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The second Sobolev best constant along the Ricci flow

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Abstract. In this work we present some properties satisfied by the second L^2-Riemannian Sobolev best constant along the Ricci flow on compact manifolds of dimensions $n \geq 4$. We prove that, along the Ricci flow $g(t)$, the second best constant $B_0(2, g(t))$ depends continuously on t and blows-up in finite time. In certain cases, the speed of the explosion is, at least, the same one of the curvature operator. We also show that, on manifolds with positive curvature operator or pointwise $1/4$-pinched curvature, one of the situations holds: $B_0(2, g(t))$ converges to an explicit constant or extremal functions there exists for t large.

Keywords: Ricci flow, blow-up, extremal functions, best constants.

Mathematical subject classification: 41A44, 53C21.

1 Introduction and main results

Best constants and sharp first-order Sobolev inequalities on compact Riemannian manifolds have been extensively studied in the last few decades and surprising results have been obtained by showing the influence of the geometry on such problems. Particularly, the arising of concentration phenomena has motivated the development of new methods in analysis, we mention [3], [13], [22] and [29] for an overview about this matter. Important advances in geometric analysis also have been obtained through the developing of an elegant theory started by Hamilton [15] in 1982 and known as the Ricci flow theory. Several mathematicians have given important contributions for the construction of this theory, see for example the works of Hamilton [15], [16], [17], [18], [19], [20], [21], of Perelman [25], [26], [27] and, more recently, of Böhm and Wilking [4] and of Brendle and Schoen [5]. The Ricci flow theory provides a powerful tool in the study of important topological and geometric questions, see [6], [8]

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and [24] for an overview of this subject and some important implications. Our main interest in this work is the study of the behavior of the second Riemannian Sobolev best constant along the Ricci flow and the discussion of some questions such as bound and asymptotic behavior of the second best constant along the flow and the existence of associated extremal functions. Our results in particular show that the Ricci flow theory can also be worked in connection with some questions of interest in best constants theory.

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Denote by $H^{1,2}(M)$ the standard first-order Sobolev space defined as the completion of $C^\infty(M)$ with respect to the norm

$$||u||_{H^{1,2}(M)} = \left(\int_M |\nabla_g u|^2 \, dv_g + \int_M u^2 \, dv_g \right)^{\frac{1}{2}} .$$

The Sobolev embedding theorem ensures that the inclusion $H^{1,2}(M) \subset L^{2^*}(M)$ is continuous for $2^* = \frac{2n}{n-2}$. Thus, there exist constants $A, B \in \mathbb{R}$ such that, for any $u \in H^{1,2}(M)$,

$$\left(\int_M |u|^{2^*} \, dv_g \right)^{\frac{2}{2^*}} \leq A \int_M |\nabla_g u|^2 \, dv_g + B \int_M u^2 \, dv_g . \tag{AB}$$

In this case, we say simply that (AB) is valid.

The first Sobolev best constant associated to (AB) is

$$A_0(2, g) = \inf \left\{ A \in \mathbb{R} : \text{ there exists } B \in \mathbb{R} \text{ such that } (AB) \text{ is valid} \right\}$$

and, by Aubin [1], its value is given by $K(n, 2)^2$, where

$$K(n, 2) = \sup_{u \in \mathcal{D}^{1,2}(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} |u|^{2^*} \, dx \right)^{\frac{1}{2^*}}}{\left(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}},$$

where $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$||u||_{\mathcal{D}^{1,2}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} .$$

In particular, the first best constant $A_0(2, g)$ does not depend on the metric g.

The first optimal Riemannain Sobolev inequality states that, for any $u \in H^{1,2}(M)$,

$$\left(\int_M |u|^{2^*} \, dv_g \right)^{\frac{2}{2^*}} \leq K(n, 2)^2 \int_M |\nabla_g u|^2 \, dv_g + B \int_M u^2 \, dv_g \tag{$I_{g,opt}$}$$

for some constant $B \in \mathbb{R}$. The validity of $(I_{g,opt})$ has been proved by Hebey and Vaugon in [23].

Define the second Sobolev best constant by

$$B_0(2,g) = \inf\left\{B \in \mathbb{R} :\ (I_{g,opt}) \text{ is valid}\right\} .$$

On the contrary of the first best constant, the second one depends on the metric. Indeed, if $\tilde{g} = \lambda g$, where $\lambda > 0$ is a constant, then $B_0(2,\tilde{g}) = \lambda^{-1} B_0(2,g)$. Note also that $B_0(2,g) \geq vol_g(M)^{-2/n}$, where $vol_g(M)$ denote the volume of M in the metric g.

Clearly, for any $u \in H^{1,2}(M)$, one has the inequality

$$\left(\int_M |u|^{2^*}\, dv_g\right)^{\frac{2}{2^*}} \leq K(n,2)^2 \int_M |\nabla_g u|^2\, dv_g + B_0(2,g) \int_M u^2\, dv_g\,. \qquad (II_{g,opt})$$

This inequality is known as the second optimal Riemannian Sobolev inequality. A function $u_0 \in H^{1,2}(M)$ is said to be an extremal of $(II_{g,opt})$ if

$$\left(\int_M |u_0|^{2^*}\, dv_g\right)^{\frac{2}{2^*}} = K(n,2)^2 \int_M |\nabla_g u_0|^2\, dv_g + B_0(2,g) \int_M u_0^2\, dv_g\,.$$

In [2], Aubin obtained the following lower bound for $B_0(2,g)$ in dimension $n \geq 4$,

$$B_0(2,g) \geq \frac{n-2}{4(n-1)} K(n,2)^2 \max_M R_g,$$

where R_g stands for the scalar curvature of g. In [11], Djadli and Druet studied the existence of extremal functions for $(II_{g,opt})$ and the explicit value of $B_0(2,g)$ for $n \geq 4$. Precisely, they show that, at least, one of the following assertions holds:

(a) $B_0(2,g) = \frac{n-2}{4(n-1)} K(n,2)^2 \max_M R_g$, or

(b) extremal functions of $(II_{g,opt})$ exists.

