

On the total curvatures of a tame function

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Abstract. Given a definable function $f: \mathbb{R}^n \mapsto \mathbb{R}$, enough differentiable, we study the continuity of the total curvature function $t \mapsto K(t)$, total curvature of the level $f^{-1}(t)$, and the total absolute curvature function $t \mapsto |K|(t)$, total absolute curvature of the level $f^{-1}(t)$. We show they admits at most finitely many discontinuities.

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1 Introduction

One of the nicest feature of o-minimal structures expanding the ordered field of real numbers is that taking the derivative of a definable function (in a given such o-minimal structure) provides a definable function (in the same given o-minimal structure). Unfortunately the inverse operation is somehow much more delicate, integration and measure lead to problems even in some of their simplest aspects.

In the setting of subanalytic geometry some measure theoretical aspects, density, Lipschitz-killing curvatures, of the subanalytic sets had been studied [BB, Com, Fu]. Still in this context, the k -dimensional volume of a global subanalytic subset of $\mathbb{R}^n$, lying in a globally subanalytic family of subsets of $\mathbb{R}^n$ of dimension at most k , is, when finite, a log-analytic function in the parameter, as proved in [LR, CLR]. This is already an issue since the logarithmic contribution cannot be avoided and so the functions carrying the quantitative aspect of the variation of the volume in the parameter of the family are already outside of the structure.

In the world of non-polynomially bounded o-minimal structures expanding the real numbers, almost nothing similar to the statement in the globally subanalytic

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context is known in whole generality. Let us nevertheless mention the results of [Le, Ka], proved independently, namely the definability of the set of the parameters at which the 2-dimensional volume of a definable family of plane definable subsets is finite.

Let us go closer to the goal of this note. Given a definable function $f: \mathbb{R}^n \mapsto \mathbb{R}$, that is C^l with $l \geq 2$, to each regular level t we associate two real numbers, namely, $K(t)$, the total curvature of the level $f^{-1}(t)$ oriented by the unitary gradient field of f , and, $|K|(t)$, the total absolute curvature of the level $f^{-1}(t)$. When n is odd and $f^{-1}(t)$ is compact and connected, the Gauss-Bonnet-Chern Theorem states that $K(t)$ is just the Euler Characteristic of $f^{-1}(t)$ (modulo a constant depending only on n). Somehow $K(t)$ and $|K|(t)$ have a connection with the topological equisingularity type of $f^{-1}(t)$, even when $f^{-1}(t)$ is no longer compact. Thus knowing the variation as a function of t of these total curvatures functions could give some information about the equisingularity of the family of the levels. This was the first motivation of this study (see [Gr] for some results in this direction).

From the measure theoretical point of view, these total curvatures are just weighted $n-1$ -dimensional volumes of a definable 1-parameter family of subsets of $\mathbb{S}^{n-1}$. So how do the total curvature functions in the parameter behave? As already said, for the level of generality we want to deal with, there is no hope (yet!) to provide some quantitative information about the variation of these functions of t . Nevertheless we propose in this note to study some qualitative properties of the functions $t \rightarrow K(t)$ and $t \rightarrow |K|(t)$, and we will find some. We actually proved

Theorem. *Let $f: \mathbb{R}^n \mapsto \mathbb{R}$ be a C^l , $l \geq 2$, definable function.*

- (1) *the function $t \rightarrow |K|(t)$ admits at most finitely many discontinuities.*
- (2) *If the function $t \rightarrow |K|(t)$ is continuous at a regular value c of f , so is the function $t \rightarrow K(t)$.*

The paper is organized as follows.

In Section 2 we provide some definitions, conventions and notations that will be used in the rest of the note.

Section 3 recalls what the total curvature and the total absolute curvature of a hypersurface are and what are the connections with linear orthogonal projections onto oriented lines.

In Section 4 we just define the Gauss map of a given function and states some of its elementary properties in the frame established in Section 3.

Section 5 is devoted to Hausdorff limits of Gauss images since they will be the key tool of our main result.

Theorem 6.4 and Proposition 6.6 are the main results of Section 6, and are proved with the help of some preliminary work.

In Section 7 we state results of the same flavor as those of Section 6, but for what we named the total λ -curvature and total absolute λ -curvature.

We finish this paper with some remarks in Section 8.

2 Notation – convention

Let $\mathbb{R}^n$ be the real n -dimensional affine space endowed with its Euclidean metric. The scalar product will be denoted by $\langle \cdot, \cdot \rangle$.

Let $\mathbb{B}_R^n$ be the open ball of $\mathbb{R}^n$ centered at the origin and of radius $R > 0$.

Let $\mathbb{S}_R^{n-1}$ be the $(n - 1)$ -sphere centered at the origin and of radius $R > 0$.

Let $\mathbb{S}^{n-1}$ be unit ball of $\mathbb{R}^n$ endowed with the induced Euclidean metric and let d_S be the intrinsic distance function on $\mathbb{S}^{n-1}$.

Let d_k be the k -dimensional Hausdorff measure of $\mathbb{R}^n$ with $k \in \{1, \dots, n\}$.

Let us recall briefly what an o-minimal structure is.

An *o-minimal structure* $\mathcal{M}$ *expanding the ordered field of real numbers* is a collection $(\mathcal{M}_p)_{p \in \mathbb{N}}$, where $\mathcal{M}_p$ is a set of subsets of $\mathbb{R}^p$ satisfying the following axioms

- 1) For each $p \in \mathbb{N}$, $\mathcal{M}_p$ is a boolean sub-algebra of subsets of $\mathbb{R}^p$.
- 2) If $A \in \mathcal{M}_p$ and $B \in \mathcal{M}_q$, then $A \times B \in \mathcal{M}_{p+q}$.
- 3) If $\pi: \mathbb{R}^{p+1} \mapsto \mathbb{R}^p$, is the projection on the first p factors, given any $A \in \mathcal{M}_{p+1}$, $\pi(A) \in \mathcal{M}_p$.
- 4) The algebraic subsets of $\mathbb{R}^p$ belongs to $\mathcal{M}_p$.
- 5) $\mathcal{M}_1$ consists exactly of the finite unions of points and intervals.

So the smallest o-minimal structure is the structure of the semi-algebraic subsets.

Assume that such an o-minimal structure $\mathcal{M}$ is given for the rest of this article.

A subset A of $\mathbb{R}^p$ is a *definable subset* (in the given o-minimal structure) of $\mathbb{R}^p$, if $A \in \mathcal{M}_p$.

A mapping $g: X \mapsto \mathbb{R}^q$, where $X \subset \mathbb{R}^p$, is a *definable mapping* (or just definable, for short) if its graph is a definable subset of $\mathbb{R}^{p+q}$.

The reader may refer to [Cos, vD1, vDM] to learn more on the properties of definable subsets and definable mappings.

Let Z be a connected definable subset of $\mathbb{R}^n$. The dimension of Z , $\dim Z$, is well defined.

