BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 39(3), 401-416

Residues of codimension one singular holomorphic distributions

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Abstract. For a codimension one locally-free singular holomorphic distribution, we give a residue formula in terms of the conormal sheaf given by Pfaffian equations. We also prove a Baum-Bott type residue formula for singular distributions.

Keywords: singular distribution, Koszul resolution, Baum-Bott residue formula.

Mathematical subject classification: Primary: 32A27; Secondary: 50A30.

1 Introduction

Let X be an n-dimensional complex manifold, and C a Riemann surface. We denote by $\Theta_\bullet$ and $\Omega_\bullet$ respectively, the tangent and the cotangent sheaves. We consider a holomorphic map $f : X \to C$ having the generic fiber M_f. Assume that the critical set of f consists only of isolated points. Then we have *the multiplicity formula* in [IV],

$$\chi(X) - \chi(M_f)\chi(C) = (-1)^n \sum_p \mu(f, p).$$

In the above, $\mu(f, p)$ is the Milnor number of f at a critical point p.

This formula follows from the facts as explained below. The differential map $df : \Theta_X \to f^{-1}\Theta_C$ of f defines a section of $\mathcal{H}om(\Theta_X, f^{-1}\Theta_C) \simeq \Omega_X \otimes f^{-1}\Theta_C$ and the critical set of f is the zero set of df. Then the top Chern class of $\Omega_X \otimes f^{-1}\Theta_C$ is localized by "the section df" at the critical point set. The corresponding residue is the Milnor number of f at the critical point.

We interpret this problem as *the transversality problem* for a foliation. If we consider the direct product $Y = X \times C$ and the natural projection $\pi_2 : Y \to C$, then we naturally have the non-singular codimension one foliation $\tilde{\mathcal{G}} = \pi_2^{-1}\Omega_C$.

Received 7 July 2007.

The map $F\colon X \to Y$ defined by the graph $F\colon x \mapsto (x, f(x))$ is non-transversal to the foliation $\tilde{\mathcal{G}}$ at the critical points of f. Let us discuss this interpretation for more general settings, that is, the case where $F\colon X \to Y$ is a general holomorphic map to an m-dimensional complex manifold Y provided with a non-singular foliation $\tilde{\mathcal{G}}$ of codimension one (see [IZ]). If we consider the pull-back $\mathcal{G} = F^{-1}\tilde{\mathcal{G}}$ via F, we have the *singular* foliation $\mathcal{G}$ on X. It is easily seen that the singular set of $\mathcal{G}$ coincides with a non-transversal locus of F with respect to $\tilde{\mathcal{G}}$. Here we also see the existence of a " section" . Since $\mathcal{G} = F^{-1}\tilde{\mathcal{G}}$ is a cotangent distribution on X, we have the injective map $\iota\colon \mathcal{G} \to \Omega_X$. Therefore we have "the section ι" of $\mathcal{H}om(\mathcal{G}, \Omega_X) \simeq \Omega_X \otimes \mathcal{G}^\vee$.

Consequently, for a codimension one distribution (or foliation) $\mathcal{G}$ on X, the *localization problem* for the sheaf $\Omega_X \otimes \mathcal{G}^\vee$ plays an important role for problems of the above sorts. In view of this, for a codimension one holomorphic distribution $\mathcal{G}$ we study the localization problem of the top Chern class of $\Omega_X \otimes \mathcal{G}^\vee$. That is to say, our approach to the study for distributions is based on the terms of *differential forms*. There are many investigations for foliation or distribution of dimension one, those are from the view point of *vector fields* (see [CL], [M], [So], and so forth). The cotangent description of the residue formula for distributions defined by Pfaffian equations behaves well for dealing with pull-back objects. The transversality problem is a typical example: as explained above, a residue for the non-transversality for a distribution is studied as the residue of the singularity of the pull-back of the distribution.

In section 2, we will define a residue for the singularity of a codimension one distribution. Particularly in isolated singular cases, we will give the following theorem:

Let X be an n-dimensional compact complex manifold and $\mathcal{G}$ a rank-one locally-free subsheaf of Ω_X. We assume that the singular supports of $\Omega_X/\mathcal{G}$ are all isolated. Then we have

$$\int_X c_n\left(\Omega_X \otimes \mathcal{G}^\vee\right) = \sum_{j=1}^{k} \operatorname{Res}_{p_j} \begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)}, \ldots, f_n^{(j)} \end{bmatrix}.$$

where $\left(f_1^{(j)}, \ldots, f_n^{(j)}\right)$ is the local coefficients of the generator of $\mathcal{G}$ at a singular point p_j.

We note that the formula was obtained by [J] in the projective case. Our result is a generarigation of this to complex analytic cases. The proof of this theorem will be given in subsection 2.3.

In section 3, we will see some geometric interpretations of the class $c_n(\Omega_X \otimes \mathcal{G}^\vee)$. We define the tangent sheaf of the distribution $\mathcal{F}$ by taking the annihilator of $\mathcal{G}$ by the dual coupling and set $\mathcal{N}_{\mathcal{F}} = \Theta_X/\mathcal{F}$. Then we will show that $c_n(\mathcal{N}_{\mathcal{F}})$ is equal to $(-1)^n(n-1)!c_n(\Omega_X \otimes \mathcal{G}^\vee)$, which means the above formula is the Baum-Bott *type* residue formula. We note that the Bott vanishing theorem based on the involutivity of $\mathcal{F}$ is not necessary for the top Chern class $c_n(\mathcal{N}_{\mathcal{F}})$.

On the complex surface, we will also see the relationship between $c_2(\Omega_X \otimes \mathcal{G}^\vee)$ and the Euler-Poincaré characteristic of the solution complex of $\mathcal{D}$-module associated to holomorphic foliations. This observation is for our long-term attempt to deal with a micro-local viewpoint (say, the conormal geometry) in the context of complex differential geometry.