As already mentioned, this work focuses the behavior of $B_0(2,g)$ along the Ricci flow $g = g(t)$ and its implications.

Given a compact Riemannian manifold (M, g_0), the Ricci flow starting at g_0 is the curve $g(t)$ in the metric-space such that $g(0) = g_0$ and satisfies the evolution equation

$$\frac{\partial g}{\partial t} = -2Ric(g), \qquad (1)$$

where $Ric(g)$ represents the Ricci curvature tensor of the metric g. As it is well known, this problem admits a unique solution $g(t)$ defined on a maximal interval $[0, T)$, see Hamilton [15] and De Turck [10]. The maximal interval may be finite or infinite depending on the metric g_0. For instance, if the scalar curvature R_{g_0} is positive on M, then T is finite. This follows from the maximum principle for parabolic equations applied to the following inequality satisfied by $R_{g(t)}$,

$$\frac{\partial R_{g(t)}}{\partial t} \geq \Delta R_{g(t)} + \frac{2}{n} R_{g(t)}^2 \, .$$

When T is finite, $g(t)$ develops singularity, i.e.

$$\max_M |Rm(g(t))| \to \infty$$

as $t \uparrow T$. Here, $Rm(g)$ denotes the Riemann tensor of the metric g, also called Riemann curvature operator.

Another flow strictly related to the Ricci flow is generated by the evolution equation

$$\frac{\partial g}{\partial t} = -2Ric(g) + \frac{2}{n}\mu g \quad \text{where} \quad \mu = \frac{\int_M R_g dv_g}{\int_M dv_g},$$

which is called the normalized Ricci flow since it preserves the volume of M in the initial metric g_0. Both flows were introduced by Hamilton in [15] and there it was proved that they differ by a change of scale in the time and a parametrization in the space. In particular, it is possible to conclude that if the maximal interval of the normalized Ricci flow starting at g_0 is finite, then the maximal interval of the Ricci flow with the same initial metric is also finite. This implies that the normalized Ricci flow also develops a singularity in finite time.

A central question in the Ricci flow theory is to know if the normalized Ricci flow there exists for all time and if converges to a metric of constant sectional curvature.

Let (M, g_0) be a Riemannian manifold of dimension $n \geq 4$. Hamilton proved in [16] that if $n = 4$ and the curvature operator is positive, then the normalized Ricci flows starting at g_0 there exists for all time and converges to a metric g of constant sectional curvature. In [7], Chen proved that the same conclusion holds when $n = 4$ and the curvature operator is 2-positive, i.e. the sum of its two smallest eigenvalues is positive. Hamilton also conjectured in [16] that its conclusion would be valid in any dimension $n \geq 4$. Recently, Böhm and Wilking [4] proved the Hamilton's conjecture requiring only 2-positivity of the curvature operator.

Another interesting situation is when the Riemannian manifold (M, g_0) has pointwise $1/4$-pinched curvature, i.e. if all sectional curvatures K are positive and, for each point $p \in M$, the ratio between the maximum and minimum sectional curvatures at p is less than 4. In other words, for any pair of planes Π_1 and Π_2 contained in the tangent space T_pM, one has $K(\Pi_1) < 4K(\Pi_2)$. In [7], Chen also showed that the normalized Ricci flow there exists for all time when $n = 4$ and the pointwise $1/4$-pinched curvature condition holds. In this year, Brendle and Schoen [5] extended this result for $n \geq 4$. Summarizing, if (M, g_0) is a Riemannian manifold of dimension $n \geq 4$ with 2-positive curvature operator or pointwise $1/4$-pinched curvature, then the normalized Ricci flow $g(t)$ is defined on all time and converges to a metric of constant sectional curvature.

Assume that the metric g_0 is Einstein, i.e.

$$Ric(g_0) = \lambda g_0$$

for some constant $\lambda \in \mathbb{R}$. In this case, the normalized Ricci flow starting at g_0 is constant on t, i.e. $g(t) = g_0$. However, the Ricci flow $g(t)$ of (1) is given by $g(t) = (1 - 2\lambda t)g_0$, so that

$$B_0(2, g(t)) = (1 - 2\lambda t)^{-1} B_0(2, g_0)$$

on the maximal interval. Moreover, if $(II_{g_0,opt})$ admits an extremal function u_0, then $(II_{g(t),opt})$ also admits an extremal function given by

$$u(x, t) = (1 - 2\lambda t)^{-\frac{n-2}{4}} u_0(x) \, .$$

In the Einstein case, note that T is finite or infinite depending on the sign of λ. For example, if (S^n, g_0) is the standard unit sphere in $\mathbb{R}^{n+1}$ of dimension $n \geq 4$, then $g(t) = (1 - 2(n-1)t)g_0$,

