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Vanishing viscosity limits and boundary layers for circularly symmetric 2D flows

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Abstract. We continue the work of Lopes Filho, Mazzucato and Nussenzveig Lopes [10] on the vanishing viscosity limit of circularly symmetric viscous flow in a disk with rotating boundary, shown there to converge to the inviscid limit in L^2-norm as long as the prescribed angular velocity $\alpha(t)$ of the boundary has bounded total variation. Here we establish convergence in stronger L^2 and L^p-Sobolev spaces, allow for more singular angular velocities α, and address the issue of analyzing the behavior of the boundary layer. This includes an analysis of concentration of vorticity in the vanishing viscosity limit. We also consider such flows on an annulus, whose two boundary components rotate independently.

Keywords: Navier-Stokes, boundary layer, vorticity.

Mathematical subject classification: Primary: 35Q30; Secondary: 76D05, 76D10, 35K05.

1 Introduction

In this paper we study the 2D Navier-Stokes equation in the disk $D = \{x \in \mathbb{R}^2 : |x| < 1\}$:

$$\partial_t u^\nu + \nabla_{u^\nu} u^\nu + \nabla p^\nu = \nu \Delta u^\nu, \quad \operatorname{div} u^\nu = 0, \tag{1.1}$$

with no-slip boundary data on a rotating boundary:

$$u^\nu(t,x) = \frac{\alpha(t)}{2\pi} x^\perp, \quad |x| = 1,\ t > 0, \tag{1.2}$$

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and with circularly symmetric initial data:

$$u^\nu(0) = u_0(x), \quad \operatorname{div} u_0 = 0, \quad u_0 \parallel \partial D. \tag{1.3}$$

In (1.2), $x^\perp = Jx$, where J is counterclockwise rotation by 90°. By definition, a vector field u_0 on D is circularly symmetric provided

$$u_0(R_\theta x) = R_\theta u_0(x), \quad \forall x \in D, \tag{1.4}$$

for each $\theta \in [0, 2\pi]$, where R_θ is counterclockwise rotation by θ. The general vector field satisfying (1.4) has the form

$$s_0(|x|)x^\perp + s_1(|x|)x, \tag{1.5}$$

with s_j scalar, but the condition $\operatorname{div} u_0 = 0$, together with the condition $u_0 \parallel \partial D$, forces $s_1 \equiv 0$, so the type of initial data we consider is characterized by

$$u_0(x) = s_0(|x|)x^\perp. \tag{1.6}$$

Another characterization of vector fields of the form (1.6) is the following. For each unit vector $\omega \in S^1 \subset \mathbb{R}^2$, let $\Phi_\omega : \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection across the line generated by ω, i.e., $\Phi_\omega(a\omega + bJ\omega) = a\omega - bJ\omega$. Then a vector field u_0 on D has the form (1.6) if and only if

$$u_0(\Phi_\omega x) = -\Phi_\omega u_0(x), \quad \forall \omega \in S^1, \ x \in D. \tag{1.7}$$

As is well known, such a vector field u_0 as given in (1.6) is a steady solution to the 2D Euler equation. In fact, a calculation gives

$$\nabla_{u_0} u_0 = -s_0(|x|)^2 x = -\nabla p_0(x), \tag{1.8}$$

with

$$p_0(x) = \tilde{p}_0(|x|), \quad \tilde{p}_0(r) = -\int_r^1 \rho s_0(\rho)^2 \, d\rho, \tag{1.9}$$

and the assertion follows. We mention that $\|p_0\|_{L^1(D)} \le C\|u_0\|^2_{L^2(D)}$.

The problem we address is the following *vanishing viscosity problem*: to demonstrate that the solution u^ν to (1.1)–(1.3) satisfies

$$\lim_{\nu \searrow 0} u^\nu(t, \cdot) = u_0, \tag{1.10}$$

and in particular to specify in what topologies such convergence holds. As has been observed, what makes this problem tractable is the following result.

Proposition 1.1. *Given that* u_0 *has the form* (1.6)*, the solution* u^ν *to* (1.1)–(1.3) *is circularly symmetric for each* $t > 0$*, of the form*

$$u^\nu(t, x) = s^\nu(t, |x|)x^\perp, \tag{1.11}$$

and it coincides with the solution to the linear PDE

$$\partial_t u^\nu = \nu \Delta u^\nu, \tag{1.12}$$

with boundary condition (1.2) *and initial condition* (1.3).

Here is a brief proof. Let u^ν solve (1.12), (1.2), and (1.3), with u_0 as in (1.6). We claim (1.11) holds. In fact, for each unit vector $\omega \in \mathbb{R}^2$,

$$-\Phi_\omega u^\nu(t, \Phi_\omega x) \tag{1.13}$$

also solves (1.12) with the same initial data and boundary condition as u^ν, so these functions coincide, and (1.11) follows. Hence $\operatorname{div} u^\nu = 0$ for each $t > 0$. Also we have an analogue of (1.8)–(1.9):

$$\begin{aligned} \nabla_{u^\nu} u^\nu &= -\nabla p^\nu, \quad p^\nu(t, x) = \tilde{p}^\nu(t, |x|), \\ \tilde{p}^\nu(t, r) &= -\int_r^1 \rho s^\nu(t, \rho)^2 \, d\rho. \end{aligned} \tag{1.14}$$

Hence this u^ν is the solution to (1.1)–(1.3). For additional discussion of this issue, in particular in the context of weak solutions, see [10], Proposition 5.1.

Previous work on the convergence problem (1.10), in the circularly symmetric context, was done by Matsui [12], who considered the case $\alpha = 0$, without assuming compatibility of the initial velocity with this boundary condition, (see also [7] for another treatment of the convergence problem in the circularly symmetric context), by Wang [17], whose general work on the convergence (building on results of Kato [6]) is applicable to the circularly symmetric case when $\alpha \in H^1_{\mathrm{loc}}(\mathbb{R})$, by three of us in [10], who treated

$$\alpha \in \mathrm{BV}(\mathbb{R}) \tag{1.15}$$

(supported in $(0, \infty)$), and also by Bona and Wu [1], who dealt with the special case

$$\alpha \equiv 0, \quad u_0\big|_{\partial D} = 0. \tag{1.16}$$

Results in these papers yield convergence in (1.10), in $L^2(D)$-norm, locally uniformly in t, when $u_0 \in L^2(D)$ has the form (1.6). (In the special case (1.16), convergence in stronger norms for more regular u_0 was obtained in [1]; cf. §10 of this paper for further discussion of this case.)

In addition to the work discussed above we should mention the results of Lombardo, Caflisch and Sammartino [9], who studied the vanishing viscosity limit for the Stokes problem in the exterior of a disk without assuming circular symmetry. In their paper they prove that the small viscosity solution is given by the solution to the Euler equations far from the boundary, adjusted using the solution to the boundary layer equation near the boundary, and they give explicit estimates for the corrector. In their work they assume the initial conditions are compatible with the boundary data and they also assume some regularity of the boundary condition. More precisely, if we consider the special case of circularly symmetric flows with given rigid rotations of the boundary, Navier-Stokes reduces to their Stokes case, and their result is valid for angular velocities with bounded second derivatives.

In this paper we sharpen the treatment of the vanishing viscosity convergence in several important respects. For one, we go beyond L^2-norm convergence, and establish norm convergence, under appropriate hypotheses, in L^q-Sobolev spaces $H^{s,q}(D)$, when $sq < 1$. This is the maximal class of Sobolev spaces for which such results could hold, since, without special compatibility hypotheses such as (1.16), the Sobolev space trace theorems forbid convergence in higher norms. The techniques we use to get such results also allow us to treat driving motions α much more singular than in (1.15); in fact, we treat

$$\alpha \in L^{p'}(\mathbb{R}), \quad p' \geq 1 \tag{1.17}$$

(supported in $(0, \infty)$).

In addition, we establish much stronger local convergence results, given more regular data u_0. On each compact subset $\overline{\Omega}$ of D, convergence in (1.10) holds in $H^k(\Omega)$ as long as u_0 is of class H^k on a neighborhood of $\overline{\Omega}$. Furthermore, we give a precise analysis of the boundary layer behavior of $u^\nu(t,x)$, as $\nu \searrow 0$, showing the transitional behavior on a layer about ∂D of thickness $\sim \nu^{1/2}$, in case $u_0 \in C^\infty(\overline{D})$, and more generally in case $u_0 \in C(\overline{D})$.

It is a classical open problem whether solutions of the Navier-Stokes equations in a bounded domain with no-slip boundary data converge to solutions of the Euler equations in the vanishing viscosity limit. The results obtained here may be regarded as an exploration of the difficulty involved in this problem by means of a nearly explicit example. In particular, we highlight two aspects of our results. First, we prove concentration of vorticity at the boundary. As is well-known, concentration of vorticity creates difficulties in treating the inviscid limit, see [13]. Second, we obtain an expression of the total mass of vorticity present in the domain in terms of the angular acceleration of the boundary, something which may be used as a sharp test for the accurate portrayal of the fluid-boundary interaction in high Reynold number numerical schemes.

The structure of the rest of this paper is as follows. In §2 we give a general description of solutions to (1.12), (1.2), and (1.3), with quite rough α. In §§3–5 we establish convergence in (1.10), first in L^2-Sobolev spaces, then in L^q-Sobolev spaces for other values of q, and then in certain Banach spaces of distributions (defined in §5), treating α of the form (1.17), first for $p' > 4$ (in §3), then for $p' > 2$ (in §4), and finally for all $p' \geq 1$ (in §5). In §6 we digress to remark on the case when α is Brownian motion.

Section 7 treats strong convergence results away from ∂D. In §8 we produce estimates on the pressure p^ν appearing in (1.1), making use of the identities in (1.14). In §9 we examine the vorticity $\omega^\nu = \operatorname{rot} u^\nu$, and contrast the local convergence to $\operatorname{rot} u_0$, on compact subsets of D, with the global behavior. In particular we analyze the concentration of vorticity on ∂D as $\nu \searrow 0$. We devote §10 to consideration of the special case (1.16), and extend results of [1]. In §§11–12 we bring the theory of layer potentials to bear on the analysis of (1.12), (1.2), and (1.3), and produce a sharp analysis of the boundary layer behavior of $u^\nu(t,x)$, in case $u_0 \in C^\infty(\overline{D})$, and more generally in case $u_0 \in C(\overline{D})$.

In §13 we extend the scope of our investigations from the setting of the disk D to an annulus $\mathcal{A} = \{x \in \mathbb{R}^2 : \rho < |x| < 1\}$, for some $\rho \in (0,1)$, allowing for independent rotations of the two components of $\partial\mathcal{A}$. Thus the boundary condition (1.2) is replaced by

$$
\begin{aligned}
u^\nu(t,x) \;=\; & \frac{\alpha_1(t)}{2\pi} x^\perp, \quad |x| = 1,\ t > 0, \\
& \frac{\alpha_2(t)}{2\pi} x^\perp, \quad |x| = \rho,\ t > 0.
\end{aligned}
\tag{1.18}
$$

We establish an analogue of Proposition 1.1, from which the extensions of most of the results of §§2–12 are straightforward, though the extension of the material of §9 requires further work.

Finally, an observation regarding notation is in order. We will denote the open interval $(0,\infty)$ by $\mathbb{R}^+$.

2 Solutions with irregular driving motion α

As explained in the introduction, the analysis of the Navier-Stokes equation in the circularly symmetric case is reduced to the analysis of the initial-boundary problem

$$\begin{aligned}
&\partial_t u^\nu = \nu \Delta u^\nu, && \text{for } (t,x) \in (0,\infty) \times D,\\
&u^\nu(0,x) = u_0(x), && \text{for } x \in D,\\
&u^\nu(t,x) = \frac{\alpha(t)}{2\pi} x^\perp, && \text{for } (t,x) \in (0,\infty) \times \partial D.
\end{aligned} \tag{2.1}$$

In passing from (1.1)–(1.3) to (2.1), it was crucial to assume that u_0 had the form (1.6), but such an hypothesis will not generally play an important role in our analysis of (2.1), with some exceptions, such as in §9.

We solve (2.1) on $(t, x) \in \mathbb{R}^+ \times D$, but it is convenient to assume α is defined on $\mathbb{R}$, with

$$\operatorname{supp} \alpha \subset [0, \infty). \tag{2.2}$$

Note that no assumption is made, or implied, on the behavior of u_0 at the boundary. Equation (2.1) will be understood in a mild sense, with the solution converging to the initial data in a suitable sense, usually strongly in L^2, as $t \to 0^+$.

As a preliminary to our main goal in this section of treating rough α in (2.1), we first dispose of the case $\alpha \equiv 0$. In this case, if $u_0 \in L^2(D)$, the solution to (2.1) is given by

$$u^\nu(t) = e^{\nu t A} u_0, \tag{2.3}$$

where A is the self-adjoint operator on $L^2(D)$, with domain $\mathcal{D}(A)$, defined by

$$\mathcal{D}(A) = H^2(D) \cap H_0^1(D), \quad Au = \Delta u \ \text{ for } u \in \mathcal{D}(A). \tag{2.4}$$

Here we supress notation recording the fact that u is vector-valued (with values in $\mathbb{R}^2$) rather than scalar-valued.