As an application of our results, we will give the residue formula for the non-transversality of a holomorphic map $F\colon X \to Y$ to a non-singular distribution on Y. As observe the above, a residue for for the non-transversality will be defined as the residue of the singularity of the lifted distribution on X. A global index formula for the residue for non-transversality will be demonstrated immediately from our main theorem.

I would like to thank T. Ohmoto and T. Suwa for helpful conversations. I also would like to thank J. Noguchi for helpful comments.

2 Residue of codimension one distribution

2.1 Singular holomorphic distribution

Let X be a complex manifold and $\mathcal{O}_X$ the sheaf of holomorphic functions on X. We define a singular holomorphic distribution $\mathcal{F}$ on X to be a coherent subsheaf of the tangent sheaf Θ_X. we call $\mathcal{F}$ the tangent sheaf of the distribution. We say $\mathcal{F}$ is dimension p if a generic stalk of $\mathcal{F}$ is rank p free $\mathcal{O}_X$-module. We also define the normal sheaf $\mathcal{N}_{\mathcal{F}}$ of $\mathcal{F}$ by the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \Theta_X \longrightarrow \mathcal{N}_{\mathcal{F}} \longrightarrow 0.$$

The singular set $S(\mathcal{F})$ of $\mathcal{F}$ is defined by

$$S(\mathcal{F}) = \left\{p \in X \mid \mathcal{N}_{\mathcal{F},p} \text{ is not } \mathcal{O}_p\text{-free}\right\}.$$

We can also give a definition of a singular holomorphic distribution $\mathcal{G}$ on X to be a coherent subsheaf of the cotangent sheaf Ω_X. We call $\mathcal{G}$ the conormal sheaf of the distribution. We also say $\mathcal{G}$ is codimension q if the generic rank is q. We also define the cotangent sheaf $\Omega_{\mathcal{G}}$ of $\mathcal{G}$ by the exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \Omega_X \longrightarrow \Omega_{\mathcal{G}} \longrightarrow 0.$$

The singular set $S(\mathcal{F})$ of $\mathcal{F}$ is also defined by

$$S(\mathcal{G}) = \left\{p \in X \mid \Omega_{\mathcal{G},p} \text{ is not } \mathcal{O}_p\text{-free}\right\}.$$

2.2 Codimension one distribution

A codimension one locally-free singular holomorphic distribution is given by a collection of 1-forms $\omega = (\omega_\alpha,\ U_\alpha)$ for an open covering $\{U_\alpha\}$ of X which has the transition relations $\omega_\beta = g_{\alpha\beta}\omega_\alpha$ on the intersection $U_\alpha \cap U_\beta$ with $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. Then the cocycle $(g_{\alpha\beta})$ defines a line bundle G. Generically at p, the covector ω_p gives an embedding of the fiber G_p into T^*_pX by $G_p \ni f_p \mapsto f_p\omega_p \in T^*_pX$. Thus G is regarded as a subbundle of T^*X on the complement of the zero locus of ω. Since the map of germs of sections $\mathcal{O}_X(G)_p \ni (f)_p \mapsto (f\omega)_p \in \Omega_{X,p}$ are injective for all $p \in X$, the sheaf $\mathcal{G} = \mathcal{O}_X(G)$ gives the subsheaf of Ω_X in the above sense. Since the quotient sheaf $\Omega_{\mathcal{F}}$ is not $\mathcal{O}$-free on the zero locus of ω on which we can not define the quotient bundle T^*X/G, we see the singular set of $\mathcal{G}$ is $S(\mathcal{G}) = \{\, p \mid \omega(p) = 0\}$.

We see that $\Omega_X \otimes \mathcal{G}^\vee$ always has a *global section* as follows. Since ω can be regarded as a homomorphism $\iota_\omega\colon \mathcal{G} \to \Omega_X$ by $\mathcal{O}_X(G)_p \ni (f)_p \mapsto (f\omega)_p \in \Omega_{X,p}$, it defines a global section

$$\iota_\omega \in H^0\big(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{G},\ \Omega_X)\big) \simeq H^0\big(X,\ \Omega_X \otimes \mathcal{G}^\vee\big). \tag{2.1}$$

Locally on U_α, ι_ω is given by $\omega_\alpha \otimes s^\vee_\alpha = \sum f_i(dx_i \otimes s^\vee_\alpha)$ for some local coordinates of X and a local frame $s^\vee_\alpha$ for $\mathcal{G}^\vee$.

2.3 Localization of the top Chern class

We describe the dual homology class of $c_n(\Omega_X \otimes \mathcal{G}^\vee)$. Our main tool is the Čech-de Rham techniques. For generalities on the integration and the Chern-Weil theory on the Čech-de Rham cohomology, see [Su3] or [IS]. Let S be an analytic set and U_1 a regular neighborhood of S.

We set $U_0 = X \setminus S$, and $U_{01} = U_0 \cap U_1$. For a covering $\mathcal{U} = \{U_0,\ U_1\}$ of X, the Čech-de Rham cohomology group $H^{2n}(\mathcal{A}^\bullet(\mathcal{U}))$ is represented by the group of cocycles of the type $(\sigma_0,\ \sigma_1,\ \sigma_{01})$ for $\sigma_0 \in Z^{2n}(U_0)$, $\sigma_1 \in Z^{2n}(U_1)$, and $\sigma_{01} \in A^{2n-1}(U_{01})$ with $d\sigma_{01} = \sigma_1 - \sigma_0$. We note that the Čech-de Rham cohomology can be regarded as the hypercohomology of the de Rham complex $\{\mathcal{A}^\bullet,\ d\}$. From the usual spectral sequence arguments for double complexes, we see that the Čech-de Rham cohomology group is canonically isomorphic to the de Rham cohomology group. If we take the subgroup $H^{2n}(\mathcal{A}^\bullet(\mathcal{U}, U_0))$ of cocycles of the form $(0, \sigma_1, \sigma_{01})$, then this is also isomorphic to the relative cohomology group $H^{2n}(X, X \setminus S; \mathbf{C})$.