A point $z_0 \in Z$ is smooth if there exists a neighborhood $U \subset \mathbb{R}^n$ of z_0 , such that $U \cap Z$ is diffeomorphic to $\mathbb{R}^k$, for an integer $k \leq \dim Z$. The property of being smooth of a given dimension is a locally open property once Z is equipped with the induced topology.

A point z_0 which is not smooth is called singular. The set of such points is definable.

Let S_1 and S_2 be respectively C^1 definable submanifolds of $\mathbb{R}^n$ and of $\mathbb{R}^p$. Let $g: S_1 \mapsto S_2$ be a C^1 definable function. A point x_0 is a smooth or regular point of g if the rank $d_{x_0}g$ is maximum in a neighborhood of x_0 .

A critical or singular point x_0 of g is a point at which $d_{x_0}g$ is not of maximum rank. The set of such points is definable, denoted by $\mathbf{crit}(g)$.

By abuse of language, we will talk about the rank of the mapping g at a point x_0 to mean the rank of the differential $d_{x_0}g$.

A *definable family* $\mathcal{E} = (E_t)_{t \in T}$ of subsets of a definable submanifold $S \subset \mathbb{R}^n$, with parameter space $T \subset \mathbb{R}^m$ does not only mean that E_t is a definable subset of S , but that the subset $\{(x, t) \in S \times T : x \in E_t\}$ is definable. Equivalently it means it is the family of the (projection onto S of the) fibers of a definable mapping.

3 Total curvature and total absolute curvature of a connected orientable hypersurface

Let M be a definable connected, C^2 hypersurface of $\mathbb{R}^n$. Assume M is orientable and the orientation is given by a C^1 map, $M \ni x \rightarrow \nu_M(x) \in \mathbb{S}^{n-1}$. The map ν_M is definable.

Let $\mathcal{U} \subset \mathbb{S}^{n-1}$ be defined as $\nu_M(M \setminus \mathbf{crit}(\nu_M))$, where $\mathbf{crit}(\nu_M)$ is the set of critical points of ν_M . The subset $\mathcal{U}$ is a definable open subset and, when not empty, for any $u \in \mathcal{U}$, $\nu_M^{-1}(u)$ is finite. By Gabrielov uniformity theorem, there exists a positive integer N_M , such that $\#\nu_M^{-1}(u) \leq N_M$ for any $u \in \mathcal{U}$.

Since $\mathcal{U}$ is a finite disjoint union of open definable subsets U_i , $i = 1, \dots, d$, let

$$M_i = \nu_M^{-1}(U_i) \setminus \mathbf{crit}(\nu_M)$$

which is a definable subset of M . Let $s(i)$ be the number of points in a fiber above any $u \in U_i$.

Let $k_M(x)$ be the Gauss curvature at $x \in M$, that is the value of the determinant of the Jacobian matrix of ν_M . We make the convention that the $(n-1)$ -dimensional volume of the empty set is 0.

Proposition 3.1. *The total absolute curvature of M is*

$$|K|_M := \int_M |k_M(x)| \, d v_{n-1}(x) = \sum_i s(i) \operatorname{vol}_{n-1}(U_i).$$

Proof. If the maximal rank of v_M is at most $n - 2$, then $\mathbf{crit}(v_M) = M$ and so we deduce $|K|_M = 0$.

So assume that the maximal rank of v_M is $n - 1$.

Let us denote by **Jac**, the Jacobian of any mapping (when it makes sense). Since the mapping v_M is of maximal rank, the set of its critical values of v_M , that is $v_M(\mathbf{crit}(v_M))$, is of codimension at least 1 in $\mathbb{S}^{n-1}$. Each U_i is open and so is each M_i in M .

We assume first that each connected component of M_i is simply connected.

Then $M_i = M_i^{(1)} \sqcup \dots \sqcup M_i^{s(i)}$ and v_M induces a diffeomorphism from $M_i^{(j)}$ onto U_i , for each $j = 1, \dots, s(i)$. Thus we find

$$\operatorname{vol}_{n-1}(U_i) = \int_{U_i} d v_{n-1} = \int_{M_i^{(j)}} \mathbf{Jac}(v_M(x)) \, d v_{n-1}(x), \text{ for each } j.$$

Thus we deduce

$$\int_{M_i} \mathbf{Jac}(v_M(x)) \, d v_{n-1}(x) = s(i) \operatorname{vol}_{n-1}(U_i).$$

As another consequence of this fact, if we define M^* as $M \setminus (\sqcup_i M_i)$, then M^* is definable, and since $v_M(M^*)$ is at most of dimension $n - 2$ (meaning that $\mathbf{Jac}(v_M)$ is zero on a definable dense open set of M^* if $\dim M^* = n - 1$), we obtain

$$\int_{M^*} \mathbf{Jac}(v_M(x)) \, d v_{n-1}(x) = 0.$$

Then

$$|K|_M = \int_M \mathbf{Jac}(v_M(x)) \, d v_{n-1}(x) = \sum_i \int_{M_i} \mathbf{Jac}(v_M(x)) \, d v_{n-1}(x),$$

which is the desired result.

In the general situation, by the cylindrical decomposition theorem, there exists a closed subset N_i of M_i of dimension at most $n - 2$, such that each connected component of $M_i \setminus N_i$ is simply connected. Then we do the same as above with $v_M(M_i \setminus N_i)$ instead of U_i . Since $\operatorname{vol}_{n-1}(v_M(N_i)) = 0$, the formula given is still true in this general case. $\square$

For each $i = 1, \dots, m$, let

$$\sigma_i^+ \text{ be } \#(v_M^{-1}(u) \cap \{\det k_M > 0\}) \quad \text{and}$$

$$\sigma_i^- \text{ be } \#(v_M^{-1}(u) \cap \{\det k_M < 0\}),$$

for any $u \in U_i$, since these numbers depend only on U_i . We deduce that $s(i) = \sigma_i^+ + \sigma_i^-$ and thus

$$|K|_M = \sum_i^{N_M} (\sigma_i^+ + \sigma_i^-) \text{vol}_{n-1}(U_i).$$

Let $\sigma_i = \sigma_i^+ - \sigma_i^-$. Note that $\sigma_i = \deg_u v_M =: \deg(v_M|_{v_M^{-1}(U_i)})$ the degree of the mapping v_M at any $u \in U_i$.

Proposition 3.2. *The total curvature of M is*

$$K_M := \int_M k_M(x) \, dv_{n-1}(x) = \sum_{i=1}^{N_M} \sigma_i \text{vol}_{n-1}(U_i).$$

Proof. It works exactly as in the proof of Proposition 3.1. □

Let us come to a more specific property of the Gauss map. For any $u \in \mathbb{S}^{n-1}$, let $\varphi_u(x) := \langle x, u \rangle$ be the orthogonal projection on the oriented vector line $\mathbb{R}u$.