The family $\{e^{tA} : t \geq 0\}$ is a strongly continuous semigroup on $L^2(D)$. As is well known, it also extends and/or restricts to a strongly continuous semigroup on a large variety of other Banach spaces of functions on D, such as $L^p(D)$ for $p \in [1, \infty)$ (but not for $p = \infty$). The maximum principle holds; $\{e^{tA} : t \geq 0\}$ is a contraction semigroup on $L^\infty(D)$, and also on $C(\overline{D})$, but these semigroups are not strongly continuous at $t = 0$. We do get a strongly continuous semigroup on $C_*(\overline{D})$, the space of functions in $C(\overline{D})$ that vanish on ∂D. We also get strongly continuous semigroups on a variety of L^q-Sobolev spaces, which we will discuss in more detail in §§3–4. Whenever $\{e^{tA} : t \geq 0\}$ acts as a strongly continuous semigroup on some Banach space X of functions on D, we get convergence in (2.3) of $u^\nu(t)$ to u_0 in X-norm, for all $u_0 \in X$.

In general, the solution to (2.1) can be written as a sum of $e^{t\nu A} u_0$ and the restriction to $t \in [0, \infty)$ of a function that it is convenient to define on $\mathbb{R} \times D$ as the solution to

$$\begin{aligned}&\partial_t v^\nu = \nu\Delta v^\nu, \quad v^\nu(t,x) = 0 \;\text{ for }\; t < 0,\\ &v^\nu(t,x) = \frac{\alpha(t)}{2\pi}x^\perp \;\text{ for }\; x \in \partial D, \quad t \geq 0.\end{aligned} \tag{2.5}$$

Recall that we are assuming (2.2). We denote v^ν in (2.5) as $S^\nu\alpha$. It is classical that

$$S^\nu : C_\flat^\infty(\mathbb{R}) \longrightarrow C_\flat^\infty(\mathbb{R} \times \overline{D}), \tag{2.6}$$

valid for each $\nu > 0$. Here and below, given a space $\mathfrak{X}$ of functions or distributions on $\mathbb{R}$ or $\mathbb{R} \times D$, we denote by $\mathfrak{X}_\flat$ the subspace consisting of elements of $\mathfrak{X}$ that vanish for $t < 0$. Thanks to the maximum principle, S^ν in (2.6) has a unique continuous extension to

$$S^\nu : C_\flat(\mathbb{R}) \longrightarrow C_\flat(\mathbb{R} \times \overline{D}). \tag{2.7}$$

Our next goal is to show that S^ν also maps $L_\flat^{p'}(\mathbb{R})$ into other function spaces, on which the boundary trace Tr is defined, and that

$$\mathrm{Tr}(S^\nu\alpha) = \frac{\alpha}{2\pi}x^\perp \tag{2.8}$$

whenever $\alpha \in L_\flat^{p'}(\mathbb{R})$.

Note that in cases (2.6) and (2.7) $S^\nu\alpha$ clearly has a boundary trace and (2.8) holds. Let us produce a variant of (2.7) as follows. Using radial coordinates (r,θ) on $\overline{D}$ (away from the center), we have

$$S^\nu : C_\flat(\mathbb{R}) \longrightarrow C([0,1], C_\flat(\mathbb{R} \times \partial D)), \tag{2.9}$$

i.e., $S^\nu\alpha(t,x)$, with $x = (r\cos\theta, r\sin\theta)$, is a continuous function of $r \in [0,1]$ with values in the space $C_\flat(\mathbb{R} \times \partial D)$. Then Tr $S^\nu\alpha \in C_\flat(\mathbb{R} \times \partial D)$ is the value of this function at $r = 1$. Noting that $C_\flat(\mathbb{R} \times \partial D) \subset L^2_{\mathrm{loc},\flat}(\mathbb{R} \times \partial D)$, we have

$$S^\nu : C_\flat(\mathbb{R}) \longrightarrow C([0,1], L^2_{\mathrm{loc},\flat}(\mathbb{R} \times \partial D)). \tag{2.10}$$

Next note that if $\beta \in C_\flat^\infty(\mathbb{R})$ then

$$\alpha = \beta' \Longrightarrow S^\nu\alpha = \partial_t S^\nu\beta. \tag{2.11}$$

From this it follows easily that

$$S^\nu\partial_t = \partial_t S^\nu : C_\flat^\infty(\mathbb{R}) \longrightarrow C([0,1], C_\flat^\infty(\mathbb{R} \times \partial D)) \tag{2.12}$$

has a unique continuous extension to

$$S^\nu \partial_t = \partial_t S^\nu : C_\flat(\mathbb{R}) \longrightarrow C([0,1], H^{-1}_{\mathrm{loc},\flat}(\mathbb{R} \times \partial D)). \tag{2.13}$$

Now, given $p' \geq 1$, each $\alpha \in L^{p'}_\flat(\mathbb{R})$ has the form $\alpha = \beta'$ with $\beta \in C_\flat(\mathbb{R})$, namely $\beta(t) = \int_{-\infty}^t \alpha(s)\, ds$. It follows that

$$S^\nu : L^{p'}_\flat(\mathbb{R}) \longrightarrow C([0,1], H^{-1}_{\mathrm{loc},\flat}(\mathbb{R} \times \partial D)), \tag{2.14}$$

for each $p' \geq 1$. Consequently we have the continuous linear map

$$\mathrm{Tr} \circ S^\nu : L^{p'}_\flat(\mathbb{R}) \longrightarrow H^{-1}_{\mathrm{loc},\flat}(\mathbb{R} \times \partial D), \tag{2.15}$$

and since (2.8) holds on the dense linear subspace $C_0^\infty(\mathbb{R}^+)$, it holds for all $\alpha \in L^{p'}_\flat(\mathbb{R})$.

The target space in (2.14) was chosen to have good trace properties, so (2.8) could be verified, but such a choice precludes establishing the convergence result

$$\lim_{\nu \searrow 0} S^\nu \alpha = 0 \tag{2.16}$$

in the strong topology of this target space. Other spaces will arise in §§3–5, for which (2.16) holds in norm (see also (2.23)). At this point we will establish some useful identities for $S^\nu \alpha$.

To begin, we will assume $\alpha \in C^\infty_\flat(\mathbb{R})$; once we have the identities we can extend the range of their validity by limiting arguments. With v^ν defined by (2.5), let us set

$$w^\nu(t,x) = v^\nu(t,x) - \frac{\alpha(t)}{2\pi} x^\perp \tag{2.17}$$

on $[0,\infty) \times D$, so w^ν solves

$$\partial_t w^\nu = \nu \Delta w^\nu - \alpha'(t) f_1, \quad w^\nu(0,x) = 0, \quad w^\nu\big|_{\mathbb{R}^+ \times \partial D} = 0, \tag{2.18}$$

with

$$f_1(x) = \frac{1}{2\pi} x^\perp. \tag{2.19}$$

We can then apply Duhamel's formula to write

$$w^\nu(t) = -\int_0^t e^{\nu(t-s)A} f_1\, \alpha'(s)\, ds. \tag{2.20}$$

Hence

$$\begin{aligned} S^\nu\alpha(t) = v^\nu(t) &= \alpha(t) f_1 - \int_0^t e^{\nu(t-s)A} f_1\, \alpha'(s)\, ds \\ &= \int_0^t \left(I - e^{\nu(t-s)A}\right) f_1\, \alpha'(s)\, ds, \end{aligned} \tag{2.21}$$

and so the solution to (2.1) can be written

$$u^\nu(t) = e^{\nu t A} u_0 + \int_0^t \left(I - e^{\nu(t-s)A}\right) f_1\, \alpha'(s)\, ds, \tag{2.22}$$

with f_1 given by (2.19).

Having (2.21) and (2.22) for $\alpha \in C_\flat^\infty(\mathbb{R})$, we can immediately extend these formulas to $\alpha \in H_\flat^{1,1}(\mathbb{R})$, i.e., α supported in $\mathbb{R}^+$ and $\alpha,\ \alpha' \in L^1(\mathbb{R})$. In fact, as in [10], we can go further, as we see below.

Proposition 2.1. *Let X be a Banach space of functions on D such that $f_1 \in X$ and $\{e^{tA} : t \geq 0\}$ is a strongly continuous semigroup on X.*

We have

$$S^\nu : \mathrm{BV}_\flat(\mathbb{R}) \longrightarrow C_\flat(\mathbb{R}, X), \tag{2.23}$$

given by

$$S^\nu\alpha(t) = \int_{I(t)} \left(I - e^{\nu(t-s)A}\right) f_1\, d\alpha(s), \quad I(t) = [0, t], \tag{2.24}$$

the integral being the Bochner integral.

We remark that we could take, for example, $X = L^2(D)$.

Proof. Using mollifiers in $C_0^\infty(\mathbb{R})$ with support contained in $(0, 1/k)$, we can approximate α by $\alpha_k \in C_\flat^\infty(\mathbb{R})$ in a fashion so that $\alpha_k' ds \to d\alpha$ weak* as Radon measures. That is to say,

$$\lim_{k\to\infty} \int_{[0,\infty)} g(s)\alpha_k'(s)\, ds = \int_{[0,\infty)} g(s)\, d\alpha(s), \tag{2.25}$$

for each compactly supported continuous function g on $[0, \infty)$. We have that $S^\nu\alpha_k$ is given by (2.22) with α replaced by α_k and that $S^\nu\alpha_k \to S^\nu\alpha$. To prove

(2.24), it suffices to show that

$$\begin{aligned}\lim_{k\to\infty}\int_0^t \left\langle \left(I - e^{\nu(t-s)A}\right) f_1, \xi \right\rangle \alpha_k'(s)\,ds \\ = \int_{I(t)} \left\langle \left(I - e^{\nu(t-s)A}\right) f_1, \xi \right\rangle d\alpha(s),\end{aligned} \tag{2.26}$$

for arbitrary $\xi \in X'$. However, (2.26) follows from (2.25) upon taking

$$\begin{aligned} g(s) = \left\langle \left(I - e^{\nu(t-s)A}\right) f_1, \xi \right\rangle \quad & \text{for } s \in [0, t], \\ 0 \qquad\qquad\quad & \text{for } s > t, \end{aligned} \tag{2.27}$$

which is continuous and compactly supported on $[0, \infty)$. Finally, the fact that the range in (2.23) is contained in $C_\flat(\mathbb{R}, X)$ is a consequence of the formula (2.24). □

Remark. Note that the continuous integrand in (2.24) vanishes at $s = t$, so one gets the same result with $I(t) = [0, t)$. Note also that

$$\|S^\nu \alpha(t)\|_X \le \|\alpha\|_{\mathrm{BV}([0,t])} \sup_{s\in[0,t]} \|e^{\nu s A} f_1 - f_1\|_X, \tag{2.28}$$

and, if $u_0 \in X$,

$$\|u^\nu(t) - u_0\|_X \le \|e^{\nu t A} u_0 - u_0\|_X + \|S^\nu \alpha(t)\|_X. \tag{2.29}$$

With $X = L^2(D)$, this gives the convergence result established in [10].

Another useful identity for $S^\nu\alpha$ arises via integration by parts. In fact, for $\alpha \in C_\flat^\infty(\mathbb{R})$ and $\varepsilon > 0$, we have

$$\begin{aligned}\int_0^{t-\varepsilon} e^{\nu(t-s)A} f_1\, \alpha'(s)\,ds \\ = \alpha(t-\varepsilon) e^{\nu\varepsilon A} f_1 + \nu \int_0^{t-\varepsilon} A e^{\nu(t-s)A} f_1\, \alpha(s)\,ds,\end{aligned} \tag{2.30}$$

justification following because $e^{\nu\sigma A} f_1 \in \mathcal{D}(A)$, given by (2.4), whenever $\sigma > 0$. Together with (2.21), this yields

$$S^\nu \alpha(t) = -\lim_{\varepsilon\searrow 0} \nu \int_0^{t-\varepsilon} A e^{\nu(t-s)A} f_1\, \alpha(s)\,ds, \tag{2.31}$$

the limit existing certainly in L^2-norm, locally uniformly in t, and as we will see in subsequent sections, also in other norms, and for more singular α.

3 L^2-Sobolev vanishing viscosity limits

The family $\{e^{tA} : t \geq 0\}$ is a strongly continuous semigroup of operators on $L^2(D)$, and on $\mathcal{D}(A)$, given by (2.4), and more generally on $\mathcal{D}((-A)^{\sigma/2})$, for each $\sigma \in \mathbb{R}^+$. As is well known,

$$\mathcal{D}((-A)^{1/2}) = H_0^1(D), \tag{3.1}$$

and, for $\sigma \in [0, 1]$,

$$\mathcal{D}((-A)^{\sigma/2}) = [L^2(D), H_0^1(D)]_\sigma, \tag{3.2}$$

the complex interpolation space. Furthermore,

$$\begin{aligned} [L^2(D), H_0^1(D)]_\sigma &= H_0^\sigma(D), \quad \frac{1}{2} < \sigma \leq 1, \\ &= H^\sigma(D), \quad 0 \leq \sigma < \frac{1}{2}. \end{aligned} \tag{3.3}$$

Cf. [8], Chapter 1, Section 11. Consequently,

$$\mathcal{D}((-A)^{\sigma/2}) = H^\sigma(D), \quad \text{for } \sigma \in \Big[0, \frac{1}{2}\Big). \tag{3.4}$$

Hence

$$\forall \sigma \in \Big[0, \frac{1}{2}\Big), \quad u_0 \in H^\sigma(D) \Longrightarrow e^{\nu t A} u_0 \to u_0 \text{ in } H^\sigma\text{-norm, as } \nu \to 0, \tag{3.5}$$

convergence holding uniformly in $t \in [0, T]$ for each $T < \infty$.