In the above settings, the top Chern class $c_n(E)$ of a vector bundle E of rank n is given by the cocycle in $H^{2n}(\mathcal{A}^\bullet(\mathcal{U}))$ as follows. For $i = 0, 1$, let ∇_i be a connection for E on U_i and $c_n(\nabla_i)$ the n-th Chern form of ∇_i. We also write by $c_n(\nabla_0, \nabla_1)$ the transgression form of $c_n(\nabla_i)$s' on U_{01}. Then $c_n(E)$ is represented by

$$\big(c_n(\nabla_1),\ c_n(\nabla_1),\ c_n(\nabla_0, \nabla_1)\big).$$

If E has a global section s with zero locus S, then we take ∇_0 as the s-trivial connection such that we have $c_n(\nabla_0) = 0$. Thus we can define the localized Chern class in $H^{2n}(X, X \setminus S; \mathbf{C})$ by a Čech-de Rham cocycle $(0,\ c_n(\nabla_1),\ c_n(\nabla_0, \nabla_1))$. If we take a compact n-dimensional manifold R which satisfies $S \subset R \subset U_1$, the integration of $c_n(E) = (0,\ c_n(\nabla_1),\ c_n(\nabla_0, \nabla_1))$ is defined by

$$\int_X c_n(E) = \int_R c_n(\nabla_1) - \int_{\partial R} c_n(\nabla_0, \nabla_1)$$

Now we apply the above arguments to our situations. Let $\mathcal{G}$ be a codimension one locally-free distribution with the singular locus $S(\mathcal{G})$ and suppose that $S(\mathcal{G})$ has connected components S_j. We set $U_0 = X \setminus S(\mathcal{G})$ and U_j is a regular neighborhood of S_j. We consider the localized class of $c_n(\Omega_X \otimes \mathcal{G}^\vee)$ in the Čech-de Rham cohomology group for the covering $\mathcal{U} = \{U_0, U_1, \dots, U_k\}$. As mentioned in 2.2, the collection ω of 1-forms ω_α defines the global section ι_ω of $\Omega_X \otimes \mathcal{G}^\vee$. If we take ∇_0 as the ι_ω-trivial, we have $c_n(\nabla_0) = 0$ as discussed above. For all $j = 1, \dots, k$, we can also take ∇_j as an arbitrary connection on U_j. So we have

$$c_n\big(\Omega_X \otimes \mathcal{G}^\vee\big) = \big(0,\ \{c_n(\nabla_j)\}_{j=1,\dots,k},\ \{c_n(\nabla_0, \nabla_j)\}_{j=1,\dots,k}\big)$$

in $H^{2n}(X, X \setminus S(\mathcal{G}); \mathbf{C})$.

We denote by R_j a compact n-dimensional manifold satistying $S_j \subset R \subset U_j$. We give the following definition of residue.

Definition 2.1. *The residue of $\mathcal{G}$ at S_j is defined by*

$$\operatorname{Res}(\mathcal{G}, S_j) = \int_{R_j} c_n(\nabla_j) - \int_{\partial R_j} c_n(\nabla_0, \nabla_j)\,.$$

We can describe the residue into precise form in isolated singular cases. Let s be a regular section of E with isolated zero $\{p\}$. Suppose that s is locally given by $(f_1, \dots, f_n)$ near p. Then we have

$$\mathrm{Res}(\mathcal{G}, p) = \mathrm{Res}_p \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \dots, f_n \end{bmatrix} \quad \text{where} \quad \mathrm{Res}_p \begin{bmatrix} df_1 \wedge \cdots \wedge df_n \\ f_1, \dots, f_n \end{bmatrix}$$

is the Grothendick residue of $(f_1, \dots, f_n)$. (See Chapter 5 of [GH], or Theorem 5.5 of [Su3].) Now the following result follows:

Theorem 2.2 [The residue formula for isolated singularity]**.** *Let ω be a codimension one singular holomorphic distribution with the cotangent sheaf $\mathcal{G}$, and $(f_1^{(j)}, \dots, f_n^{(j)})$ a local coefficients of ι_ω in* (2.1) *near p_j. Then we have*

$$\int_X c_n(\Omega_X \otimes \mathcal{G}^\vee) = \sum_{j=1}^{k} \mathrm{Res}_{p_j} \begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)}, \dots, f_n^{(j)} \end{bmatrix}.$$

3 Baum-Bott type residue formula

3.1 Koszul resolution

First we recall the definition of the Koszul complex. (See Chapter 4 of [FL], or Chapter 5 of [GH].) Let $\mathcal{E}$ be a locally-free $\mathcal{O}$-module of rank n, and $d\colon \mathcal{E} \to \mathcal{O}$ an $\mathcal{O}$-homomorphism. Then the Koszul complex of sheaves

$$0 \to \wedge^n \mathcal{E} \to \wedge^{n-1}\mathcal{E} \to \cdots \to \wedge^1 \mathcal{E} \to \mathcal{O} \to 0$$

is defined by the boundary operator

$$d_p(\varepsilon_1 \wedge \cdots \wedge \varepsilon_p) = \sum_{i=1}^{p} (-1)^{i-1} d(\varepsilon_i)\, \varepsilon_1 \wedge \cdots \wedge \hat{\varepsilon}_i \wedge \cdots \wedge \varepsilon_p.$$

This complex is exact expect for the last term. The image $\mathcal{I}_d$ of d is an ideal sheaf. If the stalk $\mathcal{I}_{d,x}$ is regular ideal for each point x, then the following complex of sheaves

$$0 \to \wedge^n \mathcal{E} \to \wedge^{n-1}\mathcal{E} \to \cdots \to \wedge^1 \mathcal{E} \to \mathcal{O} \to \mathcal{O}/\mathcal{I}_d \to 0$$

is exact. We call this exact sequence the Koszul resolution of $\mathcal{O}/\mathcal{I}_d$.