Let us consider the following

Lemma 3.3. *Let $y \in M$ be a point at which the rank of $d_y v_M$ is $n - 1$. Let $u = v_M(y)$. Then the function $\varphi_u|_M$ has a Morse singular point at y , that is the Hessian matrix $\text{Hess}_y(\varphi_u|_M)$ is non degenerate.*

Proof. Let us identify the hyperplane $T_y M$ with $\mathbb{R}^{n-1}$ with coordinates $\mathbf{x} = (x_1, \dots, x_{n-1})$ centered at y and remember that $d_y v_M$ can be seen as a reflexive endomorphism of $T_y M = T_u \mathbb{S}^{n-1}$. Since it is of rank $n - 1$, it has exactly $(n - 1)$ non-zero real eigenvalues.

By the definable implicit function theorem, there exist an open neighborhood $\mathcal{U}$ of the origin in the hyperplane $T_y M$ and a C^1 -definable map $\phi: \mathcal{U} \mapsto \mathbb{R}$ such that there exists a neighborhood $\mathcal{V}$ of y in M such that $\mathcal{V} = \{x_n = \phi(\mathbf{x}): \mathbf{x} \in \mathcal{U}\}$. Thus we find that $\varphi_u|_M(x) = \phi(\mathbf{x})$. We thus rewrite v_M as

$$v_M(x) = v_M(\mathbf{x}) = \frac{\nabla(x_n - \phi(\mathbf{x}))}{|\nabla(x_n - \phi(\mathbf{x}))|}.$$

For any $u \in T_y M$ we find that

$$\begin{aligned} d_y v_M \cdot \xi &= \frac{1}{|\nabla(x_n - \phi(\mathbf{x}))|} [\text{Hess}_y \phi \cdot \xi - \langle \text{Hess}_y \phi \cdot \xi, u \rangle u] \\ &= \frac{1}{|\nabla(x_n - \phi(\mathbf{x}))|} \text{Hess}_y \phi \cdot \xi, \end{aligned}$$

where $\text{Hess}_y \phi$ is the Hessian matrix of ϕ at y . Since $d_y v_M$ is of rank $n - 1$, we deduce that $\text{Hess}_y(\varphi_u|_M) = \text{Hess}_y \phi$, and thus is Morse at y . $\square$

Thanks to Lemma 3.3 we can then define the following

Definition 3.4. *The Morse index of M at $x \notin \mathbf{crit}(v_M)$ is the integer $\lambda_M(x)$ define as the Morse index at x of the function $\varphi_u|_M$, with $u = v_M(x)$.*

Obviously this definition depends on the choice of v_M .

Let U be a connected component of $\mathcal{U} = v_M(M) \setminus v_M(\mathbf{crit}(v_M))$. Thus v_M induces a finite C^{l-1} covering

$$v_M: M_1 \sqcup \dots \sqcup M_d \mapsto U.$$

Proposition 3.5. *With the previous notation, the function $M_j \ni x \rightarrow \lambda_M(x)$ is definable and so is constant.*

Proof. Each subset M_j is a definable and open in M . Given an orthonormal basis of $\mathbb{R}^n$, let $P(x; T) \in \mathbb{R}[T]$ be the characteristic polynomial of $\text{Hess}_x(\varphi_{v_M(x)}|_M)$. The coefficients of this polynomial are definable functions of $x \in M \setminus \mathbf{crit}(v_M)$ and C^{l-1} . As functions of x , the roots (counted with multiplicity) are continuous and definable and all non zero. Since M_j is connected, the number of negative roots is constant and so is the index $\lambda_M(x)$. $\square$

There is a straightforward corollary of Lemma 3.3 and Proposition 3.5

Corollary 3.6. *Let $y \in M$ be a point at which the rank of $d_y v_M$ is $n - 1$. Let $u = v_M(y)$. Then the function Morse index of $\varphi_u|_M$ and the index of $d_y v_M$ are the same.*

Thus we can also rewrite the total curvature as

$$K_M = \int_{v_M(M) \setminus v_M(\mathbf{crit}(v_M))} \left(\sum_{x \in v_M^{-1}(u) \setminus \mathbf{crit}(v_M)} (-1)^{\lambda_M(x)} \right) dv(u),$$

since the set of points $\nu_M^{-1}(\nu_M(\mathbf{crit}(\nu_M))) \cap (M \setminus \mathbf{crit}(\nu_M))$ is at most of dimension $n - 2$.

4 On the Gauss map of a tame function

Let $f: \mathbb{R}^n \mapsto \mathbb{R}$, be a definable function, enough differentiable, say C^l , with $l \geq 2$.

Let us denote by $K_0(f)$ the set of critical values of f .

For each t , let F_t be the level $f^{-1}(t)$.

The Gauss map of the function f is the following mapping

$$\begin{aligned} \nu_f : \mathbb{R}^n \setminus \mathbf{crit}(f) &\mapsto \mathbb{S}^{n-1} \\ x &\rightarrow \frac{\nabla f(x)}{|\nabla f(x)|} \end{aligned}$$

It is a definable and C^{l-1} mapping. We will denote by ν_t the restriction $\nu_f|_{F_t}$, providing an orientation to each (connected component of the) level F_t which is compatible with the transverse structure of the foliation (on $\mathbb{R}^n \setminus \mathbf{crit}(f)$) by the connected components of the (regular) levels of f . Note that $\mathbf{crit}(\nu_f) \cap F_t = \mathbf{crit}(\nu_t)$.

The differential mapping of ν_f at $x \notin \mathbf{crit}(f)$, is

$$d_x \nu_f: \xi \mapsto \frac{1}{|\nabla f(x)|} [\text{Hess}_x(f) \cdot \xi - \langle \text{Hess}_x(f) \cdot \xi, \nu_f(x) \rangle \nu_f(x)]$$

We recall that for any $x \in F_t$, the linear mapping $d_x \nu_t = d_x \nu_f|_{T_x F_t}$ seen as an endomorphism of $T_{\nu_f(x)} \mathbb{S}^{n-1}$ is reflexive.

Definition 4.1. Let $x \notin \mathbf{crit}(f) \cup \mathbf{crit}(\nu_f)$. The tangent Gauss index of f at x , denoted by $\lambda_f(x)$ is the index of the reflexive endomorphism $d_x \nu_f|_{T_x F_t(x)}$.

Proposition 4.2. The function $x \rightarrow \lambda_f(x)$ is a locally constant definable mapping.

Proof. Assume a fixed orthonormal basis of $\mathbb{R}^n$ is given. Let $P(x, T) \in \mathbb{R}[T]$ be the characteristic polynomial of the endomorphism $d_x \nu_f|_{T_x F_t(x)}$. From the computation of the differential $d_x \nu_f$ we deduce that the coefficients of this polynomial are definable and C^{l-2} . Given x , let $\alpha_1(x) \leq \dots \leq \alpha_{n-1}(x)$ be the roots of this polynomial. For each $i = 1, \dots, n-1$, the functions $x \rightarrow \alpha_i(x)$ is continuous and definable. For this reason the number of negative roots (that

is the tangent Gauss index) is constant on each connected components of $\mathbb{R}^n \setminus (\mathbf{crit}(f) \cup \mathbf{crit}(v_f))$, that is $x \rightarrow \lambda_f(x)$ is constant on $\mathbb{R}^n \setminus (\mathbf{crit}(f) \cup \mathbf{crit}(v_f))$.