Recall the formula (2.31), i.e.,

$$S^\nu \alpha(t) = -\lim_{\varepsilon \searrow 0} \nu \int_0^{t-\varepsilon} A e^{\nu(t-s)A} f_1 \, \alpha(s)\, ds, \tag{3.6}$$

for $\alpha \in C_\flat^\infty(\mathbb{R})$. We now seek conditions that imply that $\|Ae^{\nu(t-s)A} f_1 \alpha(s)\|_{H^\sigma}$ is integrable on $[0, t]$ and that

$$S^\nu \alpha(t) = -\nu \int_0^t A e^{\nu(t-s)A} f_1 \, \alpha(s)\, ds. \tag{3.7}$$

Note that

$$\begin{aligned} &\int_0^t \|\nu A e^{\nu(t-s)A} f_1 \, \alpha(s)\|_{H^\sigma(D)}\, ds \\ &\leq \|\alpha\|_{L^{p'}([0,t])} \Big(\int_0^t \|\nu A e^{\nu s A} f_1\|^p_{H^\sigma(D)}\, ds\Big)^{1/p}. \end{aligned} \tag{3.8}$$

Let $0 \leq \sigma < 1/2$ and choose $\sigma < \tau < 1/2$. Then

$$\begin{aligned}
&\|\nu A e^{\nu s A} f_1\|_{H^\sigma(D)} \\
&\quad \leq C\|\nu(-A)^{1+\sigma/2} e^{\nu s A} f_1\|_{L^2(D)} \\
&\quad = C\|\nu(-A)^{1-(\tau-\sigma)/2} e^{\nu s A}(-A)^{\tau/2} f_1\|_{L^2(D)} \\
&\quad = C\nu^{(\tau-\sigma)/2} s^{(\tau-\sigma)/2-1}\|(-\nu s A)^{1-(\tau-\sigma)/2} e^{\nu s A}(-A)^{\tau/2} f_1\|_{L^2(D)} \\
&\quad \leq C\nu^{(\tau-\sigma)/2} s^{(\tau-\sigma)/2-1}\|f_1\|_{H^\tau(D)}.
\end{aligned} \tag{3.9}$$

Furthermore,

$$\begin{aligned}
p\Big(1-\frac{\tau-\sigma}{2}\Big) < 1 \Longrightarrow \int_0^t (s^{(\tau-\sigma)/2-1})^p\,ds \\
= C_{p\sigma\tau} t^{p(\tau-\sigma)/2-p+1}, \quad C_{p\sigma\tau} < \infty.
\end{aligned} \tag{3.10}$$

Hence

$$\begin{aligned}
1 \leq p < \frac{2}{2-(\tau-\sigma)} \Longrightarrow \|S^\nu \alpha(t)\|_{H^\sigma(D)} \\
\leq C(t)\nu^{(\tau-\sigma)/2}\|\alpha\|_{L^{p'}([0,t])}\|f_1\|_{H^\tau(D)}.
\end{aligned} \tag{3.11}$$

Approximating a rough α by smoothing convolutions and passing to the limit, we obtain:

Proposition 3.1. *Assume $\alpha \in L^{p'}_\flat(\mathbb{R})$ where p satisfies the hypothesis in* (3.11). *Then the formula* (3.7) *holds, we have*

$$S^\nu : L^{p'}_\flat(\mathbb{R}) \longrightarrow C_\flat(\mathbb{R}, H^\sigma(D)), \ \textit{for } \sigma \in \Big[0, \frac{1}{2}\Big), \ p \in \Big[1, \frac{2}{3/2+\sigma}\Big), \tag{3.12}$$

and the estimate (3.11) *holds.*

Remark. There exist σ, τ such that $0 \leq \sigma < \tau < 1/2$ and the hypothesis above on p holds provided $1 \leq p < 4/3$, i.e., provided $p' > 4$.

4 L^q-Sobolev vanishing viscosity limits

Let $q \in (1, \infty)$. Then e^{tA} provides a strongly continuous semigroup on $L^q(D)$, indeed a holomorphic semigroup. We sometimes denote the infinitesimal generator by A_q, to emphasize the q-dependence. Now, for $\lambda > 0$,

$$R_\lambda = (\lambda - A)^{-1} = \int_0^\infty e^{tA} e^{-\lambda t}\,dt \tag{4.1}$$

has the mapping property

$$R_\lambda : L^q(D) \xrightarrow{\approx} \mathcal{D}(A_q), \tag{4.2}$$

and standard elliptic theory gives

$$\mathcal{D}(A_q) = H^{2,q}(D) \cap H_0^{1,q}(D). \tag{4.3}$$

We record the following useful known results.

Proposition 4.1. *Given $\sigma \in (0, 2)$, the operator $-(-A_q)^{\sigma/2}$ is well defined and is the generator of a holomorphic semigroup on $L^q(D)$. Furthermore,*

$$\mathcal{D}((-A_q)^{\sigma/2}) = [L^q(D), \mathcal{D}(A_q)]_{\sigma/2}, \tag{4.4}$$

where the right side is a complex interpolation space. In addition,

$$0 \le \sigma < \frac{1}{q} \Longrightarrow \mathcal{D}((-A_q)^{\sigma/2}) = H^{\sigma,q}(D). \tag{4.5}$$

Also, if $\gamma \in [0, 1]$ and $T \in (0, \infty)$,

$$\|(-tA_q)^\gamma e^{tA_q} f\|_{L^q(D)} \le C_{q\gamma T} \|f\|_{L^q(D)}, \quad \textit{for } t \in [0, T]. \tag{4.6}$$

Proof. The results (4.4)–(4.5) are proven in [14]–[15]. The result (4.6) is equivalent to

$$\|e^{tA_q} f\|_{\mathcal{D}((-A_q)^\gamma)} \le Ct^{-\gamma} \|f\|_{L^q(D)}. \tag{4.7}$$

For $\gamma = 1$ this follows from the fact that e^{tA_q} is a holomorphic semigroup. For $\gamma = 0$ it is clear. Then for $0 < \gamma < 1$ it follows from these endpoint cases, via (4.4) and the general interpolation estimate

$$\|g\|_{[L^q, \mathcal{D}(A_q)]_\gamma} \le C\|g\|^\gamma_{\mathcal{D}(A_q)} \|g\|^{1-\gamma}_{L^q(D)}. \qquad \square \tag{4.8}$$

Given this proposition, we proceed as follows on the estimation of

$$S^\nu \alpha(t) = -\int_0^t \nu A e^{\nu(t-s)A} f_1\, \alpha(s)\, ds. \tag{4.9}$$

Pick $q \in (1, \infty)$, and pick σ, τ satisfying

$$0 \le \sigma < \tau < \frac{1}{q}. \tag{4.10}$$

Then, as in (3.8), we have

$$\|S^\nu\alpha(t)\|_{H^{\sigma,q}(D)} \le \|\alpha\|_{L^{p'}([0,t])}\Big(\int_0^t \|\nu A e^{\nu s A} f_1\|^p_{H^{\sigma,q}(D)}\, ds\Big)^{1/p}. \tag{4.11}$$

Now Proposition 4.1 yields the following analogue of (3.9):

$$\begin{aligned}
&\|\nu A e^{\nu s A} f_1\|_{H^{\sigma,q}(D)} \\
&\quad = C\|\nu(-A)^{1+\sigma/2} e^{\nu s A} f_1\|_{L^q(D)} \\
&\quad = \|\nu(-A)^{1-(\tau-\sigma)/2} e^{\nu s A}(-A)^{\tau/2} f_1\|_{L^q(D)} \\
&\quad = C\nu^{(\tau-\sigma)/2} s^{(\tau-\sigma)/2-1}\|(-\nu s A)^{1-(\tau-\sigma)/2} e^{\nu s A}(-A)^{\tau/2} f_1\|_{L^q(D)} \\
&\quad \le C\nu^{(\tau-\sigma)/2} s^{(\tau-\sigma)/2-1}\|f_1\|_{H^{\tau,q}(D)}.
\end{aligned} \tag{4.12}$$

We can then use (3.10) to conclude:

Proposition 4.2. *We have*

$$\begin{aligned}
&S^\nu : L^{p'}_\flat(\mathbb{R}) \longrightarrow C_\flat(\mathbb{R}, H^{\sigma,q}(D)), \\
&\textit{for } q > 1,\ \sigma \in \Big[0, \frac{1}{q}\Big),\ p \in \Big[1, \frac{2}{2 - 1/q + \sigma}\Big),
\end{aligned} \tag{4.13}$$

and as long as (4.10) *holds,*

$$\begin{aligned}
1 \le p < \frac{2}{2-(\tau-\sigma)} &\Longrightarrow \|S^\nu\alpha(t)\|_{H^{\sigma,q}(D)} \\
&\le C(t)\nu^{(\tau-\sigma)/2}\|\alpha\|_{L^{p'}([0,t])}\|f_1\|_{H^{\tau,q}(D)}.
\end{aligned} \tag{4.14}$$

Remark. Note that, for a given p, there exist q, τ, and σ satisfying (4.10), for which the hypothesis in (4.14) holds, provided $1 \le p < 2$, i.e., provided $p' > 2$.

In the setting of Proposition 4.1, e^{tA_q} is also a strongly continuous semigroup on $\mathcal{D}((-A_q)^{\sigma/2})$, hence in the setting of (4.5), on $H^{\sigma,q}(D)$, so we also have:

Proposition 4.3. *If $q \in (1, \infty)$ and $\sigma \in [0, 1/q)$, then*

$$u_0 \in H^{\sigma,q}(D) \Longrightarrow \lim_{\nu\searrow 0} \|e^{\nu t A} u_0 - u_0\|_{H^{\sigma,q}(D)} = 0. \tag{4.15}$$

5 Generalized function space vanishing viscosity limits

Here we show that for $\alpha \in L^{p'}_\flat(\mathbb{R})$, we have $S^\nu\alpha(t) \to 0$ in various topologies weaker than the L^2-norm, even when $p' \in [1, 2]$. We will use a continuation of the scale $\mathcal{D}((-A_2)^{s/2}) = \mathcal{D}_s$. There are analogous results involving A_q, which we will not discuss here. As stated before, we have

$$\mathcal{D}_2 = H^2(D) \cap H^1_0(D). \tag{5.1}$$

Also

$$0 \leq s \leq 2 \Longrightarrow \mathcal{D}_s = [L^2(D), \mathcal{D}_2]_{s/2}, \tag{5.2}$$

and in particular

$$0 \leq s < \frac{1}{2} \Longrightarrow \mathcal{D}_s = H^s(D). \tag{5.3}$$

For $s < 0$, we set

$$\mathcal{D}_s = \mathcal{D}^*_{-s}. \tag{5.4}$$

Details on this are given in Chapter 5, Appendix A of [16]. We mention that

$$\mathcal{D}_{-1} = H^{-1}(D). \tag{5.5}$$

Also

$$s = -\sigma < 0 \Longrightarrow \mathcal{D}_s = (-A_2)^{\sigma/2} L^2(D). \tag{5.6}$$

Given this, we have in parallel with (3.8)–(3.9) that for $\sigma \in \mathbb{R}$,

$$\|S^\nu\alpha(t)\|_{\mathcal{D}_\sigma} \leq \|\alpha\|_{L^{p'}([0,t])} \Big(\int_0^t \|\nu A e^{\nu s A} f_1\|^p_{\mathcal{D}_\sigma} \, ds \Big)^{1/p}, \tag{5.7}$$

for $p \in (1, \infty)$, and

$$\|S^\nu\alpha(t)\|_{\mathcal{D}_\sigma} \leq \|\alpha\|_{L^1([0,t])} \sup_{0 \leq s \leq t} \|\nu A e^{\nu s A} f_1\|_{\mathcal{D}_\sigma}, \tag{5.8}$$

and for $-\infty < \sigma < \tau < 1/2$,

$$\|\nu A e^{\nu s A} f_1\|_{\mathcal{D}_\sigma} \leq C \nu^{(\tau-\sigma)/2} s^{(\tau-\sigma)/2-1} \|f_1\|_{\mathcal{D}_\tau}. \tag{5.9}$$

We still have (3.10), and now we can take $\sigma < 0$, as well as τ close to $1/2$. In particular, we have

$$\begin{aligned} & -2 < \sigma < -\frac{3}{2}, \ \tau = \sigma + 2 \\ & \Longrightarrow \|S^\nu\alpha(t)\|_{\mathcal{D}_\sigma} \leq C\nu \|\alpha\|_{L^1([0,t])} \, \|f_1\|_{H^\tau(D)}. \end{aligned} \tag{5.10}$$

6 A stochastic interlude

Here, instead of having α be deterministic, we consider

$$\alpha(s) = \omega(s),$$

where $\omega \in \mathcal{P}_0$, the space of continuous paths from $[0, \infty)$ to $\mathbb{R}$ (such that $\omega(0) = 0$) endowed with Wiener measure W_0, and expectation E_0.

The estimates of §3 apply to $S^\nu \omega(t)$, but we record special results for this stochastic situation.

We are dealing with

$$S^\nu(t, \omega) = \int_0^t (I - e^{\nu(t-s)A}) f_1 \, d\omega(s), \tag{6.1}$$

which is a Wiener-Ito integral. The integrand is independent of ω, so the analysis of such an integral is relatively elementary. We make the following:

Hypothesis 6.1. H is a Hilbert space of functions on D, with values in $\mathbb{R}^2$, such that $f_1 \in H$ and $\{e^{sA} : s \geq 0\}$ is a strongly continuous semigroup on H.

