots, e_n\}$ on M^n, with respect to which the shape operator S takes one of the following forms:*

(i) $S = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}$,

(ii) $S = \begin{pmatrix} \lambda & & & \\ & \mu & & \\ & & \ddots & \\ & & & \mu \end{pmatrix}$, *with* $\lambda\mu = -\cos^2\theta$ *and moreover, if* $n \geq 3$,

$T = \|T\| e_1$,

(iii) $S = \begin{pmatrix} 0 & & & & & & \\ & \lambda & & & & & \\ & & \ddots & & & & \\ & & & \lambda & & & \\ & & & & \mu & & \\ & & & & & \ddots & \\ & & & & & & \mu \end{pmatrix}$, *with* $\lambda\mu = -1$ *and* $e_1 = T = \partial_{x_{n+2}}$.

Proof. Let M^n be a hypersurface of $\mathbb{S}^n \times \mathbb{R}$ and denote by $\{e_1, \dots, e_n\}$ an orthonormal tangent frame satisfying $Se_i = \lambda_i e_i$. Assume that $T = \sum_{i=1}^n T^i e_i$. By using the equation of Gauss (6), we obtain

$$\begin{aligned} R(e_i, e_j)e_j &= (\lambda_i\lambda_j + 1 - (T^j)^2)e_i + T^i T^j e_j - T^i T, \\ R(e_i, e_j)e_k &= T^i T^k e_j - T^j T^k e_i, \end{aligned}$$

where i, j and k are assumed to be mutually different. If $n < 3$, the second formula does not make sense, but the first one is still correct. From these equations, we can compute $R \cdot h$:

$$\begin{aligned} (R \cdot h)(e_i, e_j, e_i, e_i) &= 0, \\ (R \cdot h)(e_i, e_j, e_i, e_j) &= \big(\lambda_j - \lambda_i\big)\big(\lambda_i\lambda_j + 1 - (T^i)^2 - (T^j)^2\big), \\ (R \cdot h)(e_i, e_j, e_k, e_i) &= \big(\lambda_i - \lambda_k\big)T^j T^k, \\ (R \cdot h)(e_i, e_j, e_k, e_l) &= 0. \end{aligned}$$

Again, we assume i, j, k and l to be mutually different, such that the last equations are meaningless for low dimensions.

Now assume that M^n is semi-parallel, then we obtain

$$\big(\lambda_i - \lambda_j\big)\big(\lambda_i\lambda_j + 1 - (T^i)^2 - (T^j)^2\big) = \big(\lambda_i - \lambda_k\big)T^j T^k = 0$$

for all mutually different indices i, j and k. If all the eigenvalues of S are equal, we are in case (i) of the lemma. From now on, suppose that S has at least two different eigenvalues and fix indices i and j such that $\lambda_i \neq \lambda_j$. Remark that $T \neq 0$, because for $T = 0$, we have that $M^n \subseteq \mathbb{S}^n \times \{t_0\}$ is totally geodesic. We consider two cases, namely $T \notin \mathrm{span}\{e_i, e_j\}$ and $T \in \mathrm{span}\{e_i, e_j\}$. In the first case, we have $n \geq 3$ and there exists an index k different from i and j such that $T^k \neq 0$. From $(\lambda_j - \lambda_i)T^k T^i = 0$, we find that $T^i = 0$, and from $(\lambda_i - \lambda_j)T^k T^j = 0$, we find that $T^j = 0$. Hence T is perpendicular to $\mathrm{span}\{e_i, e_j\}$ and the equation $(\lambda_i - \lambda_j)(\lambda_i\lambda_j + 1 - (T^i)^2 - (T^j)^2) = 0$ yields $\lambda_i\lambda_j = -1$. In the second case, it follows from $(\lambda_i - \lambda_j)(\lambda_i\lambda_j + 1 - (T^i)^2 - (T^j)^2) = 0$ that $\lambda_i\lambda_j + 1 - \|T\|^2 = 0$ and hence $\lambda_i\lambda_j = -\cos^2\theta$. We conclude the following: if λ_i and λ_j are different eigenvalues of S, then either $\lambda_i\lambda_j = -1$ and $T \perp \mathrm{span}\{e_i, e_j\}$ or $\lambda_i\lambda_j = -\cos^2\theta$ and $T \in \mathrm{span}\{e_i, e_j\}$.