Now we return to the case we study. As observed in (2.1), ω can be regarded as a global section $\iota_\omega \in H^0(X,\ \Omega_X \otimes \mathcal{G}^\vee)$ with local coefficents $(f_1^\alpha, \dots, f_n^\alpha)$ on U_α. If we assume that $S(\mathcal{G}) = \{\, p \in X \mid \omega_p = 0 \,\}$ consists only of isolated points, the local coefficients $(f_1, \ldots, f_n)$ of ω (or of ι_ω) is a regular sequence near p_j. If we define *the contraction operator* $\iota_\omega\colon \Theta_X \otimes \mathcal{G} \to \mathcal{O}$ by dual action of ι_ω on $\Theta_X \otimes \mathcal{G}$, the complex of sheaves

$$0 \to \wedge^n(\Theta_X \otimes \mathcal{G}) \to \wedge^{n-1}(\Theta_X \otimes \mathcal{G}) \to \cdots \to \wedge^1(\Theta_X \otimes \mathcal{G}) \to \mathcal{O} \to \mathcal{O}/\mathcal{I}_\omega \to 0$$

is exact with the boundary operator

$$d_p(e_1 \wedge \cdots \wedge e_p) = \sum_{i=1}^{p} (-1)^{i-1} f_i\, e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_p$$

where we denote by $\mathcal{I}_w$ the ideal sheaf defined by $\mathcal{I}_\omega = \mathrm{Im}\{\iota_\omega\colon \Theta_X \otimes \mathcal{G} \to \mathcal{O}\}$ and we set $e_i = \frac{\partial}{\partial x_i} \otimes s$. This complex of sheaves gives the Koszul resolution of $\mathcal{O}/\mathcal{I}_\omega$. By using this projective resolution, we can define the Chern character of the coherent sheaf $\mathcal{O}/\mathcal{I}_\omega$ by

$$\mathrm{ch}(\mathcal{O}/\mathcal{I}_\omega) = \mathrm{ch}\left(\sum_{i=0}^{n} (-1)^i \wedge^i (\Theta_X \otimes \mathcal{G})\right).$$

We have the following proposition.

Proposition 3.1.

$$\mathrm{ch}(\mathcal{O}/\mathcal{I}_\omega) = c_n(\Omega_X \otimes \mathcal{G}^\vee).$$

Proof. This follows immediately from Theorem 10.1.1 of [H] as

$$\begin{aligned} \mathrm{ch}(\mathcal{O}/\mathcal{I}_\omega) &= \mathrm{ch}\left(\sum_{i=0}^{n} (-1)^i \wedge^i (\Theta_X \otimes \mathcal{G})\right) \\ &= \mathrm{td}^{-1}\big(\Omega_X \otimes \mathcal{G}^\vee\big) c_n\big(\Omega_X \otimes \mathcal{G}^\vee\big) \\ &= c_n\big(\Omega_X \otimes \mathcal{G}^\vee\big). \end{aligned}$$

The last equality follows from the dimensional reason since td^{-1} has the form of $1 + \frac{1}{2}c_1 + \cdots$. □

Remark 3.2. We note that we can also give a parallel proof of Theorem 2.2 as an application of the Grothendick-Riemann-Roch formula for embeddings. For simplicity, we discuss the case that the singular set has only point p.

Since the sheaf $\mathcal{O}/\mathcal{I}_\omega$ is clearly sky-scraper at p, we see that the sheaf $\mathcal{O}/\mathcal{I}_\omega$ is given by the extention by zero $\iota_!(\mathcal{O}_p/\mathcal{I}_{\omega,p}) = \mathcal{O}/\mathcal{I}_\omega$ for $\iota\colon p \to X$. Let

$$\mathrm{ch}_p\colon K^0(X, X \setminus p) \longrightarrow H^*(X, X \setminus p)$$

be the local chern character. In this case, the Grothendick-Riemann-Roch formula for $\iota\colon p \to X$ is writen as

$$\mathrm{ch}_p(\mathcal{O}/\mathcal{I}_\omega) = \mathrm{ch}_p(\iota_!(\mathcal{O}_p/\mathcal{I}_{\omega,p})) = \iota_*(\mathrm{td}(N_p)^{-1}\,\mathrm{ch}(\mathcal{O}_p/\mathcal{I}_{\omega,p})).$$

Since the Chern character of $\mathbb{C}$-vector space $\mathcal{O}_p/\mathcal{I}_{\omega,p}$ is just the dimension and we also see $N_p = T_pX$ such that $\mathrm{td}(N_p) = 1$, the above implies the Alexander dual of $ch(\mathcal{O}/\mathcal{I}_\omega)$ is equals to $\dim(\mathcal{O}_p/\mathcal{I}_{\omega,p})$.

3.2 Baum-Bott type residue formula

Let $\mathcal{G}$ be a codimension one, locally-free distribution and $\mathcal{F} = \{v \in \Theta_X | \langle v, \omega\rangle = 0\}$ its annihilator. On complex surfaces, we have *the adjunction formula* which discribe the relation between $\mathcal{G}$ and $\mathcal{F}$. In general cases, we have no such an isomorphism, however, we can give the relation in the level of the Chern character.