In such a case, we have

$$E_0\Big(\|S^\nu(t, \cdot)\|_H^2\Big) = \int_0^t \|(e^{\nu s A} - I) f_1\|_H^2 \, ds. \tag{6.2}$$

This is a standard identity in the scalar case (cf. [16], Chapter 11, Proposition 7.1, where an extra factor arises due to an idiosyncratic normalization of Wiener measure made there), and extends readily to Hilbert space-valued integrands, by taking an orthonormal basis of H and examining each component. As seen in §3, this is applicable for

$$H = H^\tau(D, \mathbb{R}^2), \quad 0 \leq \tau < \frac{1}{2}. \tag{6.3}$$

It might be interesting to obtain statistical information on the boundary layers that arise for $S^\nu(t, \omega)$.

7 Local convergence results

Here we examine convergence of

$$u^\nu(t) = e^{\nu t A} u_0 + S^\nu \alpha(t) \tag{7.1}$$

as $\nu \to 0$ to u_0, on compact subsets of D. Recall that

$$S^\nu \alpha(t) = -\nu \int_0^t A e^{\nu(t-s)A} f_1\, \alpha(s)\, ds, \tag{7.2}$$

where f_1 is given by (2.19), so

$$f_1 \in C^\infty(\overline{D}). \tag{7.3}$$

We will prove the following.

Proposition 7.1. *Assume that*

$$u_0 \in L^2(D), \quad u_0\big|_{\mathcal{O}} \in H^k(\mathcal{O}), \quad \alpha \in L^1_\flat(\mathbb{R}), \tag{7.4}$$

where $\mathcal{O}$ is an open subset of D, and take $\overline{\Omega} \subset\subset \mathcal{O}$. Then

$$\lim_{\nu \searrow 0} u^\nu(t)\big|_\Omega = u_0\big|_\Omega \quad in \;\; H^k(\Omega), \tag{7.5}$$

uniformly for $t \in [0, T]$, given $T < \infty$.

Proof. First we show

$$\lim_{\nu \searrow 0} e^{\nu t A} u_0\big|_\Omega = u_0\big|_\Omega \quad \text{in} \;\; H^k(\Omega). \tag{7.6}$$

To see this, take $\varphi \in C_0^\infty(D)$, compactly supported in $\mathcal{O}$, such that $\varphi = 1$ on a neighborhood Ω_1 of $\overline{\Omega}$, and write

$$u_0 = \varphi u_0 + (1-\varphi) u_0 = u_1 + u_2, \tag{7.7}$$

so

$$u_1 \in H_0^k(D) \subset \mathcal{D}((-A)^{k/2}), \quad u_2 \in L^2(D), \quad u_2 = 0 \;\text{ on }\; \Omega_1. \tag{7.8}$$

It follows that

$$\lim_{\nu \searrow 0} e^{\nu t A} u_1 = u_1 \;\text{ in }\; \mathcal{D}((-A)^{k/2}) \subset H^k(D), \tag{7.9}$$

so we will have (7.6) if we show that

$$\lim_{\nu \searrow 0} e^{\nu t A} u_2\big|_{\overline{\Omega}} = 0 \;\text{ in }\; C^\infty(\overline{\Omega}). \tag{7.10}$$

To do this, we define $w(s,x)$ on $\mathbb{R}\times\Omega_1$ by

$$w(s,x) = \begin{cases} e^{sA}u_2(x), & s \geq 0, \\ 0, & s < 0. \end{cases} \tag{7.11}$$

Then w is a weak solution of $(\partial_s - \Delta)w = 0$ on $\mathbb{R}\times\Omega_1$, and the well known hypoellipticity of $\partial_s - \Delta$ implies

$$w \in C^\infty(\mathbb{R}\times\Omega_1). \tag{7.12}$$

This implies (7.10) and hence we have (7.6).

Now that we have (7.6), we can apply this to f_1 in place of u_0 and deduce that

$$\lim_{\nu\searrow 0} e^{\nu(t-s)A} f_1\big|_\Omega = f_1\big|_\Omega \quad \text{in} \quad H^{k+2}(\Omega), \tag{7.13}$$

uniformly on $0 \leq s \leq t \leq T$, which by (7.2) then gives

$$\lim_{\nu\searrow 0} S^\nu\alpha(t)\big|_\Omega = 0 \quad \text{in} \quad H^k(\Omega), \tag{7.14}$$

and hence proves (7.5). □

8 Pressure estimates

For u^ν given by (1.1)–(1.4), the pressure gradient ∇p^ν is given by

$$\nabla p^\nu = -\nabla_{u^\nu} u^\nu. \tag{8.1}$$

It is convenient to rewrite the right side of (8.1), using the general identity

$$\nabla_v u = \operatorname{div}(u\otimes v) - (\operatorname{div} v)u \tag{8.2}$$

(cf. [16], Chapter 17, (2.43)), which in the current context yields

$$\nabla p^\nu = -\operatorname{div}(u^\nu\otimes u^\nu). \tag{8.3}$$

Recall that

$$u^\nu = e^{\nu t A}u_0 + \int_0^t \left(I - e^{\nu(t-s)A}\right) f_1\, d\alpha(s), \tag{8.4}$$

where

$$f_1(x) = \frac{1}{2\pi}x^\perp. \tag{8.5}$$

For simplicity we will work under the assumption that α has bounded variation on each interval $[0, t]$. We will assume

$$u_0 \in L^\infty(D) \cap H^{\tau,q}(D), \tag{8.6}$$

with

$$q \in (1, \infty), \quad 0 < \tau < \frac{1}{q}. \tag{8.7}$$

We aim to prove the following.

Proposition 8.1. *Let u^ν be given by* (1.1)–(1.4)*, and assume u_0 satisfies* (8.6)–(8.7)*. Assume α has locally bounded variation on* $[0, \infty)$*. Take $T \in (0, \infty)$ and $\nu_0 > 0$. Then, uniformly for $t \in [0, T]$, we have*

$$u^\nu(t) \otimes u^\nu(t) \ \text{ bounded in } \ L^\infty(D) \cap H^{\tau,q}(D), \tag{8.8}$$

for $\nu \in (0, \nu_0]$, and, as $\nu \to 0$,

$$u^\nu(t) \otimes u^\nu(t) \longrightarrow u_0 \otimes u_0 \ \text{ weak* in } \ H^{\tau,q}(D), \tag{8.9}$$

hence in $H^{\sigma,q}$-norm, for all $\sigma < \tau$.

Proof. First note that under these hypotheses, we have

$$u^\nu(t) \ \text{ bounded in } \ L^\infty(D) \cap H^{\tau,q}(D). \tag{8.10}$$

This bound is a direct consequence of (8.4), (4.5), and the maximum principle, which implies $\|e^{sA} f\|_{L^\infty} \le \|f\|_{L^\infty}$, $s \ge 0$. From here, (8.8) is a consequence of the estimate

$$\|u \otimes v\|_{H^{\tau,q}} \le C\|u\|_{L^\infty}\|v\|_{H^{\tau,q}} + C\|u\|_{H^{\tau,q}}\|v\|_{L^\infty}; \tag{8.11}$$

cf. [16], Chapter 13, (10.52).

To proceed, we note also that the hypothesis $u_0 \in L^\infty$ plus the fact that e^{tA} is a strongly continuous semigroup on $L^p(D)$ whenever $p < \infty$ gives

$$u^\nu(t) \longrightarrow u_0 \ \text{ in } \ L^p\text{-norm}, \ \ \forall\, p < \infty. \tag{8.12}$$

Hence

$$u^\nu(t) \otimes u^\nu(t) \longrightarrow u_0 \otimes u_0 \ \text{ in } \ L^p\text{-norm}, \ \ \forall\, p < \infty. \tag{8.13}$$

The bound (8.11) implies $\{u^\nu(t) \otimes u^\nu(t)\}$ has weak* limit points in $H^{\tau,q}(D)$ as $\nu \searrow 0$, while (8.13) implies any such limit point must be $u_0 \otimes u_0$. This gives (8.9). The $H^{\sigma,q}$-norm convergence follows from the compactness of the inclusion $H^{\tau,q}(D) \hookrightarrow H^{\sigma,q}(D)$. $\square$

From here we can draw conclusions about the nature of the convergence of $p^\nu(t)$ to p_0, which satisfies

$$\nabla p_0 = -\nabla_{u_0} u_0 = -\mathrm{div}(u_0 \otimes u_0). \tag{8.14}$$

Of course, $p^\nu(t)$ and p_0 are defined only up to additive constants. We fix these by requiring

$$\int_D p^\nu(t,x)\,dx = 0 = \int_D p_0(x)\,dx. \tag{8.15}$$

Then we obtain the following:

Proposition 8.2. *In the setting of Proposition* 8.1*, we have*

$$p^\nu(t) \ \text{bounded in} \ L^\infty(D) \cap H^{\tau,q}(D), \tag{8.16}$$

for $\nu \in (0, \nu_0]$*, and, as* $\nu \to 0$*,*

$$p^\nu(t) \longrightarrow p_0 \ \text{in} \ H^{\sigma,q}\text{-norm}, \quad \forall\, \sigma < \tau. \tag{8.17}$$

By paying closer attention to the special structure of our velocity fields, we can improve Proposition 8.2 substantially. Recall that

$$u^\nu(t,x) = s^\nu(t,|x|)x^\perp, \tag{8.18}$$

with a real-valued factor $s^\nu(t,|x|)$. Now $x^\perp = x_1\partial_{x_2} - x_2\partial_{x_1} = r\partial_\theta$, and

$$\nabla_{r\partial_\theta} r\partial_\theta = -x_1\partial_{x_1} - x_2\partial_{x_2} = -x, \tag{8.19}$$

so, as noted in §1,

$$\nabla_{u^\nu} u^\nu = -s^\nu(t,|x|)^2 x = -|u^\nu(t,x)|^2 \frac{x}{r^2}, \tag{8.20}$$

and hence

$$r\nabla p^\nu = |u^\nu|^2 \frac{x}{r}. \tag{8.21}$$

Noting that

$$\frac{x}{|x|} \in L^\infty(D) \cap H^{1,p}(D), \quad \forall\, p < 2, \tag{8.22}$$

we obtain via arguments used in Proposition 8.1 the following conclusion:

Proposition 8.3. *In the setting of Proposition* 8.1*, we have*

$$r\,\nabla p^\nu(t) \ \text{bounded in} \ L^\infty(D)\cap H^{\tau,q}(D), \tag{8.23}$$

for $\nu \in (0, \nu_0]$*, and, as* $\nu \to 0$*,*

$$r\nabla p^\nu(t) \longrightarrow r\nabla p_0 \ \text{in} \ H^{\sigma,q}\text{-norm}, \quad \forall\, \sigma < \tau. \tag{8.24}$$

The results (8.23)–(8.24) are weak near $x = 0$, but strong away from this point.

In case $u_0 \in C^\infty(\overline{D})$, one can use methods of §7 to get more precise information on $p^\nu(t,x)$, from that on $u^\nu(t,x)$. One has from (8.1) the smooth convergence of $\nabla p^\nu(t,x)$ to $\nabla p_0(x)$ for $|x| \le 1 - c < 1$. Results that will be presented in §11 can be applied to (8.21) to get good control over the boundary layer behavior of $\nabla p^\nu(t,x)$ for $1 - c \le |x| \le 1$. In particular, it will follow that while the pressure gradient $\nabla p^\nu(t,x)$ varies noticeably over the boundary layer, of thickness $\sim \sqrt{\nu}$, the pressure itself $p^\nu(t,x)$ varies only slightly over this boundary layer.

9 Vorticity estimates and vorticity concentration

Here we study the vorticity

$$\omega^\nu(t,x) = \operatorname{rot} u^\nu(t,x) = \partial_{x_1} u_2^\nu(t,x) - \partial_{x_2} u_1^\nu(t,x) \tag{9.1}$$

of a solution to (2.1), i.e.,

$$u^\nu(t) = e^{\nu t A} u_0 + S^\nu \alpha(t), \tag{9.2}$$

under the hypothesis (1.6) of circular symmetry, which implies as in (1.11) that

$$u^\nu(t,x) = s^\nu(t,|x|)x^\perp. \tag{9.3}$$

As noted in §2, while (9.3) played an important role in passing from the Navier-Stokes equation (1.1) to the linear equation (2.1), it has not played a key role in much of the subsequent linear analysis. However, it will play a key role in this section. Note that (9.3) implies

$$\omega^\nu(t,x) = \varpi^\nu(t,|x|), \qquad \varpi^\nu(t,r) = \Bigl(r\frac{d}{dr} + 2\Bigr)s^\nu(t,r). \tag{9.4}$$

In particular, $\omega^\nu(t,x)$ is circularly symmetric.

Here is our first result.