Now assume that S has exactly two distinct eigenvalues, say $Se_i = \lambda e_i$ for $i \in \{1, \ldots, k\}$ and $Se_i = \mu e_i$ for $i \in \{k+1, \ldots, n\}$. If $\lambda\mu = -1$, we have that T is perpendicular to $\mathrm{span}\{e_i, e_j\}$ for every $i \in \{1, \ldots, k\}$ and for every $j \in \{k+1, \ldots, n\}$. But this implies that $T = 0$, a contradiction. Hence we have $\lambda\mu = -\cos^2\theta$, which yields that $T \in \mathrm{span}\{e_i, e_j\}$ for every $i \in \{1, \ldots, k\}$ and for every $j \in \{k+1, \ldots, n\}$. This is only possible if $k = 1$ (or $n - k = 1$, but then we switch the role of λ and μ). Moreover, if $n \geq 3$, we have that $T = \|T\|e_1$. This is case (ii) of the lemma.

Now assume that S has at least three different eigenvalues, say λ, μ and ν. If $\lambda\mu = \lambda\nu = -1$, we have $\mu = \nu$, which is a contradiction. If $\lambda\mu = -1$ and $\lambda\nu = -\cos^2\theta$, we find

$$\mu\nu = \frac{\cos^2\theta}{\lambda^2} \geq 0,$$

which is only possible if $\nu = \cos\theta = 0$. Finally, if $\lambda\mu = \lambda\nu = -\cos^2\theta$, we obtain that either $\mu = \nu$, which is a contradiction, or $\lambda = \cos\theta = 0$. We conclude that one of the eigenvalues is zero and that $\cos\theta = 0$. Assume that $\lambda_1 = 0$. Since $\lambda_1\lambda_i = 0$ for every $i \in \{2, \ldots, n\}$, we have $T \in \mathrm{span}\{e_1, e_i\}$ for every $i \in \{2, \ldots, n\}$. Hence, $T = \partial_{x_{n+2}}$ lies in the direction of e_1. Now let λ_i and λ_j be mutually different eigenvalues with $i, j \neq 1$. Since $T \perp \mathrm{span}\{e_i, e_j\}$ we have $\lambda_i\lambda_j = -1$. It follows that there can be only two different nonzero eigenvalues and that their product is -1. This is case (iii) of the lemma. □

Lemma 1 enables us to give a full description of semi-parallel hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$:

Theorem 5. *Let M^n be a semi-parallel hypersurface of $\mathbb{S}^n \times \mathbb{R}$. Then there are four possibilities:*

(i) *$n = 2$ and M^2 is flat,*

(ii) *M^n is totally umbilical,*

(iii) *M^n is an open part of a rotation hypersurface for which the profile curve is either a vertical line, or can be parametrized as*

$$\alpha(s) = \left(\cos s, 0, \ldots, 0, \sin s, \pm \int_{s_0}^{s} \sqrt{C \cos^2 \sigma - 1}\, d\sigma \right),$$

(iv) *$M^n \subseteq \bar{M}^{n-1} \times \mathbb{R}$, where $\bar{M}^{n-1}$ is a semi-parallel hypersurface of $\mathbb{S}^n(1)$.*

As mentioned above, the classification of semi-parallel hypersurfaces of $\mathbb{S}^n(1)$ is given in [5].

Proof of Theorem 5. Let M^n be a semi-parallel hypersurface of $\mathbb{S}^n \times \mathbb{R}$ with shape operator S. According to Lemma 1, there are three possible forms of S to consider.

In the first case, M^n is totally umbilical by definition. This gives case (ii) of the theorem.