Proposition 3.3.

$$\mathrm{ch}(\mathcal{N}_\mathcal{F}) = (1 - c_n(\Omega_X \otimes \mathcal{G}^\vee))\,\mathrm{ch}(\mathcal{G}^\vee).$$

Proof. Recall the exact sequences,

$$0 \longrightarrow \mathcal{G} \longrightarrow \Omega_X \longrightarrow \Omega_\mathcal{G} \longrightarrow 0, \tag{3.1}$$

$$0 \longrightarrow \mathcal{F} \longrightarrow \Theta_X \longrightarrow \mathcal{N}_\mathcal{F} \longrightarrow 0. \tag{3.2}$$

Note that $\Theta_X = \mathcal{H}om_\mathcal{O}(\Omega_X, \mathcal{O})$ and also $\mathcal{F} = \mathcal{H}om_\mathcal{O}(\Omega_\mathcal{G}, \mathcal{O})$ since $\mathcal{F}$ is the annihilator of $\mathcal{G}$. Likewise, we need to show the comparison of the Chern characters of $\mathcal{N}_\mathcal{F}$ and $\mathcal{G}^\vee = \mathcal{H}om_\mathcal{O}(\mathcal{G}, \mathcal{O})$. If we apply $\mathcal{H}om_\mathcal{O}(\quad, \mathcal{O})$ to (3.1), we have

$$0 \longrightarrow \mathcal{F} \longrightarrow \Theta_X \longrightarrow \mathcal{G}^\vee \longrightarrow \mathcal{E}xt^1_\mathcal{O}(\Omega_\mathcal{G}, \mathcal{O}) \longrightarrow 0.$$

Taking the quotient of the first two terms, we have

$$0 \longrightarrow \mathcal{N}_\mathcal{F} \longrightarrow \mathcal{G}^\vee \longrightarrow \mathcal{E}xt^1_\mathcal{O}(\Omega_\mathcal{G}, \mathcal{O}) \longrightarrow 0. \tag{3.3}$$

This implies

$$\mathrm{ch}(\mathcal{N}_{\mathcal{F}}) = \mathrm{ch}(\mathcal{G}^{\vee}) - \mathrm{ch}(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O})). \tag{3.4}$$

We also do the dual computation. Since $\mathcal{G}$ is locally-free, the functor $\otimes\mathcal{G}$ is exact. Thus if we apply $\otimes\mathcal{G}$ to (3.2) and (3.3), we have

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{G} \longrightarrow \Theta_X \otimes \mathcal{G} \longrightarrow \mathcal{N}_{\mathcal{F}} \otimes \mathcal{G} \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{N}_{\mathcal{F}} \otimes \mathcal{G} \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \otimes \mathcal{G} \longrightarrow 0.$$

Since the kernel of $\iota_\omega \colon \Theta_X \otimes \mathcal{G} \to \mathcal{O}_X$ is equal to $\mathcal{F} \otimes \mathcal{G}$, we have $\mathcal{I}_\omega = \mathrm{Im}(\iota_\omega) \simeq (\Theta_X \otimes \mathcal{G})/(\mathcal{F} \otimes \mathcal{G}) \simeq \mathcal{N}_{\mathcal{F}} \otimes \mathcal{G}$ and $\mathcal{O}/\mathcal{I}_\omega \simeq \mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \otimes \mathcal{G}$. This also implies

$$\mathrm{ch}(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O})) = \mathrm{ch}(\mathcal{O}/\mathcal{I}_\omega)\,\mathrm{ch}(\mathcal{G}^{\vee}). \tag{3.5}$$

Now this proposition follows immediately from (3.4), (3.5), and Proposition 3.1 as

$$\begin{aligned}\mathrm{ch}(\mathcal{N}_{\mathcal{F}}) &= (1 - \mathrm{ch}(\mathcal{O}/\mathcal{I}_\omega))\,\mathrm{ch}(\mathcal{G}^{\vee}) \\ &= (1 - c_n(\Omega_X \otimes \mathcal{G}^{\vee}))\,\mathrm{ch}(\mathcal{G}^{\vee}).\end{aligned}$$

□

If we take the relation of Chern classes from this proposition, we have

Proposition 3.4. *Let $\mathcal{G}$ be a codimension one, locally-free distribution and $\mathcal{F}$ its annihilator. Then we have*

$$c_n(\mathcal{N}_{\mathcal{F}}) = (-1)^n (n-1)!\, c_n(\Omega_X \otimes \mathcal{G}^{\vee}).$$

Proof. Let $\{\xi_i\}$ be the formal Chern roots of $c(\mathcal{N}_{\mathcal{F}})$ and ch_i the terms of i-th degree in ch. Then from Proposition 3.1, we have

$$\mathrm{ch}_i(\mathcal{N}_{\mathcal{F}}) = \frac{1}{i!} c_1(\mathcal{G}^{\vee})^i$$

for $i \leq n-1$ and

$$\mathrm{ch}_n(\mathcal{N}_{\mathcal{F}}) = \frac{1}{n!} c_1(\mathcal{G}^{\vee})^n - c_n(\Omega_X \otimes \mathcal{G}^{\vee}).$$

It is obvious that $\mathrm{ch}_1(\mathcal{N}_{\mathcal{F}}) = c_1(\mathcal{G}^{\vee})$. We also see that

$$\begin{aligned}\frac{1}{2!} c_1(\mathcal{G}^{\vee})^2 = \mathrm{ch}_2(\mathcal{N}_{\mathcal{F}}) &= \frac{1}{2!}\left(\xi_1^2 + \cdots + \xi_n^2\right) \\ &= \frac{1}{2!}\left\{\left(\xi_1 + \cdots + \xi_n\right)^2 - 2\sum \xi_i \xi_j\right\} \\ &= \frac{1}{2!} c_1(\mathcal{G}^{\vee})^2 - c_2(\mathcal{N}_{\mathcal{F}}),\end{aligned}$$

which implies $c_2(\mathcal{N}_{\mathcal{F}}) = 0$. If we continue the similar computations for fundamental symmetric polynomials, we have

$$c_2(\mathcal{N}_{\mathcal{F}}) = \cdots = c_{n-1}(\mathcal{N}_{\mathcal{F}}) = 0.$$