Proposition 9.1. *Assume* u^ν *has the form* (9.2)–(9.3) *with* $u_0 \in L^2(D)$ *and* $\alpha \in C_\flat^\infty(\mathbb{R})$. *Then* $\omega^\nu = \operatorname{rot} u^\nu$ *belongs to* $C^\infty((0,\infty) \times \overline{D})$ *and satisfies the following:*

$$\partial_t \omega^\nu = \nu \Delta \omega^\nu, \quad \text{on } (0,\infty) \times D, \tag{9.5}$$

and

$$n \cdot \nabla \omega^\nu(t,x) = \frac{\alpha'(t)}{2\pi\nu}, \quad \text{on } (0,\infty) \times \partial D, \tag{9.6}$$

where n *is the outward unit normal to* ∂D. *In addition,*

$$\int_D \omega^\nu(t,x)\,dx = \alpha(t), \quad \forall\, t > 0. \tag{9.7}$$

Proof. Standard regularity results yield $\omega^\nu \in C^\infty((0,\infty) \times \overline{D})$, and (9.5) is also obvious. Since $\omega^\nu = -\operatorname{div}(u^\nu)^\perp$, the divergence theorem gives

$$\int_D \omega^\nu(t,x)\,dx = -\int_{\partial D} n \cdot u^\nu(t,x)^\perp\,ds = \alpha(t), \tag{9.8}$$

since $u^\nu(t,x)^\perp = -\alpha(t)x/2\pi$ on ∂D. Next, using (9.5) and the divergence theorem,

$$\frac{d}{dt}\int_D \omega^\nu(t,x)\,dx = \nu \int_D \Delta\omega^\nu(t,x)\,dx = \nu \int_{\partial D} x \cdot \nabla \omega^\nu(t,x)\,dx, \tag{9.9}$$

which is $2\pi\nu$ times the left side of (9.6), by circular symmetry. Since the left side of (9.9) equals $\alpha'(t)$, (9.6) is proven. □

From this follows:

Proposition 9.2. *Assume*

$$u_0 \in H_0^1(D), \quad u_0(x) = s_0(|x|)x^\perp, \tag{9.10}$$

and set

$$\omega_0 = \operatorname{rot} u_0, \quad \omega^\nu(t) = \operatorname{rot} e^{\nu t A} u_0. \tag{9.11}$$

Then

$$\omega^\nu(t) = e^{\nu t A_N}\omega_0, \tag{9.12}$$

where A_N *is the self adjoint operator on (scalar functions in)* $L^2(D)$ *given by*

$$\begin{aligned} \mathcal{D}(A_N) &= \left\{\omega \in H^2(D) : n \cdot \nabla\omega|_{\partial D} = 0\right\}, \\ A_N\omega &= \Delta\omega \ \text{ for } \ \omega \in \mathcal{D}(A_N). \end{aligned} \tag{9.13}$$

Proof. Proposition 9.1 applies with $\alpha \equiv 0$, so we have $\omega^\nu \in C^\infty((0,\infty)\times\overline{D})$ satisfying

$$\begin{aligned} \partial_t \omega^\nu &= \nu \Delta \omega^\nu \qquad \text{on } (0,\infty)\times D, \\ n\cdot\nabla\omega^\nu &= 0 \qquad \text{on } (0,\infty)\times \partial D, \end{aligned} \tag{9.14}$$

hence, for $0 < s < t < \infty$,

$$\omega^\nu(t) = e^{\nu(t-s)A_N}\omega^\nu(s). \tag{9.15}$$

Also, the hypothesis $u_0 \in H_0^1(D)$ implies $e^{\nu t A}u_0 \in C([0,\infty), H_0^1(D))$, since $H_0^1(D) = \mathcal{D}((-A)^{1/2})$. It follows that

$$\omega^\nu \in C([0,\infty), L^2(D)), \quad \omega^\nu(0) = \omega_0, \tag{9.16}$$

and hence (9.12) follows from (9.15) in the limit $s \searrow 0$. □

Generally, given

$$u_0 \in H^1(D), \quad u_0(x) = s_0(|x|)x^\perp, \tag{9.17}$$

we can write

$$u_0 = u_{00} + a f_1, \quad u_{00} \in H_0^1(D), \tag{9.18}$$

with f_1 as in (2.19), i.e., $f_1(x) = x^\perp/2\pi$. Now, Proposition 9.2 is clearly applicable to u_{00}, but it is not applicable to u_0 because it is not applicable to f_1. To see this let

$$\omega^\nu(t) = \operatorname{rot} e^{\nu t A} f_1. \tag{9.19}$$

Then we still have (9.14)–(9.15), but (9.16) fails, and so does (9.12). In fact, in this case

$$\omega_0 = \operatorname{rot} f_1 = \frac{1}{\pi}, \quad \text{so } e^{\nu t A_N}\omega_0 \equiv \frac{1}{\pi}, \tag{9.20}$$

but, in view of (9.7), and since the flow $e^{\nu t A} f_1$ has $\alpha \equiv 0$, we have that, for $\omega^\nu(t)$ given by (9.19),

$$\int_D \omega^\nu(t,x)\,dx = 0, \quad \forall t > 0. \tag{9.21}$$

In this context, we note that Proposition 7.1 implies that, for each compact $\overline{\Omega} \subset D$,

$$e^{\nu t A} f_1 \longrightarrow f_1 \text{ in } C^\infty(\overline{\Omega}), \tag{9.22}$$

as $\nu \searrow 0$ (or as $t \searrow 0$), and hence, with ω^ν, ω_0 as in (9.19)–(9.20),

$$\omega^\nu(t) \longrightarrow \omega_0 \text{ in } C^\infty(\overline{\Omega}) \tag{9.23}$$

as $\nu \searrow 0$ (or as $t \searrow 0$). This implies a "concentration phenomenon" for the vorticity $\omega^\nu(t)$, in ∂D, which we discuss in a more general context below.

Regarding the formula (9.12), it is standard that $\{e^{sA_N} : s \geq 0\}$ is a strongly continuous contraction semigroup on $L^p(D)$, for each $p \in [1, \infty)$. Consequently Proposition 9.2 yields:

$$u_0 \text{ as in (9.10)} \Longrightarrow \| \operatorname{rot} e^{\nu t A} u_0 \|_{L^1(D)} \leq \| \operatorname{rot} u_0 \|_{L^1(D)}. \tag{9.24}$$

The following is a useful complement.

Proposition 9.3. *Given*

$$\alpha \in C_b^\infty(\mathbb{R}), \quad v^\nu(t) = S^\nu \alpha(t), \quad \omega^\nu(t) = \operatorname{rot} v^\nu(t), \tag{9.25}$$

we have

$$\|\omega^\nu(t)\|_{L^1(D)} \leq \|\alpha\|_{\mathrm{BV}([0,t])}. \tag{9.26}$$

Proof. Under these hypotheses, $v^\nu, \omega^\nu \in C^\infty(\mathbb{R} \times \overline{D})$. Take $\varphi_\varepsilon(y) = (\varepsilon^2 + y^2)^{1/2}$, which approximates $|y|$ as $\varepsilon \searrow 0$. Multiply the equation (9.5) by $\varphi'_\varepsilon(\omega^\nu)$ and integrate over D to obtain

$$\begin{aligned} \frac{d}{dt} \int_D \varphi_\varepsilon(\omega^\nu(t,x))\, dx &= \nu \int_D \Big[\Delta \varphi_\varepsilon(\omega^\nu(t,x)) - \varphi''_\varepsilon(\omega^\nu(t,x)) |\nabla \omega^\nu(t,x)|^2 \Big] dx \\ &\leq \nu \int_D \Delta \varphi_\varepsilon(\omega^\nu(t,x))\, dx, \end{aligned} \tag{9.27}$$

since $\varphi'' \geq 0$. Then the divergence theorem gives

$$\begin{aligned} \nu \int_D \Delta \varphi_\varepsilon(\omega^\nu(t,x))\, dx &= \nu \int_{\partial D} x \cdot \nabla \varphi_\varepsilon(\omega^\nu(t,x))\, ds \\ &= \frac{\alpha'(t)}{2\pi} \int_{\partial D} \varphi'_\varepsilon(\omega^\nu(t,x))\, ds, \end{aligned} \tag{9.28}$$

the last identity using (9.6). Since $|\varphi'|_\varepsilon \leq 1$, this yields

$$\frac{d}{dt} \int_D \varphi_\varepsilon(\omega^\nu(t,x))\, dx \leq |\alpha'(t)|. \tag{9.29}$$

Consequently, for each $\varepsilon > 0$,

$$\int_D \varphi_\varepsilon(\omega^\nu(t,x))\,dx \le \|\alpha\|_{\mathrm{BV}([0,t])} + \int_D \varphi_\varepsilon(0)\,dx. \tag{9.30}$$

Taking $\varepsilon \searrow 0$ gives (9.26). □

Returning to $e^{\nu t A} f_1$ in (9.19)–(9.20), we note that

$$f_1 - e^{\nu t A} f_1 = S^\nu \chi_{\mathbb{R}^+}(t), \quad \text{for } t > 0, \tag{9.31}$$

and hence, approximating $\chi_{\mathbb{R}^+}$ by a sequence in $C_\flat^\infty(\mathbb{R})$ and passing to the limit, we get from (9.26) that

$$\|\operatorname{rot} f_1 - \operatorname{rot} e^{\nu t A} f_1\|_{L^1(D)} \le 1. \tag{9.32}$$

By (9.20)–(9.21),

$$\int_D \Big[\operatorname{rot} f_1(x) - \operatorname{rot} e^{\nu t A} f_1(x)\Big]\,dx = 1, \tag{9.33}$$

so in fact we have identity in (9.32), and we see the integrand in (9.33) is ≥ 0 on D, i.e.,

$$\operatorname{rot} e^{\nu t A} f_1(x) \le \frac{1}{\pi}. \tag{9.34}$$

Returning to (9.32), we have

$$\|\operatorname{rot} e^{\nu t A} f_1\|_{L^1(D)} \le 2 = 2\|\operatorname{rot} f_1\|_{L^1(D)}. \tag{9.35}$$

We can put together (9.24) and (9.35) as follows. Take u_0 as in (9.17), so we have (9.18) with $a = \int_D \operatorname{rot} u_0(x)\,dx$, hence $\|\operatorname{rot} u_{00}\|_{L^1} \le 2\|\operatorname{rot} u_0\|_{L^1}$. Consequently,

$$u_0 \text{ as in (9.17)} \Longrightarrow \|\operatorname{rot} e^{\nu t A} u_0\|_{L^1(D)} \le 4\|\operatorname{rot} u_0\|_{L^1(D)}. \tag{9.36}$$

We can extend the scope of (9.36) as follows. Set

$$R^1(D) = \big\{u \in L^2(D) \colon u(x) = s(|x|)x^\perp,\ \operatorname{rot} u \in L^1(D)\big\}. \tag{9.37}$$

An element of $R^1(D)$ is continuous on $\overline{D} \setminus \{0\}$. Set

$$R_0^1(D) = \big\{u \in R^1(D) : u\big|_{\partial D} = 0\big\}. \tag{9.38}$$

The argument in (9.8) readily extends to give

$$u \in R^1_0(D) \Longleftrightarrow u \in R^1(D) \quad \text{and} \quad \int_D \operatorname{rot} u(x)\, dx = 0. \tag{9.39}$$

We mention that one can apply a circularly symmetric mollifier to $\operatorname{rot} u$ to approximate elements of $R^1(D)$ by elements of $H^1(D) \cap R^1(D)$ and elements of $R^1_0(D)$ by elements of $H^1_0(D) \cap R^1_0(D)$.

Proposition 9.4. *We have*

$$u_0 \in R^1_0(D) \Longrightarrow \| \operatorname{rot} e^{\nu t A} u_0 \|_{L^1(D)} \leq \| \operatorname{rot} u_0 \|_{L^1(D)}, \tag{9.40}$$

and

$$u_0 \in R^1(D) \Longrightarrow \| \operatorname{rot} e^{\nu t A} u_0 \|_{L^1(D)} \leq 4 \| \operatorname{rot} u_0 \|_{L^1(D)}. \tag{9.41}$$

Proof. The result (9.40) follows from (9.24) by a standard approximation argument, and then (9.41) follows by the same sort of argument as used for (9.36). □

Here is an associated convergence result.

Proposition 9.5. *We have*

$$u_0 \in R^1_0(D) \Longrightarrow \lim_{\nu \searrow 0} \| \operatorname{rot} u_0 - \operatorname{rot} e^{\nu t A} u_0 \|_{L^1(D)} = 0. \tag{9.42}$$

Furthermore, with $D_a = \{x \in \mathbb{R}^2 : |x| < a\}$, we have for each $a \in (0, 1)$,

$$u_0 \in R^1(D) \Longrightarrow \lim_{\nu \searrow 0} \| \operatorname{rot} u_0 - \operatorname{rot} e^{\nu t A} u_0 \|_{L^1(D_a)} = 0. \tag{9.43}$$

Proof. Take $u_0 \in R^1_0(D)$. Given $\varepsilon > 0$, there exists $v_0 \in H^1_0(D) \cap R^1_0(D)$ such that $\| \operatorname{rot}(u_0 - v_0) \|_{L^1(D)} \leq \varepsilon$. As in (9.16) we have

$$\operatorname{rot} e^{sA} v_0 \in C([0, \infty), L^2(D)), \tag{9.44}$$

so, using the estimate (9.40) with u_0 replaced by $u_0 - v_0$, we have

$$\limsup_{\nu \searrow 0} \| \operatorname{rot} u_0 - \operatorname{rot} e^{\nu t A} u_0 \|_{L^1(D)} \leq 2\varepsilon, \tag{9.45}$$

which gives (9.42).