Since the mapping $x \rightarrow \lambda_f(x)$ is constant on each connected component of $\mathbb{R}^n \setminus (\mathbf{crit}(f) \cup \mathbf{crit}(v_f))$, then its graph is a definable subset of $\mathbb{R}^n \times \mathbb{R}$ since $\mathbb{R}^n \setminus (\mathbf{crit}(f) \cup \mathbf{crit}(v_f))$ is a definable subset of $\mathbb{R}^n$. $\square$

This property will be very useful in Section 6.

When $\mathbf{crit}(v_f) \neq \emptyset$, let us consider the following C^{l-1} definable mapping:

$$\Psi_f: \mathbb{R}^n \setminus (\mathbf{crit}(f) \cup \mathbf{crit}(v_f)) \mapsto \mathbb{S}^{n-1} \times \mathbb{R} \text{ defined as } x \rightarrow (v_f(x), f(x)).$$

This mapping is definable and C^{l-1} . It is also a local diffeomorphism at any of the point of $\mathbb{R}^n \setminus (\mathbf{crit}(f) \cup \mathbf{crit}(v_f))$, thus its image $\tilde{\mathcal{U}}$ is open.

Definition 4.3. Let $(u, t) \in \tilde{\mathcal{U}}$. The tangent Gauss degree of f at (u, t) , denoted $\deg_f(u, t)$ is the degree of Ψ_f at (u, t) .

Then we get the following,

Proposition 4.4. The function $(u, t) \rightarrow \deg_f(u, t)$ is a locally constant definable mapping.

Proof. This comes from the fact that Ψ_f is a local diffeomorphism at each point of its definition domain. $\square$

More interestingly we also have

Proposition 4.5. For any $(u, t) \in \tilde{\mathcal{U}}$,

$$\deg_{(u,t)} \Psi_f = \deg_u v_t = \sum_{x \in v_t^{-1}(u)} (-1)^{\lambda_f(x)}.$$

Proof. Let $y \in \Psi_f^{-1}(u, c)$. Let $\mathcal{F} := (v_1, \dots, v_{n-1}, v_n)$, with $v_n = v_f$, be a C^1 and direct orthonormal frame in a neighborhood $\mathcal{Y}$ of y . With such “coordinates”, for any $x \in F_t \cap \mathcal{Y}$, both $d_x \Psi_f$ and $d_x v_t$ are considered as endomorphisms. For $x \in F_t \cap \mathcal{Y}$, writing the matrices of $d_x \Psi_f$ and $d_x v_t$ in this frame, gives

$$\det(\text{Mat}_{\mathcal{F}(x)} d_x \Psi_f) \cdot \det(\text{Mat}_{\mathcal{F}(x)} d_x v_t) > 0,$$

since the frame respects the orientations of $\mathbb{R}^n$ and of $T_x F_t$ and where $\text{Mat}_{\mathcal{F}(x)}$ denotes the matrix in the base $\mathcal{F}(x) := (v_1(x), \dots, v_{n-1}(x), v_n(x))$. Thus the signs of these determinants are the same and so is proved the lemma. $\square$

5 Hausdorff limits of Gauss images

We use notations of Section 4.

We recall that $\tilde{\mathcal{U}} = \Psi_f(\mathbb{R}^n \setminus (\mathbf{crit}(f) \cup \mathbf{crit}(v_f)))$ is open and definable in $\mathbb{S}^{n-1} \times \mathbb{R}$ and that for any $(u, t) \in \tilde{\mathcal{U}}$, $\Psi_f^{-1}(u, t)$ is finite.

From Gabrielov Uniformity theorem, there exists a positive integer N_f such that for any $(u, t) \in \tilde{\mathcal{U}}$, $\#\Psi_f^{-1}(u, t) \leq N_f$.

Let us define the following definable sets

$$\tilde{\mathcal{U}}_k = \{(u, t) \in \tilde{\mathcal{U}} : \#\Psi_f^{-1}(u, t) = k\}.$$

$$\mathcal{U}_t = \{u \in \mathbb{S}^{n-1} : (u, t) \in \tilde{\mathcal{U}}\} = v_t(F_t \setminus \mathbf{crit}(v_f)) = v_t(F_t \setminus \mathbf{crit}(v_t)).$$

$$\mathcal{U}_{k,t} = \{u \in \mathbb{S}^{n-1} : (u, t) \in \tilde{\mathcal{U}}_k\}.$$

The subsets $\mathcal{U}_t$ and $\mathcal{U}_{k,t}$ are open in $\mathbb{S}^{n-1}$.

Note that at any point of $\mathbb{R}^n \setminus (\mathbf{crit}(f) \cup \mathbf{crit}(v_f))$, Ψ_f is a local diffeomorphism.

Let us remark that both families $(\mathcal{U}_t)_{t \in f(\mathbb{R}^n) \setminus K_0(f)}$ and $(\mathcal{U}_{k,t})_{t \in f(\mathbb{R}^n) \setminus K_0(f)}$ are definable families of open subsets of $\mathbb{S}^{n-1}$.

Let $\mathcal{K}(\mathbb{R}^n)$ be the space of compact subsets of $\mathbb{R}^n$.

Given Y and Z compacts subsets of $\mathbb{R}^n$, the Hausdorff distance between Y and Z , denoted by $d_{\mathcal{K}(\mathbb{R}^n)}(Y, Z)$, is defined as

$$\begin{aligned} d_{\mathcal{K}(\mathbb{R}^n)}(Y, Z) &= \max \left(\min_{y \in Y} \text{dist}(y, Z), \min_{z \in Z} \text{dist}(z, Y) \right) \\ &= \min \left\{ r \geq 0 : \forall y \in Y \text{ and } \forall z \in Z, \text{dist}(y, Z) \leq r \right. \\ &\quad \left. \text{and } \text{dist}(z, Y) \leq r \right\} \end{aligned}$$

The space $\mathcal{K}(\mathbb{R}^n)$ equipped with the Hausdorff distance $d_{\mathcal{K}(\mathbb{R}^n)}$ becomes a complete metric space.

Bounded definable families of compacts subsets behave well under the Hausdorff limit, (see [Br, vD2, LS] for a general frame). Since we are only interested in 1-parameter families of such subsets, the statement of the next result is given in this context and in the form we will use it below in the rest of this paper.

Theorem 5.1 ([Br, vD2, LS]). *Let $(C_t)_{t \in [0, 1[}$ be a definable family of closed subsets of $\mathbb{R}^n$. Assume there exists a compact subset Q such that $C_t \subset Q$ for each $t \in [0, 1[$. Then, the Hausdorff limit $C_1 := \lim_{t \rightarrow 1} C_t$ does exist and is a definable subset of $\mathbb{R}^n$ contained in Q .*

Let $(C_t)_{t \in I}$ be a definable family of closed subsets of $\mathbb{S}^{n-1}$. Since there exists a constant C , such that given any $u, v \in \mathbb{S}^{n-1}$, $d_{\mathbb{S}}(u, v) \leq C|u - v|$, taking Hausdorff limits of closed subsets $(C_t)_{t \in I}$ of $\mathbb{S}^{n-1}$, will provide exactly the same Hausdorff limits of $(C_t)_{t \in I}$ considered as closed subsets of $\mathbb{R}^n$.