$$B_0(2, g(t)) = (1 - 2(n-1)t)^{-1} \omega_n^{-\frac{2}{n}}$$

and, by [2] and [22],

$$u_{x_0,\beta}(x, t) = (1 - 2(n-1)t)^{-\frac{n-2}{4}} \left(\beta - \cos r_{g_0}\right)^{1-\frac{n}{2}} ,$$

for $x_0 \in S^n$ and $\beta > 1$, are all extremal functions of $(II_{g(t),opt})$, where ω_n stands for the volume of S^n and r_{g_0} denotes the geodesic distance from x to x_0, both in relation to the metric g_0. Therefore, extremal functions there exist along the Ricci flow on S^n starting at the standard metric. More generally, let (M, g_0) be a homogeneous Riemannian manifold of dimension $n \geq 4$ and consider the Ricci flow $g(t)$ starting at g_0 on the maximal interval $[0, T)$. The scalar curvature $R_{g(t)}$ along this flow is constant on M at each time, so that $(II_{g(t),opt})$ admits extremal function for all $t \in [0, T)$.

In the Einstein case, remark that the second best constant $B_0(2, g(t))$ is always continuous on t. This lead us to ask if $B_0(2, g(t))$ remains continuous for any initial metric g_0 on M.

Our first result answers this question.

Theorem 1.1 (Continuous evolution). *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$ and $g(t)$ the Ricci flow (normalized or not) starting at g_0 and defined on the maximal interval $[0, T)$. Then, both assertions hold:*

(a) *$B_0(2, g(t))$ is continuous on $[0, T)$,*

(b) *if $g(t)$ converges to a metric g, then $B_0(2, g(t))$ converges to $B_0(2, g)$ as $t \uparrow T$.*

The continuity of the second best constant along the Ricci flow connected with the best constants and Ricci flow theories produce some interesting results which we state as follows.

Corollary 1.1 (Blow-up in finite time). *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$ and $g(t)$ the Ricci flow (normalized or not) starting at g_0 and defined on the maximal interval $[0, T)$. If T is finite, then $B_0(2, g(t))$ blows up as $t \uparrow T$. Moreover, if the curvature operator of g_0 is positive, then there exists a positive constant $a(n)$, depending only on n, such that*

$$\frac{B_0(2, g(t))}{\max_M |Rm(g(t))|} \geq a(n)$$

for all $t \in [0, T)$.

Corollary 1.2 (Bound in infinite time). *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$ with 2-positive curvature operator or pointwise 1/4-pinched curvature and let $g(t)$ be the normalized Ricci flow starting at g_0 and defined on all time. Then, $B_0(2, g(t))$ is uniformly bounded on t.*

Corollary 1.3 (Asymptotic behavior or extremal existence). *Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 4$ with 2-positive curvature operator or pointwise 1/4-pinched curvature and let $g(t)$ be the normalized Ricci* *flow starting at g_0 and defined on all time. Then, at least, one of the assertions holds:*

(a) *there exists $R > 0$ such that $B_0(2, g(t))$ converges to $R^{-1}\left(\frac{1}{\omega_n}\right)^{2/n}$ as $t \to \infty$, or*

(b) *there exists $t_0 \geq 0$ such that $(II_{g(t),opt})$ admits extremal function for all $t \geq t_0$.*

The proof of Theorem 1.1 is made by contradiction. In this case, we find two possible alternatives. One of them is directly eliminated according to the definition of second best constant. The other alternative implies the existence of minimizers of certain functionals which concentrate in some point. The proof then consists in obtaining estimates of these minimizers around a concentration point and in combining them in order to find a contradiction. These ideas are inspired in the work of Djadli and Druet [11]. The proofs of the remaining results are based on Theorem 1.1 and on best constants and Ricci flow theories.

2 Proof of Theorem 1.1

Let $g(t)$ be the Ricci flow on M starting at g_0 defined on a maximal interval $[0, T)$. We prove here only the part (a), since that the ideas involved in proof of the part (b) are similar. Suppose, by contradiction, that $g(t)$ is discontinuous in some time $t_0 \in [0, T)$. Then, there exist $\varepsilon_0 > 0$ and a sequence $(t_k) \subset [0, T)$ such that $t_k \to t_0$ and

$$|B_0(2, g(t_k)) - B_0(2, g(t_0))| > \varepsilon_0$$

for all k. Then, at least, one of the cases holds:

$$B_0(2, g(t_0)) - B_0(2, g(t_k)) > \varepsilon_0$$

or

$$B_0(2, g(t_k)) - B_0(2, g(t_0)) > \varepsilon_0$$

for infinitely many k. If the first one holds, for any $u \in H^{1,2}(M)$, one has

$$\left(\int_M |u|^{2^*} \, dv_{g(t_k)}\right)^{\frac{2}{2^*}} \leq K(n,2)^2 \int_M |\nabla_{g(t_k)} u|^2 \, dv_{g(t_k)}$$
$$+ (B_0(2, g(t_0)) - \epsilon_0) \int_M u^2 \, dv_{g(t_k)} \, .$$

Taking the limit in this inequality as $k \to \infty$, one finds

$$\left(\int_M |u|^{2^*} dv_{g(t_0)}\right)^{\frac{2}{2^*}} \leq K(n,2)^2 \int_M |\nabla_{g(t_0)} u|^2 \, dv_{g(t_0)}$$
$$+(B_0(2, g(t_0)) - \varepsilon_0) \int_M u^2 \, dv_{g(t_0)},$$

which contradicts the definition of $B_0(2, g(t_0))$.