Thus for n-th term, we have

$$\begin{aligned}
\frac{1}{n!}c_1(\mathcal{G}^\vee)^n - c_n(\Omega_X \otimes \mathcal{G}^\vee) = \mathrm{ch}_n(\mathcal{N}_{\mathcal{F}}) &= \frac{1}{n!}\left(\xi_1^n + \cdots + \xi_n^n\right) \\
&= \frac{1}{n!}\left\{\left(\xi_1 + \cdots + \xi_n\right)^n - (-1)^n n\, \xi_1 \cdots \xi_n\right\} \\
&= \frac{1}{n!}c_1(\mathcal{G}^\vee)^n - \frac{(-1)^n}{(n-1)!}c_n(\mathcal{N}_{\mathcal{F}}),
\end{aligned}$$

from which the result follows. □

Now the following formula is an immediate consequence of Theorem 2.2 and Proposition 3.4.

Theorem 3.5 [Baum-Bott type residue formula]. *Let ω be a codimension one distribution with conormal sheaf $\mathcal{G}$, and $\mathcal{F}$ the anihilator of $\mathcal{G}$. We suppose that $S(\mathcal{G}) = \{p_1, \ldots, p_k\}$ and we write $\omega = \sum f_i^{(j)}(dx_i \otimes s^\vee)$ near p_j. Then we have*

$$\int_X c_n(\mathcal{N}_{\mathcal{F}}) = (-1)^n(n-1)! \sum_j \mathrm{Res}\begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)}, \ldots f_n^{(j)} \end{bmatrix}.$$

Proof. This is simply given by

$$\begin{aligned}
\int_X c_n(\mathcal{N}_{\mathcal{F}}) &= (-1)^n(n-1)! \int_X c_n(\Omega_X \otimes \mathcal{G}^\vee) \\
&= (-1)^n(n-1)! \sum_j \mathrm{Res}\begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)}, \ldots, f_n^{(j)} \end{bmatrix}.
\end{aligned}$$

□

Remark 3.6. If we assume the integrability condition on $\mathcal{G}$, the above formula is the Baum-Bott residue formula for singular holomorphic foliations. The Baum-Bott residue for $c_n(\mathcal{N}_{\mathcal{F}})$ is given by

$$(-1)^n(n-1)!\ \dim \mathrm{Ext}^1_{\mathcal{O}_p}(\Omega_{\mathcal{G},p}, \mathcal{O}_p) = (-1)^n(n-1)!\ \dim \mathcal{O}_p/\mathcal{I}_{\omega,p}.$$

(See [Su1].) Thus the right-hand side of Theorem 3.5 coincides with the Baum-Bott residue. (See also [O].)

4 Some remarks on the Riemann-Roch formula for $\mathcal{D}$-modules associated to holomorphic foliations

In this section, we observe the other interpretation of the class $c_n(\Omega_X \otimes \mathcal{G}^\vee)$. First we recall *the global index formula* for singular foliation due to Suwa (Theorem 5.5 of [Su2]). On the definition of regular foliation, see also [Su2].

For a regular foliation $\mathcal{F}$ of dimension p on a compact complex manifold X, we have

$$\chi(\mathrm{R}\Gamma(X, Sol\mathcal{D}_{\mathcal{F}})) = \int_X \mathrm{td}(\mathcal{N}_{\mathcal{F}})c_p(\mathcal{F}).$$

In codimension one cases, we compute the right-hand side of the index formula. Recall that the exact sequence (3.3) of subsection 3.2.

$$0 \longrightarrow \mathcal{N}_{\mathcal{F}} \longrightarrow \mathcal{G}^\vee \longrightarrow \mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}) \longrightarrow 0.$$

If we take the Todd genus "td" of each terms in (3.3), then we have

$$\mathrm{td}(\mathcal{N}_{\mathcal{F}}) = \mathrm{td}(\mathcal{G}^\vee)\,\mathrm{td}(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}))^{-1}.$$

The right-hand side of the above is expanded as

$$\begin{cases} \mathrm{td}(\mathcal{G}^\vee) = 1 + \mathrm{td}_1 + \mathrm{td}_2 + \cdots, \\ \mathrm{td}(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}))^{-1} = 1 - (\mathrm{td}_1 + \cdots + \mathrm{td}_n) + (\mathrm{td}_1 + \cdots + \mathrm{td}_n)^2 - \cdots. \end{cases}$$

Taking the multiplication $c_{n-1}(\mathcal{F})\,\mathrm{td}(\mathcal{N}_{\mathcal{F}})$ and we have

$$\mathrm{td}_1(\mathcal{N}_{\mathcal{F}})c_{n-1}(\mathcal{F}) = \mathrm{td}_1(\mathcal{G}^\vee)c_{n-1}(\mathcal{F}) - \mathrm{td}_1(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O}))c_{n-1}(\mathcal{F}).$$

Since we have seen $\mathrm{ch}(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O})) = c_n(\Omega_X \otimes \mathcal{G}^\vee)$ in (3.5), we have

$$\mathrm{td}_1(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O})) = \frac{1}{2}\,\mathrm{ch}_1(\mathcal{E}xt^1_{\mathcal{O}}(\Omega_{\mathcal{G}}, \mathcal{O})) = 0.$$

Therefore we have

$$\mathrm{td}_1(\mathcal{N}_{\mathcal{F}})c_{n-1}(\mathcal{F}) = \mathrm{td}_1(\mathcal{G}^\vee)c_{n-1}(\mathcal{F}) = \frac{1}{2}c_1(\mathcal{G}^\vee)c_{n-1}(\mathcal{F}).$$

Now we obtain the following result.