Let us return to our topic. Given a value c let us denote respectively by $\mathcal{V}_c^+$ and by $\mathcal{V}_c^-$ the following Hausdorff limits

$$\mathcal{V}_c^+ := \lim_{t \rightarrow c^+} \mathbf{clos}(\mathcal{U}_t) \quad \text{and} \quad \mathcal{V}_c^- := \lim_{t \rightarrow c^-} \mathbf{clos}(\mathcal{U}_t).$$

For each $k = 1, \dots, N_f$ and for $* = +, -$, let $\mathcal{V}_k^*$ be the Hausdorff limit $\lim_{t \rightarrow c^*} \mathbf{clos}(\mathcal{U}_{k,t})$.

Proposition 5.2. *Let $U_1, \dots, U_{d_c}$ be the connected components of $\mathcal{U}_c$. For each $i = 1, \dots, d_c$, there exist positive integers $l_- = l_-(i) \geq s(i)$ and $l_+ = l_+(i) \geq s(i)$ such that $U_i \cap \mathcal{V}_c^- \subset \mathcal{V}_{l_-}^-$ and $U_i \cap \mathcal{V}_c^+ \subset \mathcal{V}_{l_+}^+$.*

Proof. Let $u \in \mathbb{S}^{n-1}$. When not empty, the subset

$$\Psi_f^{-1}(\{u\} \times \mathbb{R}) = \{x \notin \mathbf{crit}(f) \cup \mathbf{crit}(v_f) : v_f(x) = u\}$$

is a C^{l-1} definable curve since Ψ_f is a local diffeomorphism. Let Γ be a connected component of $\Psi_f^{-1}(\{u\})$ which intersects with F_c . Since the function $f|_{\Gamma}$ is strictly monotonic, there exists $\varepsilon > 0$ such that for any $t \in]c - \varepsilon, c[\cup]c, c + \varepsilon[$, the curve Γ also intersects with F_t , that is $u \in \mathcal{U}_t$. Since the number of connected components of $\Psi_f^{-1}(\{u\} \times \mathbb{R})$ meeting F_c does depend only on U_i , the Proposition is proved. $\square$

As a consequence of Proposition 5.2 we obtain

Corollary 5.3. *Let c be a value such that $\mathcal{U}_c \neq \emptyset$. Then $\mathcal{U}_c \subset \mathcal{V}_c^+ \cap \mathcal{V}_c^-$.*

Proof. It is contained in the proof of Proposition 5.2. $\square$

The statement of Corollary 5.3 would be wrong for a general definable family $(C_t)_t$ of closed subsets of $\mathbb{S}^{n-1}$. As read in the proof, the transverse structure given by the function f is somehow also carried in the family $(\mathcal{U}_t)_t$, and thus explain this result.

We end this section with an elementary result about the volume of a Hausdorff limit of a 1 parameter definable family.

Proposition 5.4. *Let $C := \cup_{t \in [0,1]} C_t$ be a definable family of closed connected subsets of dimension n of Q a compact subset of $\mathbb{R}^n$ or of $\mathbb{S}^n$. Let C_1 be the Hausdorff limit $\lim_{t \rightarrow 1} C_t$, then*

$$\lim_{t \rightarrow 1} \text{vol}_n(C_t) = \text{vol}_n(C_1).$$

Proof. According to [LS, Theorem 1] there exist at most finitely many subsets of $\mathcal{K}(\mathbb{R}^n)$ that belong to $\text{clos}(C) \setminus C \subset \mathcal{K}(\mathbb{R}^n)$. By [vD2, Proposition 3.2]), C_1 is well defined and definable, and its dimension is at most n . Note that C_1 is necessarily connected.

Assume first that $Q \subset \mathbb{R}^n$.

Let $T_\varepsilon(C_t)$ be the ε -neighborhood of C_t in $\mathbb{R}^n$ for $\varepsilon > 0$.

Let r be the dimension of C_1 . Using a generalized Weyl-Steiner's tube formula for compact definable subsets of $\mathbb{R}^n$ ([BK]), we deduce that for small $\varepsilon > 0$ there exists a positive constant L such that

$$\left| \text{vol}_r(C_1) - \frac{\text{vol}_n(T_\varepsilon C_1)}{\sigma_{n-r,n} \varepsilon^{n-r}} \right| < L\varepsilon,$$

where $\sigma_{n-r,n}$ is the volume of the unit ball $\mathbb{B}_{n-r}$ of $\mathbb{R}^{n-r}$.

By definition of C_1 , for any $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon)$ such that for $t > 1 - \eta$ we get $d_{\mathcal{K}(\mathbb{R}^n)}(C_t, C_1) < \varepsilon$, that is $C_t \subset T_\varepsilon(C_1)$.

If $r = n$, we deduce that $\text{vol}_n(C_t) - \text{vol}_n(C_1) \rightarrow 0$ as $\varepsilon \rightarrow 0$, since $C_t \subset T_\varepsilon(C_1)$.

When $r < n$ we know $\text{vol}_n(T_\varepsilon(C_1)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. So $\text{vol}_n(C_t)$ must tend to 0 as $t \rightarrow 1$ (otherwise $\limsup_{t \rightarrow 1} \text{vol}_n(C_t) > 0$, which would contradict $\lim_{\varepsilon \rightarrow 0} \text{vol}_n(T_\varepsilon(C_1)) = 0$).

Assume now $Q \subset \mathbb{S}^n$. The generalized Weyl's tube formula ([BB],[BK]) works also in spaces of constant curvatures. Thus the proof above in the flat case, adapts almost readily to the case of $\mathbb{S}^n$. $\square$

6 Continuity of the total curvature and of the total absolute curvature of a definable function

We use the notations of Section 4.

Given $t \notin K_0(f)$, let k_t be the Gauss curvature of F_t with respect to the orientation ν_t . Let $\mathcal{E}_t$ be the set of the connected components of F_t .

When $t \notin K_0(f)$, we define the *total absolute curvature* of F_t as

$$|K|(t) := \sum_{E \in \mathcal{E}_t} |K|_E,$$

and the *total curvature* of F_t as

$$K(t) := \sum_{E \in \mathcal{E}_t} K_E.$$

Assume c is a value taken by f such that $\mathcal{U}_c$ is not empty. Let $U_1, \dots, U_{d_c}$, be the connected components of $\mathcal{U}_c$. For each $i = 1, \dots, d_c$, let $s(i)$ be the number of points in the fiber $v_c^{-1}(u)$ above any $u \in U_i$, that is $U_i \subset \mathcal{U}_{s(i),c}$, and let $\sigma_i(c)$ be the degree of v_c on U_i (see Section 3). We recall that

$$|K|(c) = \sum_{l=1}^{N_f} l \cdot \text{vol}_{n-1}(\mathbf{clos}(\mathcal{U}_{l,c})) = \sum_{i=1}^{d_c} s(i) \text{vol}_{n-1}(\mathbf{clos}(U_i)). \quad (6.1)$$

$$K(c) = \sum_{i=1}^{d_c} \sigma_i(c) \text{vol}_{n-1}(\mathbf{clos}(U_i)) \quad (6.2)$$

As a consequence of the result of the existence of definable Hausdorff Limits ([vD2],[LS]) and of the definition of the total curvature we deduce the following

Corollary 6.1. *The following limits,*

$$\lim_{t \rightarrow c^-} K(t), \quad \lim_{t \rightarrow c^+} K(t), \quad \lim_{t \rightarrow c^-} |K|(t) \quad \text{and} \quad \lim_{t \rightarrow c^+} |K|(t)$$

do exist, once given a value c .