Suppose then that the second case holds, i.e. $B_0(2, g(t_0)) + \varepsilon_0 < B_0(2, g(t_k))$ for infinitely many k. For each k, consider the functional

$$J_k(u) = \int_M |\nabla_{g(t_k)} u|^2 \, dv_{g(t_k)} + (B_0(2, g(t_0)) + \varepsilon_0) K(n,2)^{-2} \int_M u^2 \, dv_{g(t_k)}$$

defined on $\Lambda_k = \left\{u \in H^{1,2}(M) : \int_M |u|^{2^*} dv_{g(t_k)} = 1\right\}$. From the definition of $B_0(2, g(t_k))$, it follows directly that

$$\lambda_k := \inf_{\Lambda_k} J_k(u) < K(n,2)^{-2} \, .$$

But this implies the existence of a nonnegative minimizer $u_k \in \Lambda_k$ for λ_k. The Euler-Lagrange equation for u_k is then

$$-\Delta_{g(t_k)} u_k + (B_0(2, g(t_0)) + \varepsilon_0) K(n,2)^{-2} u_k = \lambda_k u_k^{2^*-1}, \qquad (E_k)$$

where $\Delta_{g(t_k)} = \mathrm{div}_{g(t_k)}(\nabla_{g(t_k)})$ is the Laplacian operator with respect to the metric $g(t_k)$. By the standard elliptic theory, u_k belongs to $C^\infty(M)$ and, by the strong maximum principle, $u_k > 0$ on M. Moreover,

$$\int_M u_k^{2^*} \, dv_{g(t_k)} = 1 \, .$$

Our goal now is to study the sequence $(u_k)_k$ as $k \to \infty$. First, note that

$$\int_M |\nabla_{g(t_k)} u_k|^2 \, dv_{g(t_k)} + (B_0(2, g(t_0)) + \varepsilon_0) K(n,2)^{-2} \int_M u_k^2 \, dv_{g(t_k)}$$
$$= \lambda_k < K(n,2)^{-2}$$

and there exists a constant $c > 0$, independent of k, such that

$$\int_M u_k^2 \, dv_{g(t_0)} \leq c \int_M u_k^2 \, dv_{g(t_k)}$$

and

$$\int_M |\nabla_{g(t_0)} u_k|^2 \, dv_{g(t_0)} \le c \int_M |\nabla_{g(t_k)} u_k|^2 \, dv_{g(t_k)}$$

for all k. This implies that $(u_k)_k$ is bounded in $H^{1,2}(M)$ with respect to the metric $g(t_0)$. So, there exists $u \in H^{1,2}(M)$, $u \ge 0$, such that $u_k \rightharpoonup u$ weakly in $H^{1,2}(M)$ and $\lambda_k \to \lambda$ as $k \to \infty$, up to a subsequence. Moreover, by the Sobolev embedding compactness theorem, one easily finds

$$\int_M u_k^q \, dv_{g(t_k)} \to \int_M u^q \, dv_{g(t_0)} \tag{2}$$

for any $1 \le q < 2^*$. So, letting $k \to \infty$ in the equation (E_k), one concludes that u satisfies

$$-\Delta_{g(t_0)} u + (B_0(2, g(t_0)) + \varepsilon_0) K(n, 2)^{-2} u = \lambda u^{2^*-1} \,. \tag{E}$$

Assume that $u \ne 0$. In this case, by $(II_{g(t_0),opt})$ and (E), one has

$$\begin{aligned}\left(\int_M u^{2^*} \, dv_{g(t_0)} \right)^{\frac{2}{2^*}} &< K(n,2)^2 \int_M |\nabla_{g(t_0)} u|^2 \, dv_{g(t_0)} \\ &+ (B_0(2, g(t_0)) + \varepsilon_0) \int_M u^2 \, dv_{g(t_0)} \\ &= K(n,2)^2 \lambda \int_M u^{2^*} \, dv_{g(t_0)} \le \int_M u^{2^*} \, dv_{g(t_0)}\end{aligned}$$

since $0 \le \lambda \le K(n,2)^{-2}$. This implies that $\int_M |u|^{2^*} \, dv_{g(t_0)} > 1$. But this inequality contradicts

$$\int_M u^{2^*} dv_{g(t_0)} \le \liminf_{k \to \infty} \int_M u_k^{2^*} dv_{g(t_k)} = 1 \,.$$

We then assume that $u = 0$ on M and prove that this assumption leads us to an contradiction. We claim that, in this case, $\lambda_k \to K(n,2)^{-2}$ as $k \to \infty$. In fact, the optimal inequality furnishes

$$\begin{aligned}\left(\int_M u_k^{2^*} \, dv_{g(t_0)} \right)^{\frac{2}{2^*}} &\le K(n,2)^2 \int_M |\nabla_{g(t_0)} u_k|^2 \, dv_{g(t_0)} \\ &+ B_0(2, g(t_0)) \int_M u_k^2 \, dv_{g(t_0)} \,.\end{aligned}$$

Note that

$$\int_M u_k^{2^*}\, dv_{g(t_0)} \to 1$$

since $u_k \in \Lambda_k$, and

$$\lim_{k\to\infty} \int_M u_k^2\, dv_{g(t_k)} = 0$$

by (2). So, letting $k \to \infty$ in the Sobolev inequality above, one finds

$$\liminf_{k\to\infty} \int_M |\nabla_{g(t_0)} u_k|^2\, dv_{g(t_0)} \geq K(n,2)^{-2},$$

so that

$$\liminf_{k\to\infty} \int_M |\nabla_{g(t_k)} u_k|^2\, dv_{g(t_k)} \geq K(n,2)^{-2}\ .$$

Therefore, combining this last inequality with

$$\int_M |\nabla_{g(t_k)} u_k|^2\, dv_{g(t_k)} \leq \lambda_k,$$

it follows directly that $\lambda = K(n,2)^{-2}$.