More generally, given $u_0 \in R^1(D)$, write $u_0 = u_{00} + b f_1$, with $u_{00} \in R^1_0(D)$. Then (9.42) applies to u_{00}, while, as we have noted,

$$e^{\nu t A} f_1 \longrightarrow f_1 \ \text{ in } \ C^\infty(\overline{D}_a), \tag{9.46}$$

for each $a < 1$, so (9.43) follows. □

We now delve further into the concentration phenomenon mentioned after (9.23). To state it, we bring in the space of finite Borel (signed) measures on $\overline{D}$:

$$\mathcal{M}(\overline{D}) = C(\overline{D})'. \tag{9.47}$$

Proposition 9.6. *Given $u_0 \in R^1(D)$, set*

$$b = \int\limits_D \operatorname{rot} u_0(x)\, dx, \tag{9.48}$$

and let μ be the rotationally invariant Borel measure in $\mathcal{M}(\overline{D})$, supported on ∂D, of mass 1, i.e., $1/2\pi$ times arc length on ∂D. Then, for each $t > 0$,

$$\lim_{\nu \searrow 0} \operatorname{rot} e^{\nu t A} u_0 = \operatorname{rot} u_0 - b\mu, \quad \textit{weak}^* \textit{ in } \mathcal{M}(\overline{D}). \tag{9.49}$$

Proof. The bound (9.41) implies $\{\operatorname{rot} e^{\nu t A} u_0\}$ has weak* limit points in $\mathcal{M}(\overline{D})$ as $\nu \searrow 0$. The result (9.43) implies any such weak* limit must be of the form $\operatorname{rot} u_0 - \lambda$, where λ is a measure supported on ∂D. Of course, λ must be rotationally invariant. Then the fact that $\int_D \operatorname{rot} e^{\nu t A} u_0\, dx = 0$ for each $\nu > 0$ uniquely specifies such λ as $b\mu$. ☐

Remark. Given $u_0 \in R^1(D)$, we have

$$b = 2\pi s_0(1). \tag{9.50}$$

We now pass from $\alpha \in C_\flat^\infty(\mathbb{R})$ to $\alpha \in \mathrm{BV}_\flat(\mathbb{R})$, and establish the following complement to Propositions 9.5–9.6.

Proposition 9.7. *Assume $\alpha \in \mathrm{BV}_\flat(\mathbb{R})$ and set $v^\nu(t) = S^\nu \alpha(t)$. Then, for each $t > 0$,*

$$\| \operatorname{rot} v^\nu(t) \|_{L^1(D)} \le \|\alpha\|_{\mathrm{BV}([0,t))}, \tag{9.51}$$

and

$$\int\limits_D \operatorname{rot} v^\nu(t, x)\, dx = \alpha(t-). \tag{9.52}$$

Furthermore, with D_a as in Proposition 9.5, we have for each $a \in (0, 1)$,

$$\lim_{\nu \searrow 0} \int\limits_{D_a} |\operatorname{rot} v^\nu(t, x)|\, dx = 0. \tag{9.53}$$

Therefore, with μ as in Proposition 9.6,

$$\lim_{\nu \searrow 0} \operatorname{rot} v^\nu(t) = \alpha(t-)\mu, \quad \textit{weak}^* \textit{ in } \mathcal{M}(\overline{D}). \tag{9.54}$$

Proof. We use the representation

$$v^{\nu}(t) = \int\limits_{[0,t)} \big(I - e^{\nu(t-s)A}\big) f_1 \, d\alpha(s); \tag{9.55}$$

cf. (2.24) and the remark following its proof. Then (9.51) follows from (9.32), (9.52) follows from (9.33), since we then have

$$\int\limits_{D} \operatorname{rot} v^{\nu}(t,x)\, dx = \int\limits_{[0,t)} d\alpha(s) = \alpha(t-), \tag{9.56}$$

(9.53) follows from (9.46), and (9.54) then follows by the same argument as used for (9.49). □

We make the following remarks regarding the concentration of vorticity exhibited in (9.49) and (9.54). First, it implies that in considering the inviscid limit for the Navier-Stokes equations in domains with boundary, one has to deal analytically with regularity at the level of vortex sheets. This confirms an observation made in [11]; cf. Remark 2 in §3 of that paper. Furthermore, for the factor $\alpha(t-)$ in (9.54) to be nonzero, acceleration must be applied to ∂D; it is this acceleration that is responsible for the formation of the vortex sheet.

10 Variants of results of Bona-Wu

In [1], J. Bona and J. Wu studied the small ν behavior of solutions to (1.1)–(1.4) in the case $\alpha \equiv 0$. Their hypotheses on the initial data were expressed in terms of the vorticity $\omega_0 = \operatorname{rot} u_0$, which, for u_0 satisfying (1.4), is radially symmetric, i.e., $\omega_0(x) = \varpi(|x|)$. They assumed ϖ is continuous on $[0, 1)$, integrable on $[0, 1]$, and satisfies $\int_0^1 r\omega(r)\, dr = 0$. Under such hypotheses, it was shown that $u^{\nu}(t,\cdot) \to u_0$ uniformly on $\overline{D}$. Here we produce several extensions of that result.

To begin, we note that since (by (9.4))

$$u_0(x) = \Big(\frac{1}{r^2}\int_0^r \rho\varpi(\rho)\, d\rho\Big)x^{\perp}, \quad r = |x|, \tag{10.1}$$

these hypotheses imply that

$$u_0 \in C_*(\overline{D}), \tag{10.2}$$

where

$$C_*(\overline{D}) = \big\{u \in C(\overline{D}) : u\big|_{\partial D} = 0\big\}. \tag{10.3}$$

Here is our first extension:

Proposition 10.1. *Let u^ν satisfy* (1.1)–(1.4) *with* $\alpha \equiv 0$ *and assume that* $u_0 \in C_*(\overline{D})$. *Given* $T_0 \in (0, \infty)$, *we have, uniformly in* $t \in [0, T_0]$, *as* $\nu \to 0$,

$$u^\nu(t, \cdot) \longrightarrow u_0, \quad \textit{uniformly on } \overline{D}. \tag{10.4}$$

Proof. Again we have (2.1) and hence, in this situation,

$$u^\nu(t) = e^{\nu t A} u_0. \tag{10.5}$$

The conclusion (10.4) follows from the well known fact that e^{sA} is a strongly continuous semigroup on $C_*(\overline{D})$. □

Before pursuing other results that involve the hypothesis $u_0 = 0$ on ∂D, we present the following extension of Proposition 10.1.

Proposition 10.2. *Let u^ν satisfy* (1.1)–(1.4) *with* $\alpha \equiv 0$ *and*

$$u_0 \in C(\overline{D}). \tag{10.6}$$

Given $T_0 \in (0, \infty)$, *we have, uniformly in* $t \in [0, T_0]$, *as* $\nu \to 0$,

$$u^\nu(t, \cdot) \longrightarrow u_0 \quad \textit{locally uniformly on the interior of } D, \tag{10.7}$$

and

$$\|u^\nu(t, \cdot)\|_{L^\infty(D)} \leq \|u_0\|_{L^\infty(D)}. \tag{10.8}$$

Proof. Again we have (10.5). Now e^{sA} is a contraction semigroup on $C(\overline{D})$, so we have (10.8), but it is not strongly continuous at $s = 0$. To get (10.7), we argue as in the proof of Proposition 7.1. Let $K \subset D$ be compact. Using a partition of unity, write

$$u_0 = u_a + u_b, \quad u_a \in C_*(\overline{D}), \ \ u_b = 0 \quad \text{on a neighborhood } U \text{ of } K. \tag{10.9}$$

Then $u^\nu(t) = e^{\nu t A} u_a + e^{\nu t A} u_b$, and $e^{sA} u_a \to u_a$ uniformly on $\overline{D}$ as $s \searrow 0$. It remains to show that

$$e^{sA} u_b \longrightarrow 0 \ \text{ uniformly on } \ K, \ \text{ as } \ s \searrow 0. \tag{10.10}$$

To see this, set

$$\begin{aligned} w_b(s, x) = e^{sA} u_b(x), &\quad s \geq 0, \ x \in U, \\ 0, &\quad s < 0, \ x \in U. \end{aligned} \tag{10.11}$$

Then w_b solves $(\partial_s - \Delta) w_b = 0$ on $\mathbb{R} \times U$, so hypoellipticity of $\partial_s - \Delta$ implies

$$w_b \in C^\infty(\mathbb{R} \times U). \tag{10.12}$$

This immediately implies (10.10) and hence (10.7). □

We can use the layer potential construction which will be carried out in §11, especially (11.35)–(11.40), to sharpen Proposition 10.2 further, obtaining:

Proposition 10.3. *In the setting of Proposition* 10.2, *we have*

$$u^\nu(t,x) \longrightarrow u_0(x) \tag{10.13}$$

uniformly on

$$\big\{(x,\nu)\colon |x| \le 1-\delta(\nu)\big\}, \tag{10.14}$$

as $t \to 0$, *whenever* $\delta(\nu)$ *satisfies*

$$\frac{\delta(\nu)}{\nu^{1/2}} \longrightarrow \infty \;\; as \;\; \nu \searrow 0. \tag{10.15}$$

Let us return to the setting $u_0|_{\partial D} = 0$, as hypothesized in [1]. Another result of Theorem 2 of [1] is that, under the hypotheses made in the first paragraph of this section, $\operatorname{rot} u^\nu(t,\cdot) \to \omega_0$ in $L^2(D)$-norm; equivalently, $u^\nu(t,\cdot) \to u_0$ in H^1-norm. (Actually, to get such a conclusion one needs to strengthen the hypothesis $\varpi \in L^1([0,1])$ to $\varpi \in L^2([0,1])$.) An alternative route to such a conclusion is to note that, since

$$\mathcal{D}((-A)^{1/2}) = H_0^1(D), \tag{10.16}$$

it follows that

$$u_0 \in H_0^1(D) \Longrightarrow e^{\nu t A}u_0 \to u_0 \quad \text{in} \quad H^1\text{-norm, as } \nu \to 0. \tag{10.17}$$

The following is an extension of this observation.

Proposition 10.4. *Let* u_ν *satisfy* (1.1)–(1.4) *with* $\alpha \equiv 0$. *Take* $\sigma \in [1, 5/2)$ *and assume*

$$u_0 \in H_0^1(D) \cap H^\sigma(D). \tag{10.18}$$

Then, given $T_0 \in (0,\infty)$, *we have, uniformly in* $t \in [0, T_0]$, *as* $\nu \to 0$,

$$u^\nu(t,\cdot) \longrightarrow u_0 \quad in \quad H^\sigma\text{-}norm. \tag{10.19}$$

Proof. This is a consequence of the fact that

$$\mathcal{D}((-A)^{\sigma/2}) = H_0^1(D) \cap H^\sigma(D), \quad \text{for } \sigma \in \Big[1, \frac{5}{2}\Big), \tag{10.20}$$

which, like (4.4)–(4.5), is a special case of results of [14]–[15]. The result (10.20) plus what have by now become familiar arguments yields (10.19). □

11 Boundary layer analysis of $e^{\nu t A}u_0$

In this section we make a detailed analysis, uniformly near ∂D, of the small ν behavior of $e^{\nu t A}u_0$, uniformly on $t \in [0, T]$, in case $u_0 \in C^\infty(\overline{D})$. This is equivalent to understanding the small t behavior of $e^{tA}u_0$.

To start, we emphasize the case $u_0(x) = f_1(x) = x^\perp/2\pi$, later making note of the minor modifications involved in examining the more general case. Note that

$$\begin{aligned} V(t,x) = f_1(x) - e^{tA} f_1(x) \quad &\text{for} \quad t \geq 0,\ x \in D, \\ 0 \qquad\qquad &\text{for} \quad t < 0,\ x \in D \end{aligned} \tag{11.1}$$

solves

$$\begin{aligned} \partial_t V = \Delta V \quad &\text{on} \ \mathbb{R} \times D \\ V\big|_{\mathbb{R}\times\partial D} = \chi_{\mathbb{R}^+}(t) f_1\big|_{\partial D} &= g(t,x). \end{aligned} \tag{11.2}$$

Our task is equivalent to determining the behavior of $V(t,x)$ as $t \searrow 0$.

An argument from §7 is again useful here. Namely, the hypoellipticity of $\partial_t - \Delta$ guarantees interior regularity:

$$V \in C^\infty(\mathbb{R} \times D). \tag{11.3}$$

In particular, since $V = 0$ for $t < 0$, we have for each $m \in \mathbb{N},\ K \subset D$ compact,

$$|V(t,x)| \leq C_{m,K} t^m, \quad \text{for} \quad x \in K. \tag{11.4}$$

Of course, $V(t,x) = f_1(x)$ for $t > 0$ and $x \in \partial D$, so there is a "boundary layer" on which (11.4) fails.

We tackle the problem of analyzing V using the method of layer potentials. Given h supported in $\mathbb{R}^+ \times \partial D$, we set

$$\mathcal{D}h(t,x) = \int_0^\infty \int_{\partial D} h(s,y) \frac{\partial H}{\partial n_y}(t-s,x,y)\, dS(y)\, ds, \quad t \in \mathbb{R},\ x \in D. \tag{11.5}$$

where

$$H(t,x,y) = (4\pi t)^{-1} e^{-|x-y|^2/4t}\, \chi_{\mathbb{R}^+}(t), \tag{11.6}$$

and n_y is the unit outward normal to ∂D at y. It is known that

$$\mathcal{D}h\big|_{\mathbb{R}\times\partial D} = \Big(\frac{1}{2}I + N\Big)h, \tag{11.7}$$

where

$$\begin{aligned} Nh(t,x) \ = \ &\int_0^\infty \int_{\partial D} h(s,y) \frac{\partial H}{\partial n_y}(t-s,x,y)\, dS(y)\, ds, \\ &t \in \mathbb{R},\ x \in \partial D. \end{aligned} \tag{11.8}$$

Cf. [16], Chapter 7, (13.50)–(13.55). Then V in (11.2) is given by

$$V = \mathcal{D}h, \tag{11.9}$$

where h solves

$$\Big(\frac{1}{2}I + N\Big)h = g, \tag{11.10}$$

with g given in (11.2). Such a solution h is in fact given by

$$h = 2(I - 2N + 4N^2 - \cdots)g. \tag{11.11}$$

Not only is this convergent (at least for small t), but

$$N \in OPS^{-1/2}_{1/2,0}(\mathbb{R} \times \partial D), \tag{11.12}$$

so the various terms in the series are progressively smoother. In fact N has further structure as a singular integral operator, exposed in [2] and [3], which implies it has order $-1/2$ on L^p-Sobolev spaces.