Proof. It is just a consequence of the formulae for the total curvatures given above and of Proposition 5.4. $\square$

Let us recall that given $*$ = $-$, $+$, we denote by $\mathcal{V}_c^*$, the Hausdorff limit $\lim_{t \rightarrow c^*} \mathbf{clos}(\mathcal{U}_t)$ and given any integer $1 \leq k \leq N_f$, $\mathcal{V}_k^*$ stands for the Hausdorff limit $\lim_{t \rightarrow c^*} \mathbf{clos}(\mathcal{U}_{k,t})$

We can write $\mathcal{V}_c^- \cap \mathcal{V}_c^+$ as the partition $\bigcup_{0 \leq k, l \leq N_f} \mathcal{V}_{k,l}$, where $\mathcal{V}_{k,l} = \mathcal{V}_k^- \cap \mathcal{V}_l^+$, if k, l are both positive, $\mathcal{V}_{k,0}$ stands for $\mathcal{V}_k^-$ if the intersection with $\mathcal{V}_c^+$ is empty, $\mathcal{V}_{0,l}$ stands for $\mathcal{V}_l^+$ if the intersection with $\mathcal{V}_c^-$ is empty, and $\mathcal{V}_{0,0} = \emptyset$.

Our goal is still to try to understand the continuity of the functions $|K|$ and K . Regarding the results of Sections 3 and 5, the non zero contribution to $|K|(t)$ only comes from the subset $\mathcal{U}_t$.

Proposition 6.2. *Let c be a value. Then*

$$|K|(c) \leq \min \left\{ \lim_{t \rightarrow c^-} |K|(t), \lim_{t \rightarrow c^+} |K|(t) \right\}.$$

Proof. If $\mathcal{U}_c$ is empty this means that $|K|(c) = 0$, and so the statement is trivially true.

Let $U_1, \dots, U_{d_c}$ be the connected components of $\mathcal{U}_c$. For each $i \in \{1, \dots, d_c\}$ and each $* = +, -$, let $l_*(i)$ be the corresponding integer of Proposition 5.2. Observe that for $* = +$ and $* = -$

$$|K|(c) = \sum_i s(i) \text{vol}_{n-1}(U_i) \leq \sum_i l_*(i) \text{vol}_{n-1}(U_i \cap \mathcal{V}_c^*),$$

and the right hand side term is $\leq \sum_l l \text{vol}_{n-1}(\mathcal{V}_l^*)$. By Proposition 5.4, we conclude the proof. $\square$

The continuity of $|K|$ at c implies that each of these inequalities is an equality. Using notation of Proposition 5.2 we actually obtain

Corollary 6.3. *If $|K|$ is continuous at c , then*

- (1) *for each pair k, l with $k \neq l$, $\text{vol}_{n-1}(\mathcal{V}_{k,l}) = 0$, or equivalently $\mathcal{V}_{k,l}$ is of dimension at most $n - 2$.*
- (2) *for each k and each $* = +, -$, $\text{vol}_{n-1}(\mathcal{V}_k^*) = \text{vol}_{n-1}(\mathcal{U}_{k,c})$.*

Proof. Using Proposition 5.2, equation 6.1 and Proposition 6.2, and writing down the continuity at c provides the different statements. $\square$

Now we can come to the first main result of this section

Theorem 6.4. *The function $|K|$ has at most finitely many discontinuities.*

Proof. This is a consequence of Proposition 5.4 and Proposition 5.2.

Assume k is given. Let $C := \cup_{t \in f(\mathbb{R}) \setminus K_0(f)} C_t$, where $C_t = \mathbf{clos}(\mathcal{U}_{k,t})$.

Assume first that $f(\mathbb{R}) \setminus K_0(f)$ is connected, so that we can assume that $f(\mathbb{R}) \setminus K_0(f) =]0, 1[$. Then C is a definable family of subsets of $\mathbb{S}^{n-1}$, and then is definable in $\mathcal{K}(\mathbb{R}^n)$. Thus its closure $\mathbf{clos}(C)$ in $\mathcal{K}(\mathbb{R}^n)$ consists of C and finitely many closed definable subsets of $\mathbb{S}^{n-1}$ by [LS, Theorem 1]. This means there exist at most finitely many $c \in f(\mathbb{R}) \setminus K_0(f) =]0, 1[$ such that the Hausdorff limit $\lim_{t \rightarrow c^-} C_t$ or the Hausdorff limit $\lim_{t \rightarrow c^+} C_t$ is not C_c . For any value $c \in]0, 1[$ such that $\lim_{t \rightarrow c^-} C_t = \lim_{t \rightarrow c^+} C_t = C_c$, Proposition 5.4 states that the function $t \rightarrow \text{vol}_{n-1}(C_t)$ is continuous at such a c . Since there are only finitely many k and finitely many connected components of $f(\mathbb{R}) \setminus K_0(f)$, the theorem is proved. $\square$

Remark 6.5. Working with globally subanalytic functions only, Theorem 6.4 is then just a consequence of Lion-Rolin's Theorem [LR].

Now let us investigate the continuity of K , $t \rightarrow K(t) = \int_{F_t} k_t$.

Let us pick a value c . Let $\{U_i\}_{i=1}^{d_c}$ be the set of connected components of $\mathcal{U}_c$. We recall that $\deg_u v_c$, the degree of v_c at $u \in U_i$ is only dependent on i , and is equal to $\sigma_i(c)$. Thus $K(c) = \sum_{i=1}^{d_c} \sigma_i(c) \text{vol}_{n-1}(U_i)$.

Proposition 6.6. *If $|K|$ is continuous at c , then K is continuous at c .*

Proof. If $|K|(c) = 0$, then we immediately get $K(c) = 0$, and so K is continuous at 0.