In the sequel, we divide the proof into four steps. Several possibly different positive constants independent of k are denoted by c.

Let $x_k \in M$ be a maximum point of u_k, i.e. $u_k(x_k) = ||u_k||_\infty$.

Step 1. For each $R > 0$, we have

$$\lim_{k\to\infty} \int_{B_{g(t_k)}(x_k, R\mu_k)} u_k^{2^*}\, dv_{g(t_k)} = 1 - \varepsilon_R \tag{3}$$

where $\mu_k = ||u_k||_\infty^{-\frac{2^*}{n}}$ and $\varepsilon = \varepsilon(R) \to 0$ as $R \to \infty$.

Proof. First, note that

$$1 = \int_M u_k^{2^*}\, dv_{g(t_k)} \leq ||u_k||_\infty^{2^*-2} \int_M u_k^2\, dv_{g(t_k)}$$

implies that $||u_k||_\infty \to \infty$ as $k \to \infty$, since

$$\int_M u_k^2\, dv_{g(t_k)} \to 0.$$

In particular, $\mu_k \to 0$ as $k \to \infty$. Consider the exponential map $\exp_{(x_k, g(t_k))}$ at x_k with respect to the metric $g(t_k)$. Clearly, there exists $\delta > 0$, independent of k, such that $\exp_{(x_k, g(t_k))}$ map $B(0, \delta) \subset \mathbb{R}^n$ onto $B_{g(t_k)}(x_k, \delta)$. For each $x \in B(0, \delta\mu_k^{-1})$, we set

$$\tilde{g}(t_k)(x) = \left(\exp^*_{(x_k, g(t_k))} g(t_k)\right)(\mu_k x)$$

and

$$\varphi_k(x) = \mu_k^{n/2^*} u_k\left(\left(\exp_{(x_k, g(t_k))}\right)(\mu_k x)\right) .$$

Clearly, $\tilde{g}(t_k)$ converges to ξ as $k \to \infty$, where ξ denotes the Euclidean metric on $\mathbb{R}^n$. Moreover, as one easily checks,

$$-\Delta_{\tilde{g}(t_k)}\varphi_k + (B_0(2, g(t_0)) + \varepsilon_0)K(n, 2)^{-2}\mu_k^2\varphi_k = \lambda_k\varphi_k^{2^*-1} . \qquad (\tilde{E}_k)$$

Since $0 \le \varphi_k \le 1$ and the coefficients of $(\tilde{E}_k)$ are bounded, from the standard elliptic theory, it follows that $\varphi_k \to \varphi$ in $C^2_{loc}(\mathbb{R}^n)$, up to a subsequence. Clearly, $\varphi \not\equiv 0$ since $\varphi_k(0) = 1$ for all k. In addition, φ satisfies

$$-\Delta\varphi = K(n, 2)^{-2}\varphi^{2^*-1},$$

since $\lambda_k \to K(n, 2)^{-2}$, $\mu_k \to 0$ and $\tilde{g}(t_k) \to \xi$ as $k \to \infty$. So,

$$\int_{\mathbb{R}^n} |\nabla\varphi|^2\, dx = K(n, 2)^{-2} \int_{\mathbb{R}^n} \varphi^{2^*}\, dx .$$

The Euclidean Sobolev inequality furnishes

$$K(n, 2)^{-2}\left(\int_{\mathbb{R}^n} \varphi^{2^*}\, dx\right)^{2/2^*} \le \int_{\mathbb{R}^n} |\nabla\varphi|^2\, dx = K(n, 2)^{-2} \int_{\mathbb{R}^n} \varphi^{2^*}\, dx,$$

so that

$$\int_{\mathbb{R}^n} \varphi^{2^*}\, dx \ge 1 .$$

Combining this fact with the inequality

$$\int_{B(0, \delta\mu_k^{-1})} \varphi_k^{2^*}\, dv_{\tilde{g}(t_k)} = \int_{B_{g(t_k)}(x_k, \delta)} u_k^{2^*}\, dv_{g(t_k)} \le 1,$$

it follows that $\int_{\mathbb{R}^n} \varphi^{2^*}\, dx = 1$. Thus, from the convergence

$$\int_{B_{g(t_k)}(x_k, R\mu_k)} u_k^{2^*}\, dv_{g(t_k)} = \int_{B(0,R)} \varphi_k^{2^*}\, dv_{\tilde{g}(t_k)} \to \int_{B(0,R)} \varphi^{2^*}\, dx,$$

we end the proof of the step 1. $\square$

Step 2. There exists a constant $c > 0$, independent of k, such that

$$d_{g(t_k)}(x, x_k)^{n/2^*} u_k(x) \le c,$$

where $d_{g(t_k)}$ stands for the distance with respect to the metric $g(t_k)$.