Proposition 4.1 [The numerical adjunction formula]. *Let $\mathcal{G}$ be a codimension one singular locally-free foliation. If the annihilator $\mathcal{F}$ of $\mathcal{G}$ is regular, we have*

$$\chi(\mathrm{R}\Gamma(X, \mathcal{S}ol\mathcal{D}_{\mathcal{F}})) = \frac{1}{2}\int_X c_1(\mathcal{G}^\vee)c_{n-1}(\mathcal{F}).$$

Remark 4.2. Though the anihilator of locally-free foliation is not always regular foliation, the assumption is natural if we consider the case where X is a complex surface. As given in Remark (4.3) of [Su2], a locally-free foliation $\mathcal{F}$ is regular if $\dim(\mathcal{F}) \leq 2$.

On complex surfaces, we have the following.

Theorem 4.3 [The Riemann-Roch formula for codimension one foliation]. *Let X be a compact complex surface and $\mathcal{F}$ a codimension one regular foliation. Then we have*

$$\chi(R\Gamma(X, \mathcal{S}ol\mathcal{D}_{\mathcal{F}})) = \pi(\mathcal{F}) - 1.$$

where denote by $\pi(\mathcal{L}) = 1 + \frac{1}{2}(\mathcal{L}\mathcal{L} + \mathcal{L}K_X)$ the virtual genus of an invertible sheaf $\mathcal{L}$.

Proof. Let $K_X = \wedge^2\Omega_X$ be the canonical sheaf of X. On a complex surface X, we have *the adjunction formula*: $\mathcal{G} = \mathcal{F} \otimes K_X$. Thus we also have

$$\frac{1}{2}\int_X c_1(\mathcal{G}^\vee)c_1(\mathcal{F}) = \frac{1}{2}\int_X c_1(\mathcal{F})c_1(\mathcal{G}) = \frac{1}{2}\int_X c_1(\mathcal{F})(c_1(\mathcal{F}) + c_1(K_X))$$

$$= \frac{\mathcal{F}\mathcal{F} + \mathcal{F}K_X}{2} = \pi(\mathcal{F}) - 1.$$

□

On a complex surface X, let $\mathcal{G}$ be a codimenson one locally-free foliation, $\mathcal{F}$ the annihilator of $\mathcal{G}$. Suppose that $\mathcal{G}$ only has isolated singular points $\{p_1, \ldots, p_k\}$. we denote by (f^j, g^j) the local coefficients of (the generator of) $\mathcal{G}$ near p_j. Then the left-hand side of Theorem 2.2 is written in the form of

$$\int_X c_2(\Omega_X \otimes \mathcal{G}^\vee) = \int_X c_2(X) + \int_X c_1(X)c_1(\mathcal{G}) + \int_X c_1^2(\mathcal{G}).$$

If we use the adjunction formula, then we see

$$\begin{aligned} c_1(X)c_1(\mathcal{G}) + c_1^2(\mathcal{G}) &= c_1(X)(c_1(\mathcal{F}) + c_1(K_X)) + (c_1(\mathcal{F}) + c_1(K_X))^2 \\ &= -c_1(K_X)c_1(\mathcal{F}) - c_1(K_X)^2 + c_(\mathcal{F})^2 + c_1(\mathcal{F})c_1(K_X) \\ &\quad + c_1(K_X)c_1(\mathcal{F}) + c_1(K_X)^2 \\ &= c_(\mathcal{F})^2 + c_1(\mathcal{F})c_1(K_X), \end{aligned}$$

which implies

$$\int_X (c_1(X)c_1(\mathcal{G}) + c_1^2(\mathcal{G})) = 2(\pi(\mathcal{F}) - 1) = 2\chi(\mathrm{R}\Gamma(X, Sol\mathcal{D}_{\mathcal{F}})).$$

Namely the integration of the class $c_2(\Omega_X \otimes \mathcal{G}^\vee)$ is related to the sum of the topological Euler-Poincaré characteristic of X and the characteristic of the solution complex of F as

$$\int_X c_2(\Omega_X \otimes \mathcal{G}^\vee) = \chi(X) + 2\chi(\mathrm{R}\Gamma(X, Sol\mathcal{D}_{\mathcal{F}}))$$

On the other hand, the localization of $c_2(\Omega_X \otimes \mathcal{G}^\vee)$ is

$$\sum_{j=1}^{k} \mathrm{Res}_{p_j} \begin{bmatrix} df^j \wedge dg^j \\ f^j, g^j \end{bmatrix}.$$

So we see the sum of the two characaristics $\chi(X) + 2\chi(\mathrm{R}\Gamma(X, Sol\mathcal{D}_{\mathcal{F}}))$ can be localized. Now we have

Theorem 4.4 [Poincaré-Hopft type formula]**.**

$$\chi(X) + 2\chi(\mathrm{R}\Gamma(X, Sol\mathcal{D}_{\mathcal{F}})) = \sum_{j=1}^{k} \mathrm{Res}_{p_j} \begin{bmatrix} df^j \wedge dg^j \\ f^j, g^j \end{bmatrix}.$$

We note that if $\mathcal{F}$ is defined by a global vector field, then the Euler-Poincaré characteristic of the solution complex is equal to zero. Thus the above formula coincides with the Poincaré-Hopf index formula.