Now assume that we are in the second case of Lemma 1. If $n = 2$, then M^2 is a general flat surface in $\mathbb{S}^2 \times \mathbb{R}$ and we are in case (i) of the theorem. If $n \geq 3$, the form of S is similar to the one given in Theorem 2. In the present case, the relation $\lambda\mu = -\cos^2\theta$ is not a functional relation in the strict sense, because θ can be a non-constant function. But from (8) we see that θ does not vary in directions orthogonal to T. By looking at the proof in [6], we see that this is actually enough to obtain that M^n is a rotation hypersurface. Moreover, the equality $\lambda\mu = -\cos^2\theta$ determines the profile curve of the rotation hypersurface. First remark that formulae (10) yield that this equality is satisfied in the case that α is a vertical line. In this case we are in case (iv) of the theorem, where $\bar{M}^{n-1}$ is a hypersphere of $\mathbb{S}^n(1)$. If α is not a vertical line, formulae (9) give

$$\lambda\mu = \frac{a'(s)a''(s)\cot s}{(1 + a'(s)^2)^2}.$$

On the other hand, we have

$$\begin{aligned} -\cos^2\theta &= \sin^2\theta - 1 = \left\langle \partial_{x_{n+2}}, \frac{T}{\|T\|} \right\rangle^2 - 1 = \left\langle \partial_{x_{n+2}}, \frac{\alpha'}{\|\alpha'\|} \right\rangle^2 - 1 \\ &= \frac{a'(s)^2}{1+a'(s)^2} - 1 = -\frac{1}{1+a'(s)^2}. \end{aligned}$$

Thus the equation $\lambda\mu = -\cos^2\theta$ is equivalent to $(a'(s)^2)' + 2\tan s\, a'(s)^2 = -2\tan s$, for which the general solution is given by $a'(s)^2 = C\cos^2 s - 1$, with $C \in \mathbb{R}$. This covers case (iii) of the theorem.

In the last case of Lemma 1, $\partial_{x_{n+2}}$ is everywhere tangent to M^n and hence we are dealing with an open part of a product hypersurface $\bar{M}^{n-1} \times \mathbb{R}$, where $\bar{M}^{n-1}$ is a hypersurface of $\mathbb{S}^n(1)$. Since $\mathbb{S}^n(1)$ is a totally geodesic hypersurface of $\mathbb{S}^n \times \mathbb{R}$, we have that the shape operator $\bar{S}$ of $\bar{M}^{n-1}$ in $\mathbb{S}^n(1)$ satisfies $\bar{S}X = SX$ for X tangent to $\bar{M}^{n-1}$, such that $\bar{S}$ takes the form

$$\bar{S} = \begin{pmatrix} \lambda & & & & & \\ & \ddots & & & & \\ & & \lambda & & & \\ & & & \mu & & \\ & & & & \ddots & \\ & & & & & \mu \end{pmatrix},$$

with $\lambda\mu = -1$. It was proven in [5] that this condition is equivalent to the condition that $\bar{M}^{n-1}$ is semi-parallel in $\mathbb{S}^n(1)$. $\square$

5 Parallel hypersurfaces

We will now give a full classification of parallel hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$. One can easily verify that a parallel hypersurface has to be semi-parallel and hence in view of Theorem 5, there are only four cases to consider.

Theorem 6. *Let M^n be a parallel hypersurface of $\mathbb{S}^n \times \mathbb{R}$. Then there are two possibilities:*

(i) *M^n is an open part of a totally geodesic hypersurface $\mathbb{S}^n \times \{t_0\}$,*

(ii) *M^n is an open part of a Riemannian product $\bar{M}^{n-1} \times \mathbb{R}$, where $\bar{M}^{n-1}$ is a parallel hypersurface of $\mathbb{S}^n(1)$.*

Proof. Since M^n has to be semi-parallel, we just have to find the parallel hypersurfaces in the four families of Theorem 5.

Case (i). Parallel surfaces in $\mathbb{S}^2 \times \mathbb{R}$ are classified in [1]. The flat ones are open parts of Riemannian products of a circle in $\mathbb{S}^2(1)$ and $\mathbb{R}$. This is a special case of the second case in the theorem.

Case (ii). If M^n is totally umbilical in $\mathbb{S}^n \times \mathbb{R}$, with shape operator $S = \lambda\,\mathrm{id}$, a straightforward computation shows that M^n is parallel if and only if λ is constant. But then the first equation of (13) yields that either $\cos\theta = 0$ or $\sin\theta = 0$. This means hat M^n is an open part of either $\bar{M}^{n-1} \times \mathbb{R}$ or of $\mathbb{S}^n \times \{t_0\}$. In the latter case, M^n is totally geodesic and we are in the first case of the theorem. In the former case, the second equation of (13) gives $\lambda = 0$ and hence we obtain that $\bar{M}^{n-1}$ has to be a totally geodesic hypersurface of $\mathbb{S}^n(1)$. This is again a special case of the second case of the theorem.