Proof. Set $\omega_k(x) = d_{g(t_k)}(x, x_k)^{n/2^*} u_k(x)$ and suppose, by contradiction, that the conclusion of this step is false. In this case, one has

$$\lim_{k\to\infty} ||\omega_k||_\infty = \infty$$

for some subsequence. We prove that this leads to a contradiction. Let $y_k \in M$ be a maximum point of ω_k. From the inequality

$$\frac{d_{g(t_k)}(y_k, x_k)}{\mu_k} = \frac{\omega_k(y_k)^{2^*/n}}{\mu_k u_k(y_k)^{2^*/n}} \ge \omega_k(y_k)^{2^*/n},$$

one has

$$\lim_{k\to\infty} \frac{d_{g(t_k)}(y_k, x_k)}{\mu_k} = \infty\ . \tag{4}$$

Fix $\delta > 0$ small enough. Set

$$\Omega_k = u_k(y_k)^{2^*/n} \exp^{-1}_{(y_k, g(t_k))}(B_{g(t_k)}(x_k, \delta))\ .$$

For each $x \in \Omega_k$, define

$$\psi_k(x) = u_k(y_k)^{-1} u_k\big(\exp_{(y_k, g(t_k))}\big(u_k(y_k)^{-2^*/n} x\big)\big)$$

and

$$\hat{g}(t_k)(x) = \big(\exp^*_{(y_k, g(t_k))} g(t_k)\big)(u_k(y_k)^{-2^*/n} x)\ .$$

Then, ψ_k satisfies

$$-\Delta_{\hat{g}(t_k)} \psi_k + B_k \psi_k = \lambda_k \psi_k^{2^*-1} \ \text{ in } \ \Omega_k$$

for a certain constant $B_k > 0$, so that

$$-\Delta_{\hat{g}(t_k)} \psi_k \le \lambda_k \psi_k^{2^*-1} \ \text{ in } \ \Omega_k\ . \tag{5}$$

On the other hand, for $x \in B(0, 2)$, one finds

$$\begin{aligned} d_{g(t_k)}\Big(x_k, \exp_{(y_k, g(t_k))}\big(u_k(y_k)^{-2^*/n} x\big)\Big) &\ge d_{g(t_k)}(x_k, y_k) - 2u_k(y_k)^{-2^*/n} \\ &\ge \Big(1 - 2\omega_k(y_k)^{-2^*/n}\Big)\, d_{g(t_k)}(x_k, y_k)\ . \end{aligned}$$

Since $\omega_k(y_k) \to \infty$ as $k \to \infty$, it follows that

$$d_{g(t_k)}\left(x_k, \exp_{(y_k,g(t_k))}\left(u_k(y_k)^{-2^*/n}x\right)\right) \geq \frac{1}{2}d_{g(t_k)}(x_k, y_k) \tag{6}$$

for k large. Hence,

$$\psi_k(x) \leq 2^{n/2^*}d_{g(t_k)}(x_k, y_k)^{-n/2^*}u_k(y_k)^{-1}\omega_k(y_k) = 2^{n/2^*},$$

so that, for k large,

$$||\psi_k||_{L^\infty(B(0,2))} \leq 2^{n/2^*} . \tag{7}$$

In addition, by (4) and (6), for any $R > 0$ and k large, one has

$$B_{g(t_k)}\left(y_k, 2u_k(y_k)^{-2^*/n}\right) \cap B_{g(t_k)}(x_k, R\mu_k) = \emptyset . \tag{8}$$

In fact, this inequality is implied by

$$\begin{aligned} w_k(y_k)^{2^*/n} &= d_{g(t_k)}(x_k, y_k)u_k(y_k)^{2^*/n} \geq 2 + Ru_k(y_k)^{2^*/n}\mu_k \\ &= 2 + R\, u_k(y_k)^{2^*/n}\, ||u_k||_\infty^{-2^*/n}, \end{aligned}$$

which clearly holds for k large. Note that the step 1 and (8) imply that

$$\int_{B_{g(t_k)}(y_k,u_k(y_k)^{-2^*/n})} u_k^{2^*}\, dv_{g(t_k)} \to 0$$

as $k \to \infty$. On the other hand, applying De Giorgi-Nash-Moser iterative scheme in (5) and using (7), one obtains

$$\psi_k(0) \leq \sup_{B(0,1)} \psi_k(x) \leq c\int_{B(0,2)} \psi_k^{2^*}\, dv_{\hat{g}(t_k)} = c\int_{B_{g(t_k)}(y_k,2u_k(y_k)^{-2^*/n})} u_k^{2^*}\, dv_{g(t_k)}$$

for some constant $c > 0$ independent of k and this contradicts the fact of $\psi_k(0) = 1$ for all k. This concludes the proof of the step 2. □

Let $x_0 \in M$ be such that $d_{g(t_0)}(x_k, x_0) \to 0$ as $k \to \infty$, up to a subsequence. In particular, $d_{g(t_k)}(x_k, x_0) \to 0$ as $k \to \infty$.