Assume that $\mathcal{U}_c$ is not empty. Let U_c be a connected component of $\mathcal{U}_c$. Then for each $u \in U$, the number of connected components of $\Psi_f^{-1}(\{u\} \times \mathbb{R})$ meeting F_c is constant, say equal to $l \geq 1$. Let $\Gamma_1(u), \dots, \Gamma_l(u)$ be these connected components. Given $u \in \mathcal{U}_c$, we know that there exists $\varepsilon > 0$ such that for any $t \in]c - \varepsilon, c[\cup]c, c + \varepsilon[$, we get

$$\Psi_f^{-1}(\{u\} \times \mathbb{R}) \cap F_t = \bigcup_{i=1}^l (\Gamma_i(u) \cap F_t).$$

From Proposition 4.4, we know that the tangent Gauss degree of f , that is the degree of v_t at u is a locally constant function of (u, t) . Thus for $u \in \mathcal{U}_c$ given, there is $\varepsilon > 0$ as above such that for any $t \in]c - \varepsilon, \varepsilon[$, $\deg_u v_t = \deg_u v_c$. Let $\tilde{V}$ be the connected component of $\tilde{\mathcal{U}}$ that contains (u, c) . Thus $\tilde{V} \cap \mathbb{S}^{n-1} \times \{c\} = U_c \times \{c\}$ is connected. So there exists $\varepsilon > 0$ such that for any $t \in]c - \varepsilon, c + \varepsilon[$, $\tilde{V} \cap \mathbb{S}^{n-1} \times \{t\} = U_t \times \{t\}$ is connected, so U_t is connected. Since $|K|$ is continuous, this implies that $\lim_{t \rightarrow c} \text{vol}_{n-1}(U_t) = \text{vol}_{n-1}(U_c)$. Since the Gauss tangent degree is constant on $\tilde{V}$, this implies the continuity at c of the function total curvature. $\square$

The converse of this result is not true as shown in the following example: take $f(x, y) = y(2x^2y^2 - 9xy + 12)$ as a function on the real plane. Then we find that

$$\lim_{t \rightarrow c^-} |K|(t) = \lim_{t \rightarrow c^+} |K|(t) = 2\pi, \text{ while } |K|(0) = 0,$$

but K is continuous at 0.

Remark 6.7. Talking about the discontinuity of $|K|$ or K makes only sense at values c that are taken by the function f .

Let us return to the continuity of the curvature of f . Let c be a regular value. For each $l = 1, \dots, N_f$, the Hausdorff limit of $\lim_{t \rightarrow c^*} \mathbf{clos}(\mathcal{U}_{l,c})$ is denoted by $\mathcal{V}_l^*$, where $*$ = + or $*$ = -.

Proposition 6.8. *Let c be a regular value at which $|K|$ is not continuous. There exists an open subset $U \subset \mathbb{S}^{n-1}$, such that for any $u \in U$, there exists a connected component Γ of $\Gamma_u^+(f) := \Psi_f^{-1}(\{u\} \times \mathbb{R})$, such that $\Gamma \cap f^{-1}(c)$ is empty and one of the two cases below happens*

- (i) *If c is the infimum of f along Γ , then for any $\varepsilon > 0$ small enough, $\Gamma \cap f^{-1}(]c, c + \varepsilon[)$ is not bounded.*
- (ii) *If c is the supremum of f along Γ , then for any $\varepsilon > 0$ small enough, $\Gamma \cap f^{-1}(]c - \varepsilon, c[)$ is not bounded.*

Proof. Assume that $\lim_{t \rightarrow c^-} |K|(t) > |K|(c)$. The other situation is absolutely similar.

Let $\mathcal{U}_c = \sqcup_{i=1}^{d_c} U_i$, where U_i is a connected component of $\mathcal{U}_c$. For each i , we have $U_i \subset \mathcal{U}_{s(i),c}$. Since $|K|(t) = \sum_{l=1}^{N_f} l \text{vol}_{n-1}(\mathcal{U}_{l,t})$, we thus find that

$$\lim_{t \rightarrow c^-} |K|(t) = \sum_{l=1}^{N_f} l \text{vol}_{n-1}(\mathcal{V}_l^-).$$

For each $i = 1, \dots, d_c$, there exists a positive integer $l(i) \geq s(i)$ such that $U_i \subset \mathcal{V}_{l(i)}^-$. Thus the following is happening:

either

- (a) there exists $i \in \{1, \dots, d_c\}$ such that $l(i) > s(i)$

or

- (b) there exists $i \in \{1, \dots, d_c\}$ such that $\text{vol}_{n-1}(\mathcal{V}_{l(i)}^-) > \text{vol}_{n-1}(U_i)$.

Assume that a phenomenon of type (a) contributes to the discontinuity of $|K|$ at c . Then for each $u \in U_i$, there exists $\varepsilon > 0$ such that $\Gamma_u^+(f) \cap f^{-1}(]c - \varepsilon, c + \varepsilon[)$ has $m(i) \geq l(i)$, connected components $\mathcal{G}_1, \dots, \mathcal{G}_{m(i)}$ such that, for $j = 1, \dots, s(i)$, $f(\mathcal{G}_j) =]c - \varepsilon, c + \varepsilon[$, for $j = s(i) + 1, \dots, l(i)$, $f(\mathcal{G}_j) =]c - \varepsilon, c[$ and for $j = l(i) + 1, \dots, m(i)$, $f(\mathcal{G}_j) =]c, c + \varepsilon[$.

Let $j \in \{s(i) + 1, \dots, l(i)\}$. If $\mathcal{G}_j$ was bounded, its closure in $\mathbb{R}^n$ would then be $\mathbf{clos}(\mathcal{G}_j) = \mathcal{G}_j \cup \{x_{c-\varepsilon}, x_c\}$, where $x_{c-\varepsilon} \in F_{c-\varepsilon}$ and $x_c \in F_c$. Since c is a regular value, this would mean that $\nabla f(x_c) = u|\nabla f(x_c)|$. Such a u is a regular

value of $v_f|_{f^{-1}([c-\varepsilon, c+\varepsilon])}$, thus G_j could be extended to $\{f > c\}$ to a regular curve through x_c , which would contradict $f(G_j) =]c - \varepsilon, c[$. So we get that G_j never meet $f^{-1}(c)$. The same works for $j \in \{l(i) + 1, \dots, m(i)\}$.

Case (b) is proved similarly.

There is a last thing to say about the case $\mathcal{U}_c$ is empty, meaning that v_c is of rank at most $n - 2$. In this situation the discontinuity of $|K|$ is created by phenomenon of type (b) only. $\square$

When the discontinuity of $|K|$ arises at a critical value c , it is almost impossible to say anything similar to the previous statement, in the general frame we are given.

7 Total λ -curvature and total absolute λ -curvature

Given a compact connected manifold M of dimension m , given a Morse function $g: M \mapsto \mathbb{R}$, let $C_\lambda(g)$ be the set of critical points of index λ . The weak Morse inequalities state

$$\#C_\lambda(g) \geq b_\lambda(M), \text{ the } \lambda\text{-th Betti number.}$$

Let us consider now that M is an orientable connected definable C^l , $l \geq 2$, hypersurface of $\mathbb{R}^n$, with orientation map v_M .

Using Definition 3.4, given $\lambda \in \{0, \dots, n - 1\}$, let us define

$$\mathcal{I}_M(\lambda) := \{x \in M : \lambda_M(x) = \lambda\}.$$

Since $\mathcal{I}_M(\lambda) \subset M \setminus \mathbf{crit}(v_M)$, and $M \setminus \mathbf{crit}(v_M) \ni x \mapsto \lambda_M(x)$ is definable and locally constant, the subset $\mathcal{I}_M(\lambda)$ is definable.