Case (iii). Assume that M^n is a rotation hypersurface with $\lambda\mu = -\cos^2\theta$. We may assume that $n \geq 3$, because for $n = 2$, we are dealing with a flat surface and this case was treated above. Take X and Y perpendicular to T. The equation of Codazzi for X and Y gives $X[\mu]Y - Y[\mu]X = 0$, yielding that μ is constant in directions perpendicular to T. Now let $\alpha(s)$ be the profile curve of M^n and choose a vector field $X(s)$ along $\alpha(s)$ which satisfies the following conditions: $X(s) \perp \alpha'(s)$ (or equivalently $X(s) \perp T$), $X(s)$ is parallel along $\alpha(s)$ in $\mathbb{E}^{n+2}$ and $\|X(s)\| = 1$. Such a vector field clearly exists, since it is sufficient to choose X orthogonal to the subspace P^3, appearing in the construction of the rotation hypersurface, and tangent to M^n. The formula of Gauss (1) yields that $D_T X = \nabla_T X$, such that X is also parallel along $\alpha(s)$ in M^n. If we now assume M^n to be parallel, we obtain

$$0 = (\nabla h)(T, X, X) = T[h(X, X)] - 2h(\nabla_T X, X) = T[\mu].$$

This implies that μ is constant.

Now take X perpendicular to T, then the equation of Codazzi for X and T gives

$$\begin{aligned}
&\nabla_X ST - \nabla_T SX - S[X, T] = \cos\theta \|T\|^2 X \\
&\quad\Rightarrow\quad \nabla_X(\lambda T) - \nabla_T(\mu X) - S(\nabla_X T - \nabla_T X) = \cos\theta \sin^2\theta\, X \\
&\quad\Rightarrow\quad X[\lambda]T + \lambda\nabla_X T - \mu\nabla_T X - S(\cos\theta\, SX) + S(\nabla_T X) = \cos\theta \sin^2\theta\, X \\
&\quad\Rightarrow\quad X[\lambda]T + \lambda\mu\cos\theta\, X - \mu\nabla_T X - \mu^2\cos\theta\, X + \mu\nabla_T X = \cos\theta \sin^2\theta\, X
\end{aligned}$$

$$\Rightarrow \quad X[\lambda]T - \cos\theta(\mu^2+1)X = 0$$
$$\Rightarrow \quad \cos\theta = 0.$$

This means that that M^n is an open part of a Riemannian product $\bar{M}^{n-1} \times \mathbb{R}$. The second equation of (8) yields $ST = 0$ and hence $\lambda = 0$. Remark that the condition $\lambda\mu = -\cos^2\theta = 0$ is automatically satisfied. The shape operators of $\bar{M}^{n-1} \subset \mathbb{S}^n(1)$ as a submanifold of $\mathbb{E}^{n+1}$ are $S_1 = \mu\,\mathrm{id}$ and $S_2 = \mathrm{id}$. If we change the basis of the normal plane by an appropriate rotation, they become $\bar{S}_1 = \sqrt{\mu^2+1}\,\mathrm{id}$ and $\bar{S}_2 = 0$. Hence $\bar{M}^{n-1}$ is an open part of a sphere of radius $1/\sqrt{\mu^2+1}$ and we are in a special case of of the second case of the theorem.

Case (iv). Finally, we assume that M^n is an open part of $\bar{M}^{n-1} \times \mathbb{R}$, where $\bar{M}^{n-1}$ is a semi-parallel hypersurface of $\mathbb{S}^n(1)$. It is easy to see that $\bar{M}^{n-1} \times \mathbb{R}$ is parallel in $\mathbb{S}^n \times \mathbb{R}$ if and only if $\bar{M}^{n-1}$ is parallel in $\mathbb{S}^n(1)$. This is the second case of the theorem. □

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