Step 3. For any $\delta > 0$ small enough,

$$\lim_{k\to\infty} \frac{\int_{M\setminus B_{g(t_k)}(x_0,\delta)} u_k^2\, dv_{g(t_k)}}{\int_M u_k^2\, dv_{g(t_k)}} = 0 . \tag{9}$$

Proof. The De Giorgi-Nash-Moser iterative scheme applied to (E_k) furnishes

$$\int_{M\setminus B_{g(t_k)}(x_0,\delta)} u_k^2\, dv_{g(t_k)} \le c\left(\int_M u_k\, dv_{g(t_k)}\right)\left(\int_M u_k^2\, dv_{g(t_k)}\right)^{1/2}, \tag{10}$$

where $c > 0$ is a constant independent of k. Let ξ_k be the solution of the problem

$$-\Delta_{g(t_k)}\xi_k + (B_0(g(t_0)) + \varepsilon_0)\xi_k = 1\ .$$

By the standard elliptic theory, there exists a constant $c > 0$, independent of k, such that $0 \le \xi_k \le c$ on M. Then,

$$\begin{aligned}\int_M u_k\, dv_{g(t_k)} &= \int_M \left(-\Delta_{g(t_k)}\xi_k + (B_0(g(t_0)) + \varepsilon_0)\xi_k\right) u_k\, dv_{g(t_k)}\\ &= \int_M \left(-\Delta_{g(t_k)}u_k + (B_0(g(t_0)) + \varepsilon_0)u_k\right) \xi_k\, dv_{g(t_k)}\\ &\le c\int_M u_k^{2^*-1}\, dv_{g(t_k)}\ .\end{aligned}$$

As one easily checks, this estimate combined with (10) and an interpolation inequality give (9) for $n \ge 5$. For $n = 4$, we use the step 1 as follows. First, write

$$\begin{aligned}\frac{\int_M u_k^3\, dv_{g(t_k)}}{\left(\int_M u_k^2 dv_{g(t_k)}\right)^{1/2}} &\le ||u_k||_{L^\infty(M\setminus B_{g(t_k)}(x_k,\delta))}\left(\int_M u_k^2\, dv_{g(t_k)}\right)^{1/2}\\ &\quad + \frac{\int_{B(0,\delta\mu_k^{-1})} \varphi_k^3\, dv_{\tilde g(t_k)}}{\left(\int_{B(0,\delta\mu_k^{-1})} \varphi_k^2\, dv_{\tilde g(t_k)}\right)^{1/2}}\ .\end{aligned}$$

For $R > 0$ fixed, Holder inequality and the step 1 lead us to

$$\int_{B(0,\delta\mu_k^{-1})} \varphi_k^3\, dv_{\tilde g(t_k)} \le \int_{B(0,R)} \varphi_k^3\, dv_{\tilde g(t_k)} + \varepsilon_R\left(\int_{B(0,\delta\mu_k^{-1})} \varphi_k^2\, dv_{\tilde g(t_k)}\right)^{1/2},$$

where $\varepsilon_R \to 0$ as $R \to \infty$. By the step 2, this implies

$$\lim_{k\to\infty} \frac{\int_M u_k^3\, dv_{g(t_k)}}{\left(\int_M u_k^2\, dv_{g(t_k)}\right)^{1/2}} \le \varepsilon_R + \frac{\int_{\mathbb{R}^n} \varphi^3\, dx}{\left(\int_{B(0,R)} \varphi^2\, dx\right)^{1/2}}, \tag{11}$$

Noting that

$$\lim_{R\to\infty} \int_{B(0,R)} \varphi^2\, dx = \infty$$

for $n = 4$ and letting $R \to \infty$ in (11), we end the proof of (9). □

Step 4. This is the final step. Combining the local isoperimetric inequality of [12] and the co-area formula, as done recently in [9], for any $\varepsilon > 0$, we easily find $\delta_\varepsilon > 0$, independent of k, such that

$$\left(\int_M |u|^{2^*}\, dv_{g(t_k)}\right)^{\frac{2}{2^*}} \le K(n,2)^2 \int_M |\nabla_{g(t_k)} u|^2\, dv_{g(t_k)} + B_\varepsilon(g(t_k)) \int_M u^2\, dv_{g(t_k)} \tag{12}$$

for all $u \in C_0^\infty(B_{g(t_k)}(x_0, \delta_\varepsilon))$, where

$$B_\varepsilon(g(t_k)) = \frac{n-2}{4(n-1)} K(n,2)^2 \left(Scal_{g(t_k)}(x_0) + \varepsilon\right) .$$

Fix $0 < \varepsilon < \varepsilon_0$ and consider a smooth cutoff function η_k such that $0 \le \eta_k \le 1$, $\eta_k = 1$ in $B_{g(t_k)}(x_0, \delta_\varepsilon/4)$ and $\eta_k = 0$ in $M \setminus B_{g(t_k)}(x_0, \delta_\varepsilon/2)$. Taking $u = \eta_k u_k$ in (12), using the identity

$$\int_M |\nabla_{g(t_k)}(\eta_k u_k)|^2\, dv_{g(t_k)} = -\int_M \eta_k^2 u_k \Delta_{g(t_k)} u_k\, dv_{g(t_k)} + \int_M |\nabla_{g(t_k)} \eta_k|^2 u_k^2\, dv_{g(t_k)},$$

the equation (E_k) and the step 3, one obtains

$$\left(\int_M (\eta_k u_k)^{2^*}\, dv_{g(t_k)}\right)^{2/2^*} + (B_0(g(t_0)) - B_\varepsilon(g(t_k)) + \varepsilon_0) \int_M \eta_k^2 u_k^2\, dv_{g(t_k)} \le \int_M \eta_k^2 u_k^{2^*}\, dv_{g(t_k)} + c \int_M |\nabla_{g(t_k)} \eta_k|^2 u_k^2\, dv_{g(t_k)} .$$