To see in an elementary manner that N is a weakly singular integral operator, we note that for $y \in \partial D$, $x \in \overline{D}$, $t > 0$,

$$\partial_{n_y} H(t,x,y) = \frac{1}{\pi}\frac{1}{(4t)^2}\, n(y)\cdot(x-y)e^{-|x-y|^2/4t}. \tag{11.13}$$

This has a relatively weak singularity on $\mathbb{R}^+ \times \partial D \times \partial D$, which when D is the disk can be given rather explicitly, using the fact that

$$n(y) = y \tag{11.14}$$

for $y \in \partial D$, so $n(y)\cdot(x-y) = x\cdot y - 1$. Also $|x-y|^2 = 2 - 2x\cdot y$, so $n(y)\cdot(x-y) = -|x-y|^2/2$, for $x, y \in \partial D$. Hence

$$\begin{aligned}\partial_{n_y} H(t,x,y) &= \frac{1}{8\pi t}\,\frac{|x-y|^2}{4t}\,e^{-|x-y|^2/4t}\\ &= \frac{1}{8\pi t}\,\Phi\Big(\frac{|x-y|}{\sqrt{4t}}\Big), \quad \text{for} \quad x, y \in \partial D,\ t > 0,\end{aligned} \tag{11.15}$$

where

$$\Phi(\lambda) = \lambda^2 e^{-\lambda^2}. \tag{11.16}$$

Note that this is less singular than its counterpart where $y \in \partial D$ and $x \in D$ approaches y radially, by a factor of $|x-y|$. From here on, we denote by

$N(t, x, y)$ the function on $\mathbb{R} \times \overline{D} \times \partial D$ given by (11.13) for $t > 0$, and vanishing for $t < 0$.

Clearly we have

$$\|N(t, x, \cdot)\|_{L^1(\partial D)} \le C t^{-1/2}, \quad t \in \mathbb{R}^+, \ x \in \partial D. \tag{11.17}$$

Hence, with g as in (11.2), $t > 0$,

$$|Ng(t, x)| \le C t^{1/2} \|g\|_{L^\infty}. \tag{11.18}$$

By contrast,

$$\|N(t, x, \cdot)\|_{L^1(\partial D)} \le C t^{-1}, \quad t \in \mathbb{R}^+, \ x \in \overline{D}. \tag{11.19}$$

We can deduce that the solution h to (11.10) satisfies

$$h = 2g + h^b, \tag{11.20}$$

with $h^b(t, x)$ supported in $t \in \mathbb{R}^+$ and (at least for small t)

$$|h^b(t, x)| \le C t^{1/2}. \tag{11.21}$$

Hence

$$V(t, x) = \mathcal{D}h(t, x) = 2\mathcal{D}g(t, x) + \mathcal{D}h^b(t, x). \tag{11.22}$$

To estimate $\mathcal{D}h^b(t, x)$, (11.19) is not so useful; instead we argue as follows. Denote by PI the solution operator to (11.2), so (11.7) gives

$$\mathrm{PI}\, g = \mathcal{D}h, \quad g = \Big(\frac{1}{2} I + N\Big) h. \tag{11.23}$$

Similarly

$$\mathcal{D}h^b = \mathrm{PI}\, g^b, \quad g^b = \Big(\frac{1}{2} I + N\Big) h^b. \tag{11.24}$$

As usual, $g^b(t, x)$ is supported in $t \in \mathbb{R}^+$. Also, (11.18) and (11.21) give

$$|g^b(t, x)| \le C t^{1/2}, \tag{11.25}$$

for $t > 0$. Then the maximum principle for solutions to the heat equation gives

$$|\,\mathrm{PI}\, g^b(t, x)| \le C t^{1/2}, \tag{11.26}$$

and by (11.24) this is the estimate we have on $\mathcal{D}h^b(t, x)$. We have established the following.

Proposition 11.1. *For V given by* (11.1) *and g in* (11.2)*, we have*

$$|V(t,x) - 2\mathcal{D}g(t,x)| \leq Ct^{1/2}, \quad \forall x \in \overline{D}. \tag{11.27}$$

To get finer approximations to $V(t,x)$, we use more terms in (11.11). Write

$$h = 2g_k + h_k^b, \tag{11.28}$$

where

$$\begin{aligned} g_k &= \sum_{j=0}^{k} (-2N)^j g, \\ h_k^b &= 2(-2N)^{k+1} \sum_{j=0}^{\infty} (-2N)^j g. \end{aligned} \tag{11.29}$$

We have, for small $t > 0$,

$$|N^j g(t,x)| \leq (Ct^{1/2})^j \|g\|_{L^\infty}, \quad |h_k^b(t,x)| \leq (Ct^{1/2})^{k+1} \|g\|_{L^\infty}. \tag{11.30}$$

Furthermore, N^j has smoothing properties, leading to the fact that $N^j g$ and h_k^b are supported in $t \in \mathbb{R}^+$ and

$$N^j g \in C^{j/2-\varepsilon}(\mathbb{R} \times \partial D), \quad h_k^b \in C^{(k+1)/2-\varepsilon}(\mathbb{R} \times \partial D). \tag{11.31}$$

Now, parallel to (11.22), we have

$$V(t,x) = 2\mathcal{D}g_k(t,x) + \mathcal{D}h_k^b(t,x), \tag{11.32}$$

where $\mathcal{D}h_k^b(t,x)$ is supported on $\mathbb{R}^+ \times \overline{D}$ and, parallel to (11.24),

$$\mathcal{D}h_k^b = \mathrm{PI}\, g_k^b, \quad g_k^b = \Big(\frac{1}{2}I + N\Big) h_k^b. \tag{11.33}$$

Thus $g_k^b \in C^{(k+1)/2-\varepsilon}(\mathbb{R} \times \partial D)$, and consequently $\mathrm{PI}\, g_k^b$ has this degree of regularity on $\mathbb{R} \times \overline{D}$. In conclusion:

Proposition 11.2. *In the setting of Proposition* 11.1*, define g_k by* (11.29)*. Then, for each $k \in \mathbb{N}$, we have* (11.32)*, with remainder*

$$\mathcal{D}h_k^b \in C^{(k+1)/2-\varepsilon}(\mathbb{R} \times \overline{D}), \tag{11.34}$$

supported in $\mathbb{R}^+$.

Of course there is a similar treatment of $e^{tA}u_0$ when $u_0 \in C^\infty(\overline{D})$. One can take an extension $u_0 \in C^\infty(\mathbb{R}^2)$ (polynomially bounded, say), set

$$U_0(t,x) = e^{t\Delta}u_0(x), \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^2, \tag{11.35}$$

and then note that

$$\begin{aligned} W(t,x) = U_0(t,x) - e^{tA}u_0(x) &\quad \text{for } t \geq 0,\ x \in D \\ 0 \qquad\qquad &\quad \text{for } t < 0,\ x \in D \end{aligned} \tag{11.36}$$

solves

$$\begin{aligned} &\partial_t W = \Delta W \quad \text{on } \mathbb{R} \times D, \\ &W\big|_{\mathbb{R}\times\partial D} = \chi_{\mathbb{R}^+}(t)U_0(t,x)\big|_{\partial D} = \tilde{g}(t,x), \end{aligned} \tag{11.37}$$

from which one has straightforward parallels of (11.3)–(11.34). One has (for t small)

$$W = \mathcal{D}\tilde{h}, \tag{11.38}$$

where $\tilde{h}$ solves

$$\Big(\frac{1}{2}I + N\Big)\tilde{h} = \tilde{g}, \tag{11.39}$$

so

$$\tilde{h} = 2(I - 2N + 4N^2 - \cdots)\tilde{g}. \tag{11.40}$$

As with g, we have that $\tilde{g}$ is piecewise smooth, with a jump across $\{t = 0\}$.

We can go further, and make the construction (11.35)–(11.40) for more general u_0, such as $u_0 \in C(\overline{D})$. In this case, one takes a polynomially bounded extension $u_0 \in C(\mathbb{R}^2)$ to define U_0 in (11.35). Then $\tilde{g}$ in (11.37) is piecewise continuous, with a jump across $\{t = 0\}$. With $\tilde{h}$ given by (11.38)–(11.40), estimates on $\tilde{h}$ parallel to those on h (given by (11.11)) hold. In particular, parallel to (11.20) we have $\tilde{h} = 2\tilde{g} + \tilde{h}^b$, and $\tilde{h}^b$ has a treatment analogous to (11.21)–(11.26). Thus we have the following analogue of Proposition 11.1, which we state explicitly, since it implies Proposition 10.3, as advertized in §10.

Proposition 11.3. *Given $u_0 \in C(\overline{D})$, define $W(t,x)$ by* (11.35)–(11.36)*, and define $\tilde{g}(t,x)$ by* (11.37)*. Then we have*

$$|W(t,x) - 2\mathcal{D}\tilde{g}(t,x)| \leq Ct^{1/2}, \quad \forall x \in \overline{D}. \tag{11.41}$$

Similarly Proposition 11.2 extends to this setting, with minimal changes in the proof.

12 Boundary layer analysis of $S^\nu\alpha(t)$

Let us assume $\alpha \in \mathrm{BV}_\flat(\mathbb{R})$. We can apply (11.22)–(11.27) to analyze

$$S^\nu\alpha(t) = \int_0^t (I - e^{\nu(t-s)A}) f_1\, d\alpha(s) = \int_0^t V(\nu(t-s), x)\, d\alpha(s), \tag{12.1}$$

obtaining

$$S^\nu\alpha(t) = 2\int_0^t \mathcal{D}g(\nu(t-s), x)\, d\alpha(s) + \mathcal{R}_\nu(t), \tag{12.2}$$

with g given by (11.2) and

$$\|\mathcal{R}_\nu(t)\|_{L^\infty(D)} \le Ct^{1/2}\nu^{1/2}\|\alpha\|_{\mathrm{TV}([0,t])}. \tag{12.3}$$

Next, we can apply (11.32)–(11.34) to analyze

$$\begin{aligned} S^\nu\alpha(t) &= -\int_0^t \nu A e^{\nu(t-s)A} f_1\, \alpha(s)\, ds \\ &= -\nu\Delta f_1 \int_0^t \alpha(s)\, ds + \nu \int_0^t \Delta V(\nu(t-s), x)\alpha(s)\, ds. \end{aligned} \tag{12.4}$$

Note that $\Delta f_1 = 0$, so we have

$$S^\nu\alpha(t) = 2\nu \int_0^t \Delta\mathcal{D}g_k(\nu(t-s), x)\alpha(s)\, ds + \nu\mathcal{R}^2_{\nu,k}(t, x), \tag{12.5}$$

with

$$\mathcal{R}^2_{\nu,k} \in C^{(k+1)/2-2-\varepsilon}(\mathbb{R} \times \overline{D}), \tag{12.6}$$

supported in $t \in \mathbb{R}^+$, given $\alpha \in L^1_b(\mathbb{R})$. In particular,

$$\|\nu\mathcal{R}^2_{\nu,k}(t, \cdot)\|_{L^\infty(D)} \le c\nu^{(k+1)/2-1-\varepsilon}\|\alpha\|_{L^1([0,t])}. \tag{12.7}$$

The significance of these estimates is that the principal term on the right side of (12.2) is an explicit integral (and its counterpart in (12.5) is more or less explicit).

13 Concentric rotating circles

Here we extend the analysis of the previous sections to the case where the disk D is replaced by the annulus

$$\mathcal{A} = \{x \in \mathbb{R}^2 : \rho < |x| < 1\}, \tag{13.1}$$

for some $\rho \in (0, 1)$. The results in this section are an extension of work described in [4].