Remark 4.5. In [Su2], Suwa pointed out that, for $\mathcal{F} = \Theta_X$ (or $\mathcal{G} = 0$) of the highest dimensional foliation, the global index formula for $\mathcal{D}_{\mathcal{F}}$ gives the Gauss-Bonnet formula and, for $\mathcal{F} = 0$ (or $\mathcal{G} = \Omega_X$) of the lowest dimension, it also gives the Hirzebruch-Riemann-Roch formula for the structure sheaf $\mathcal{O}_X$. As mentioned in [GH], if X admits a global holomorphic vector field, the Hirzeburch-Riemann-Roch formula for the structure sheaf can be demonstrated as the combination of the Lefchetz fixed point formula for the Dolbeault complex and the Bott residue formula for holomorphic vector fields. (The same is true for Gauss-Bonnet formula with Lefchetz fixed point formula for de Rham complex and Hopf index formula. But the point is that there always exists a global differential vector field.) On a complex surface, if we assume, as *an auxiliary*, the existence of a codimension one (or dimension one) singular foliation, it gives an *intermediate form* of the Riemann-Roch type formula between the Gauss-Bonnet formula and the Riemann-Roch formula.

5 Applications

5.1 Residue for the non-transversal locus of a holomorphic map

Let $F\colon X^n \to Y^m$ be a holomorphic map between n and m dimensional compact complex manifolds. If Y admits a non-singular distribution $\tilde{\mathcal{G}} = \mathcal{O}_Y(G)$, the inverse image $\mathcal{G} = F^{-1}\tilde{\mathcal{G}}$ gives a distribution of X which is possibly singular. In codimension one case, if a distribution $\tilde{\mathcal{G}}$ on Y is given by a collection of 1-forms $\tilde{\omega} = (\tilde{\omega}_\alpha)$, the inverse image $\mathcal{G} = F^{-1}\tilde{\mathcal{G}}$ of the invertible sheaf $\tilde{\mathcal{G}}$ is given by the collection of 1-forms $\omega = (F^*\tilde{\omega}_\alpha)$. If the image of the differential DF_p dose not contain the normal space G_p^*, we see that covector ω_p is zero. In such cases we say F is non-transversal to $\tilde{\mathcal{G}}$. By definition the non-transversal locus of F to $\tilde{\mathcal{G}}$ is

$$S(\mathcal{G}) = \left\{\, p \in X \,:\, F^*\tilde{\omega}_\alpha(p) = 0 \right\}$$

Now we give the residue formula for the non-transversality of F to $\tilde{\mathcal{G}}$. We assume that $S(\mathcal{G})$ consists of isolated points $\{\, p_1, \ldots, p_k \,\}$. We set that, near p_j, $f_i^{(j)}$ are the coefficients of $F^*\tilde{\omega}_\alpha^{(j)}$ such that we write $F^*\omega_\alpha^{(j)} = f_1^{(j)}dx_1 + \cdots + f_n^{(j)}dx_n$. Then we have

$$\begin{aligned}\int_X c_n(\Omega_X \otimes \mathcal{G}^\vee) &= \sum_{l=0}^{n} \int_X c_{n-l}(\Theta_X) c_1(\mathcal{G})^l \\ &= \sum_{j=1}^{k} \mathrm{Res}_{p_j} \begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)}, \ldots, f_n^{(j)} \end{bmatrix}.\end{aligned}$$

Now we have the following result.

Theorem 5.1 [Residue formula for non-transversality]. *Let $F\colon X^n \to Y^m$ be a holomorphic map of generic* rank r *and $\tilde{\mathcal{G}}$ a codimension one non-singular distribution on Y. We assume that the non-transversal points of F to $\tilde{\mathcal{G}}$ are $\{p_1, \ldots, p_k\}$, then we have*

$$\chi(X) + \sum_{l=1}^{r} \int_{F_*(c_{n-l}(X) \frown [X])} c_1(\tilde{\mathcal{G}})^l = \sum_{j=l}^{k} \mathrm{Res}_{p_j} \begin{bmatrix} df_1^{(j)} \wedge \cdots \wedge df_n^{(j)} \\ f_1^{(j)}, \ldots, f_n^{(j)} \end{bmatrix}.$$

Proof. We denote by X^* the set of generic points where F has rank k. By using projection formula,

$$\int_X c_{n-l}(\Theta_X) c_1(\mathcal{G})^l = \int_{X^*} c_{n-l}(\Theta_X) F^*(c_1(\tilde{\mathcal{G}})^l) = \int_{F_*(c_{n-l}(X) \frown [X])} c_1(\tilde{\mathcal{G}})^l.$$

It is obvious that the above terms are zero for $k \leq l$. $\square$

As an example let us consider the case where $F: X^n \to C$ is a map onto a curve C and $\tilde{\mathcal{G}} = \Omega_C$ is the point distribution. Then the above formula implies the multiplicity formula. (See [IS], [F].)

Theorem 5.2 [The multiplicity formula]. *Let $F: X^n \to C$ be a holomorphic map onto a compact complex curve C with the generic fiber M_F. If F has finite number of isolated critical points $\{p_1, \ldots, p_k\}$, then we have*

$$\chi(X) - \chi(M_F)\chi(C) = (-1)^n \sum_{j=1}^{k} \mu(F,\ p_j),$$

where $\mu(F,\ p_j)$ is the Milnor number of F at p_j.

Remark 5.3. The one dimensional cases of Theorem 5.2 is the classical Riemann-Hurwitz formula for a morphism of Riemann surfaces $F: C \to \tilde{C}$. We note that it cannot be deduced from the Baum-Bott type formula for $c_1(\mathcal{N}_{\mathcal{F}})$, however we can still apply Theorem 2.2 for deducing the residue formula for $\mathcal{G}$. By taking the anihilator of the inverse image $\mathcal{G}$ of $\Omega_{\tilde{C}}$, the given tangent sheaf $\mathcal{F}$ of the lifted foliation turn outs to be reduced. Since one dimensional manifolds only admits point foliations, the zero schemes of singularities are the points with multiplicities. Thus those kinds of singularities become non-singular by taking reduction. Therefore in our pull-back situation, the normal sheaf $\mathcal{N}_{\mathcal{F}}$ is always locally-free and only $\mathcal{G}$ itself keeps the informations of singularities of F.

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