By Hölder inequality,

$$\int_M \eta_k^2 u_k^{2^*}\, dv_{g(t_k)} \le \left(\int_M (\eta_k u_k)^{2^*}\, dv_{g(t_k)}\right)^{2/2^*} \left(\int_M u_k^{2^*}\, dv_{g(t_k)}\right)^{1-2/2^*} = \left(\int_M (\eta_k u_k)^{2^*}\, dv_{g(t_k)}\right)^{2/2^*},$$

so that

$$(B_0(g(t_0)) - B_\varepsilon(g(t_k)) + \varepsilon_0) \int_M \eta_k^2 u_k^2\, dv_{g(t_k)} \le c \int_M |\nabla_{g(t_k)} \eta_k|^2 u_k^2\, dv_{g(t_k)} .$$

This inequality imply

$$\frac{n-2}{4(n-1)} K(n,2)^2 \left(Scal_{g(t_0)} - Scal_{g(t_k)} \right)(x_0) + \varepsilon_0 - \varepsilon$$

$$\leq c \frac{\int_{M \setminus B_{g(t_k)}(x_0, \delta_\varepsilon/2)} u_k^2 \, dv_{g(t_k)}}{\int_M u_k^2 \, dv_{g(t_k)}} .$$

Letting $k \to \infty$ and using the step 3, one finds the desired contradiction, since $0 < \varepsilon < \varepsilon_0$ and

$$\frac{n-2}{4(n-1)} K(n,2)^2 \left(Scal_{g(t_0)} - Scal_{g(t_k)} \right)(x_0) = o(1) . \qquad \square$$

3 Proof of the Corollaries

Proof of Corollary 1.1. The Ricci flow $g(t)$ develops a singularity in finite time, i.e.

$$\max_M |Rm(g(t))| \to \infty$$

as $t \uparrow T$. Since $R_{g(t)}$ satisfies

$$\frac{\partial R_{g(t)}}{\partial t} \geq \Delta R_{g(t)} + \frac{2}{n} R^2_{g(t)},$$

it follows from the maximum principle applied to parabolic equations that $R_{g(t)} \geq \min_M R_{g_0}$ for all $t \in [0, T)$. So, $\max_M R_{g(t)} \to +\infty$ as $t \uparrow T$. Thus, by [2], it follows that

$$B_0(2, g(t)) \geq \frac{n-2}{4(n-1)} K(n,2)^2 \max_M R_{g(t)} \to +\infty$$

as $t \uparrow T$. Furthermore, when the curvature operator of g_0 is positive, Hamilton [16] proved that it remains positive along the Ricci flow. In this case, there exists a constant $c(n) > 0$, depending only on n, such that $R_{g(t)} \geq c(n)|Rm(g(t))|$. So,

$$\frac{B_0(2, g(t))}{\max_M |Rm(g(t))|} \geq \frac{n-2}{4(n-1)} K(n,2)^2 \max_M \left(\frac{R_{g(t)}}{|Rm(g(t))|} \right) \geq a(n)$$

for all $t \in [0, T)$ and this concludes the proof. $\square$

Proof of Corollary 1.2. The proof follows immediately from the part (b) of Theorem 1.1. $\square$

Proof of Corollary 1.3. By [4] and [5], the normalized Ricci flow $g(t)$ converges to a metric g of constant sectional curvature K. If M is simply connected, then (M, g) is isometric to (S^n_R, h_0), where S^n_R is the n-sphere with radius R, $K = \frac{1}{R^2}$ and h_0 is the induced metric by the standard metric of $\mathbb{R}^{n+1}$. So, $B_0(2, g) = B_0(2, h_0)$. Consider the diffeomorphism $\varphi : x \mapsto \sqrt{R}x$ from the unit n-sphere S^n onto S^n_R. It is easy to see that $\varphi^* h_0 = Rh$, where h is the standard metric on S^n. Hence,

$$\omega_n^{-\frac{2}{n}} = B_0(2, h) = RB_0(2, \varphi^* h_0) = RB_0(2, h_0)$$

and

$$B_0(2, g(t)) \to R^{-1}\omega_n^{-\frac{2}{n}} .$$

Now, suppose that M is not simply connected. In this case, (M, g) is not conformally diffeomorphic to (S^n, h). As proved in [2] and [28],

$$\inf_{u\in H^{1,2}(M)\setminus\{0\}} \left[\frac{\int_M |\nabla u|^2\, dv_g + \frac{n-2}{4(n-1)} R_g \int_M u^2\, dv_g}{\left(\int_M |u|^{2^*}\, dv_g\right)^{\frac{2}{2^*}}} \right] < K(n, 2)^{-2},$$

so that

$$B_0(2, g) > \frac{n-2}{4(n-1)} K(n, 2)^2 R_g .$$

From the part (b) of Theorem 1.1, it follows then that

$$B_0(2, g(t)) > \frac{n-2}{4(n-1)} K(n, 2)^2 R_{g(t)}$$

for all t large enough, so that the conclusion follows from [11]. $\square$

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