Now we can define

Definition 7.1. Given $\lambda \in \{0, \dots, n - 1\}$,

- (1) the total λ -curvature of M is $K_M(\lambda) := \int_{\mathcal{I}_M(\lambda)} k_M(x) \, dv_{n-1}(x)$,
- (2) the total absolute λ -curvature of M is $|K|_M(\lambda) := \int_{\mathcal{I}_M(\lambda)} |k_M|(x) \, dv_{n-1}(x)$.

If M is compact then averaging on the restriction to M of all the oriented linear projections we deduce

$$|K|_M(\lambda) \geq \text{vol}_{n-1}(v_M(\mathcal{I}_M(\lambda))) \cdot b_\lambda(M).$$

Now let us come to the case of definable functions. Let $f: \mathbb{R}^n \mapsto \mathbb{R}$ still be a C^l , definable function with $l \geq 2$. We use the notations of Section 4 and

Section 5. Given any $t \notin K_0(f)$, let v_t be the restriction of v_f to F_t , and let $k_t(x)$ be the Gauss curvature at $x \in F_t$.

Let us define the open definable subset

$$\mathcal{I}_f(\lambda) := \{x \notin \mathbf{crit}(f) \cup \mathbf{crit}(v_f) : \text{the index of } d_x v_f|_{T_x F_{f^{-1}(x)}} \text{ is } \lambda\}$$

Let $\mathcal{E}_t$ still be the set of connected components of F_t . Given $t \notin K_0(f)$ and $\lambda \in \{0, \dots, n-1\}$, let

$$\mathcal{I}_t(\lambda) = \cup_{E \in \mathcal{E}_t} \mathcal{I}_E(\lambda) = \mathcal{I}_f(\lambda) \cap F_t.$$

The family $(v_t(\mathcal{I}_t(\lambda)))_{t \notin K_0(f)}$ is a definable family of subsets of $\mathbb{S}^{n-1}$.

We define two new functions of t , the *total λ -curvature* of f :

$$K(\lambda; t) := \sum_{E \in \mathcal{E}_t} K_E(\lambda) = \int_{\mathcal{I}_t(\lambda)} k_t(x) \, d v_{n-1}(x),$$

and the *total absolute λ -curvature* of f :

$$|K|(\lambda; t) := \sum_{E \in \mathcal{E}_t} |K|_E(\lambda) = \int_{\mathcal{I}_t(\lambda)} |k_t|(x) \, d v_{n-1}(x).$$

Now we can state the main result of this section

Theorem 7.2. *Let $\lambda \in \{1, \dots, n-1\}$ be given.*

(1) *Let c be a value. Then $\lim_{t \rightarrow c^-} K(\lambda; t)$, $\lim_{t \rightarrow c^+} K(\lambda; t)$, $\lim_{t \rightarrow c^-} |K|(\lambda; t)$ and $\lim_{t \rightarrow c^+} |K|(\lambda; t)$ exist.*

(2) *Let c be a value. Then*

$$|K|(\lambda; c) \leq \min \left\{ \lim_{t \rightarrow c^-} |K|(\lambda; t), \lim_{t \rightarrow c^+} |K|(\lambda; t) \right\}.$$

(3) *The function $t \rightarrow |K|(\lambda; t)$ admits at most finitely many discontinuities.*

(4) *If the function $t \rightarrow |K|(\lambda; t)$ is continuous at c , then so is the function $t \rightarrow K(\lambda; t)$.*

Proof. The proof of each point works as the proof of the similar statement given for the functions $t \rightarrow K(t)$ and $t \rightarrow |K|(t)$. The reason for that is to consider the function $\Psi_\lambda := \Psi_f|_{\mathcal{I}_f(\lambda)}$, and then do exactly the same work as that done in Section 5 and 6

8 Remarks, comments and suggestions

In the introduction we mentioned that the original motivation was to try to find some equisingularity conditions on the family of levels (F_t) that could be read through the continuity of these total curvature functions. We were especially interested in the problem caused by the regular values that are also bifurcation values. This phenomenon is without any doubt caused by some curvature, in the broader sense of Lipschitz-Killing curvature, accumulation at infinity (i.e. on the boundary of the domain). Without any bound on the complexity of the singularity phenomenon occurring at infinity, it would be very naive and wrong to hope that the continuity of these curvature functions we dealt with is a sufficiently fine measure of the equisingularity of the levels F_t nearby a regular value. In some simple cases, see [Gr], they can provide sufficient conditions to ensure the equisingularity in a neighborhood of a given regular asymptotic critical value. But in whole generality we may also have to consider these higher order curvatures.

Consider again the situation of Section 4: That is of $f: \mathbb{R}^n \mapsto \mathbb{R}$ a definable function enough differentiable. We have associated to each regular value t of f , two real numbers, namely $K(t)$ and $|K|(t)$.

Let t be a regular value of f and let $x \in F_t$. Let $q \in \{1, \dots, n-1\}$ be a given integer. Let N be a q -dimensional sub-vector space of $T_x F_t$. Let us denote $k_t(N, x)$ the Gauss curvature at x of $F_t \cap (x + (\mathbb{R}v_t(x) \oplus N))$ a hypersurface of the $q+1$ -dimensional affine subspace $x + (\mathbb{R}v_t(x) \oplus N)$. The q -th Lipschitz-Killing curvature of F_t at x is the real number

$$LK_q(x) := \int_{G(q, T_x F_t)} k_t(N, x) dN,$$

where dN is the volume form of $G(q, T_x F_t)$ the Grassmann manifold of q -dimensional sub-vector spaces of $T_x F_t$.

So we would like to define two real numbers:

– the total q -th Lipschitz-Killing curvature,

$$L_q(t) = c_{n,q} \int_{F_t} LK_q(x) dv_{n-1}(x), \quad \text{and}$$

– the total absolute q -th Lipschitz-Killing curvature, also called total q -th length

$$|L|_q(t) = c_{n,q} \int_{F_t} |LK_q(x)| dv_{n-1}(x),$$

where $c_{n,q}$ is a universal constant depending only on q and n .

In general, once $q < n - 1$, with the level F_t non compact, the number $|L|_q(t)$ is likely to be infinite!

A more relevant question would be to estimate the first (and may be the second) dominant term of the asymptotic of

$$|L|_q(t; R) = c_{n,q} \int_{F_t \cap \mathbb{B}_R^n} |LK_q(x)| \, dv_{n-1}(x)$$

$$L_q(t; R) = c_{n,q} \int_{F_t \cap \mathbb{B}_R^n} LK_q(x) \, dv_{n-1}(x)$$

as $R \rightarrow +\infty$ and to see how these dominant terms varies in t or in (t, R) . But as already said, even in the subanalytic category where results from [CLR, LR] provide some general information on the nature of such terms, it is likely to be a very difficult question!

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