Thus we consider solutions to the Navier-Stokes equations (1.1) on $\mathbb{R}^+ \times \mathcal{A}$, with no-slip boundary data on the two components of $\partial\mathcal{A}$, which might be rotating independently:

$$\begin{aligned} u^\nu(t,x) &= \frac{\alpha_1(t)}{2\pi} x^\perp, \quad |x| = 1,\ t > 0, \\ &\frac{\alpha_2(t)}{2\pi} x^\perp, \quad |x| = \rho,\ t > 0, \end{aligned} \tag{13.2}$$

and with circularly symmetric initial data

$$u^\nu(0,x) = u_0(x), \quad \operatorname{div} u_0 = 0, \quad u_0 \parallel \partial\mathcal{A}, \tag{13.3}$$

where again circular symmetry is defined by (1.4), and entails the formulas (1.6) and (1.7). Proposition 1.1 immediately extends to this setting, so $u^\nu(t,x)$ satisfies (1.11) for all $t > 0$ and is specified as the solution to the linear PDE

$$\partial_t u^\nu = \nu \Delta u^\nu, \tag{13.4}$$

on $\mathbb{R}^+ \times \mathcal{A}$, with boundary data (13.2) and initial data (13.3). The material of §2 easily extends. We can represent the solution to (13.2)–(13.4) as

$$u^\nu(t) = e^{\nu t \Delta} u_0 + S^\nu(\alpha_1, \alpha_2)(t), \tag{13.5}$$

where A is given by

$$\mathcal{D}(A) = H^2(\mathcal{A}) \cap H_0^1(\mathcal{A}), \quad Au = \Delta u \quad \text{for} \quad u \in \mathcal{D}(A), \tag{13.6}$$

and

$$S^\nu : C_\flat^\infty(\mathbb{R}) \oplus C_\flat^\infty(\mathbb{R}) \longrightarrow C_\flat^\infty(\mathbb{R} \times \overline{\mathcal{A}}) \tag{13.7}$$

is defined as $S^\nu(\alpha_1, \alpha_2) = v^\nu$, where v^ν is the solution to (13.2)–(13.4) that vanishes for $t < 0$ (as in (2.5)). As before, we have extensions of S^ν such as

$$S^\nu : C_\flat(\mathbb{R}) \oplus C_\flat(\mathbb{R}) \longrightarrow C_\flat(\mathbb{R} \times \overline{\mathcal{A}}), \tag{13.8}$$

and also analogues of (2.10)–(2.15). We obtain analogues of (2.17)–(2.22) as follows. With $v^\nu = S^\nu(\alpha_1, \alpha_2)$ defined above, set

$$w^\nu(t,x) = v^\nu(t,x) - \Phi(t,x) \tag{13.9}$$

on $[0, \infty) \times \mathcal{A}$, where $\Phi(t, x)$ is defined for each $t \geq 0$ by

$$\Delta\Phi(t, \cdot) = 0 \ \text{ on } \ \mathcal{A}, \qquad \Phi(t, x) = \frac{\alpha_j(t)}{2\pi} x^{\perp} \ \text{ on } \ B_j, \tag{13.10}$$

where $B_1 = \{x \in \mathbb{R}^2 \colon |x| = 1\}$, $B_2 = \{x \in \mathbb{R}^2 \colon |x| = \rho\}$. Then w^ν solves

$$\partial_t w^\nu = \nu \Delta w^\nu - \partial_t \Phi, \qquad w^\nu(0, x) = 0, \qquad w^\nu\big|_{\mathbb{R}^+ \times \partial\mathcal{A}} = 0, \tag{13.11}$$

so by Duhamel's formula we have the following variant of (2.20):

$$w^\nu(t) = -\int_0^t e^{\nu(t-s)A} \partial_s \Phi(s, x)\, ds. \tag{13.12}$$

Hence

$$\begin{aligned} S^\nu(\alpha_1, \alpha_2)(t) &= \Phi(s, x) - \int_0^t e^{\nu(t-s)A} \partial_s \Phi(s, x)\, ds \\ &= \int_0^t \big(I - e^{\nu(t-s)A}\big) \partial_s \Phi(s, x)\, ds. \end{aligned} \tag{13.13}$$

As in §2, we first get these identities for $\alpha_j \in C_\flat^\infty(\mathbb{R})$, and then we can extend the validity of these formulas via limiting arguments. We can obtain formulas more closely resembling (2.21) as follows. Vector fields on $\mathcal{A}$ of the form $s_0(|x|)x^\perp$ that are harmonic are linear combinations of $x^\perp$ and $|x|^{-2}x^\perp$, so $\Phi(t, x)$, defined by (13.5), is given by

$$\Phi(t, x) = \beta_1(t) f_1(x) + \beta_2(t) f_2(x), \tag{13.14}$$

with

$$f_1(x) = \frac{x^\perp}{2\pi}, \qquad f_2(x) = \frac{x^\perp}{2\pi |x|^2}, \tag{13.15}$$

and $\beta_j(t)$ given by

$$\beta_1(t) + \beta_2(t) = \alpha_1(t), \qquad \beta_1(t) + \frac{\beta_2(t)}{\rho^2} = \alpha_2(t). \tag{13.16}$$

Solving for β_j and plugging into (13.13), we obtain

$$\begin{aligned} &S^\nu(\alpha_1, \alpha_2)(t) \\ &= \int_0^t \Big(I - e^{\nu(t-s)A}\Big) \Big[\frac{f_1 - \rho^2 f_2}{1 - \rho^2} \alpha_1'(s) - \frac{\rho^2 (f_1 - f_2)}{1 - \rho^2} \alpha_2'(s)\Big]\, ds, \end{aligned} \tag{13.17}$$

first for $\alpha_j \in C_\flat^\infty(\mathbb{R})$, then in more general cases. For example, parallel to Proposition 2.1, we have

$$S^\nu : \mathrm{BV}_\flat(\mathbb{R}) \oplus \mathrm{BV}_\flat(\mathbb{R}) \longrightarrow C_\flat(\mathbb{R}, X), \tag{13.18}$$

whenever X is a Banach space of functions on $\mathcal{A}$ such that $f_1, f_2 \in X$ and $\{e^{tA} : t \geq 0\}$ is a strongly continuous semigroup on X. In such a case,

$$\begin{aligned} &S^\nu(\alpha_1, \alpha_2)(t) \\ &= \int_{I(t)} \Big(I - e^{\nu(t-s)A}\Big)\Big[\frac{f_1 - \rho^2 f_2}{1-\rho^2}\, d\alpha_1(s) - \frac{\rho^2(f_1 - f_2)}{1-\rho^2}\, d\alpha_2(s)\Big], \end{aligned} \tag{13.19}$$

where we can take $I(t) = [0, t]$ or $I(t) = [0, t)$. Results of §§3–8 extend in a straightforward way to the current setting.

To extend the results of §9, we need to do a little more work. First we have the following variant of Proposition 9.1.

Proposition 13.1. *Assume $u^\nu(t, x) = s^\nu(t, |x|)x^\perp$ has the form* (13.5) *with $u_0 \in L^2(\mathcal{A})$ and $\alpha_j \in C_\flat^\infty(\mathbb{R})$. Then $\omega^\nu = \operatorname{rot} u^\nu$ belongs to $C^\infty((0, \infty) \times \overline{\mathcal{A}})$ and satisfies the following:*

$$\partial_t \omega^\nu = \nu \Delta \omega^\nu \ \text{ on } \ (0, \infty) \times \mathcal{A}, \tag{13.20}$$

and

$$n \cdot \nabla \omega^\nu(t, x) = (-1)^{j-1}|x|\frac{\alpha_j'(t)}{2\pi\nu} \ \text{ on } \ B_j, \tag{13.21}$$

with B_j as in (13.10) *and n the outward unit normal to $\partial\mathcal{A}$. Also*

$$\int_{\mathcal{A}} \omega^\nu(t, x)\, dx = \alpha_1(t) - \rho\alpha_2(t). \tag{13.22}$$

Proof. The results (13.20) and (13.22) are proven just as in Proposition 9.1. However, (13.21) does not follow as easily as (9.6), because $\partial\mathcal{A}$ has two components. Instead, we calculate as follows. With $\tilde{n} = x/|x|$, we have

$$\begin{aligned} \tilde{n} \cdot \nabla \omega^\nu(t, x) &= \frac{\partial}{\partial r}\varpi^\nu(t, r) \\ &= \frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r} + 2\Big)s^\nu(t, r) \\ &= \Big(r\frac{\partial^2}{\partial r^2} + 3\frac{\partial}{\partial r}\Big)s^\nu(t, r). \end{aligned} \tag{13.23}$$

Note that $\tilde{n} = (-1)^{j-1} n$ on B_j. Also we have

$$\begin{aligned} \Delta u^\nu &= (\Delta s^\nu) x^\perp + 2\nabla s^\nu \cdot \nabla x^\perp \\ &= \Big(\Delta s^\nu + \frac{2}{r}\partial_r s^\nu\Big) x^\perp \\ &= \Big(\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r}\Big) s^\nu \, x^\perp, \end{aligned} \tag{13.24}$$

while

$$\Delta u^\nu\big|_{\mathbb{R}^+ \times B_j} = \frac{1}{\nu}\partial_t u^\nu\big|_{\mathbb{R}^+\times B_j} = \frac{\alpha_j'(t)}{2\pi\nu}\, x^\perp. \tag{13.25}$$

Comparison of (13.23)–(13.25) yields (13.21). □

From here we readily extend Proposition 9.2, obtaining

$$\begin{aligned} u_0 \in H_0^1(\mathcal{A}), \quad u_0(x) = s_0(|x|)x^\perp, \quad \omega_0 = \operatorname{rot} u_0, \\ \omega^\nu(t) = \operatorname{rot} e^{\nu t A} u_0 \Longrightarrow \omega^\nu(t) = e^{\nu t A_N}\omega_0. \end{aligned} \tag{13.26}$$

and the results described in Propositions 9.3–9.5 readily extend to the current setting. In particular, with

$$R^1(\mathcal{A}) = \big\{u_0 \in L^2(\mathcal{A}) : u_0(x) = s_0(|x|)x^\perp,\ \operatorname{rot} u_0 \in L^1(\mathcal{A})\big\}, \tag{13.27}$$

we have

$$u_0 \in R^1(\mathcal{A}) \Longrightarrow \|\operatorname{rot} e^{\nu t A} u_0\|_{L^1(\mathcal{A})} \le 4 \|\operatorname{rot} u_0\|_{L^1(\mathcal{A})}. \tag{13.28}$$

Consequently, given $u_0 \in R^1(\mathcal{A})$, the family $\{\operatorname{rot} e^{\nu t A} u_0\}$ has weak* limit points in $\mathcal{M}(\overline{\mathcal{A}})$ as $\nu \searrow 0$, for each $t \in (0, \infty)$. If we compare the integral

$$\int_{\mathcal{A}} \operatorname{rot} u_0(x)\, dx = 2\pi\big[s_0(1) - \rho s_0(\rho)\big] \tag{13.29}$$

which need not be zero, with

$$\int_{\mathcal{A}} \operatorname{rot} e^{\nu t A} u_0 \, dx = 0, \quad \forall\, \nu, t > 0, \tag{13.30}$$

which follows from (13.22), we see there is a concentration phenomenon, such as described in Propositions 9.6–9.7. We will establish the following variant of Proposition 9.6.

Proposition 13.2. *Let μ_j be the rotationally invariant Borel measures of mass 1 on the components B_j of $\partial\mathcal{A}$. Then, given $u_0 \in R^1(\mathcal{A})$, we have for each $t > 0$,*

$$\lim_{\nu\searrow 0} \operatorname{rot} e^{\nu t A} u_0 = \operatorname{rot} u_0 - 2\pi s_0(1)\mu_1 + 2\pi\rho s_0(\rho)\mu_2, \tag{13.31}$$

weak in $\mathcal{M}(\overline{\mathcal{A}})$.*

We see from (13.29)–(13.30) that the measure concentrated on $\partial\mathcal{A}$ on the right side of (13.31) has the correct integral against 1. The fact that $\partial\mathcal{A}$ has two connected components requires us to devote greater effort than was needed in Proposition 9.6 to proving (13.31). We will prove (13.31) with the aid of the following localization result, which is of independent interest. To state it, let A_D stand for the operator denoted A in §9:

$$\mathcal{D}(A_D) = H^2(D) \cap H_0^1(D), \quad A_D v = \Delta v \quad \text{for} \quad v \in \mathcal{D}(A_D). \tag{13.32}$$

Proposition 13.3. *Consider $u_0 \in R^1(\mathcal{A})$ and $v_0 \in R^1(D)$, and assume*

$$u_0(x) = v_0(x) \ \textit{for} \ x \in \mathcal{O} = \big\{x \in \mathcal{A} : |x| > (1+\rho)/2\big\}. \tag{13.33}$$

Also set $\mathcal{O}_1 = \big\{x \in \mathcal{A} : |x| > (2+\rho)/3\big\}$. Then, for each $t > 0$,

$$e^{\nu t A} u_0 - e^{\nu t A_D} v_0 \longrightarrow 0 \ \textit{in} \ C^\infty(\overline{\mathcal{O}}_1), \tag{13.34}$$

as $\nu \searrow 0$.

Proof. Define W on $\mathbb{R} \times \mathcal{O}$ by

$$\begin{aligned} W(t,x) = e^{tA}u_0 - e^{tA_D}v_0, &\quad t \geq 0, \\ 0, &\quad t < 0. \end{aligned} \tag{13.35}$$

Then $e^{\nu t A}u_0(x) - e^{\nu t A_D}v_0(x) = W(\nu t, x)$. Note that W in (13.35) solves

$$\partial_t W = \Delta W \ \text{ on } \ \mathbb{R} \times \mathcal{O}, \quad W\big|_{\mathbb{R}\times B_1} = 0. \tag{13.36}$$

Standard results on regularity up to the boundary give

$$W \in C^\infty(\mathbb{R} \times \overline{\mathcal{O}}_1), \tag{13.37}$$

which in turn gives (13.34). □

We use Proposition 13.3 to prove Proposition 13.2. Any weak* limit point of $\{\text{rot}\, e^{\nu t A} u_0\}$ as $\nu \searrow 0$ must have the for $\text{rot}\, u_0 + \lambda$, where λ is a signed measure supported on $\partial \mathcal{A}$. If we take v_0 as in Proposition 13.3 and apply Proposition 9.6, we have $\text{rot}\, e^{\nu t A_D} v_0$ tending to $\text{rot}\, v_0 - 2\pi s_0(1)\mu_1$ weak* in $\mathcal{M}(\overline{D})$. By (13.34) we see that $\text{rot}\, u_0 + \lambda$ must coincide with this measure when restricted to $\overline{\mathcal{O}}_1$. Given this, (13.31) now follows from the previous comments about the integral against 1, plus rotational invariance.

In a similar fashion we have the following variant of Proposition 9.7.

Proposition 13.4. *Assume* $\alpha_j \in \text{BV}_\flat(\mathbb{R})$ *and set* $v^\nu(t) = S^\nu(\alpha_1, \alpha_2)(t)$. *Then we have, weak* in* $\mathcal{M}(\overline{\mathcal{A}})$,

$$\lim_{\nu \searrow 0} \text{rot}\, v^\nu(t) = \alpha_1(t-)\mu_1 - \rho\alpha_2(t-)\mu_2, \tag{13.38}$$

for each $t > 0$, *with* μ_j *as in Proposition* 13.2.

This concludes our discussion of extensions of results of §9. Extensions of results of §§10–12 to the current setting are straightforward.

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