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On the solutions of a parametric family of cubic Thue equations

Alain Togbé*

Abstract. In this paper, we solve a family of Diophantine equations associated with families of number fields of degree 3. In fact, we use Baker's method find all solutions to the Thue equation

$$\Phi_n(x, y) = x^3 - n(n^2+n+3)(n^2+2)x^2y - (n^3+2n^2+3n+3)xy^2 - y^3 = \pm 1,$$

for $n \geq 0$.

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1 Introduction

A *Thue equation* is a Diophantine equation of the form

$$F(x, y) = k,$$

where $F \in \mathbb{Z}[X, Y]$ is an irreducible binary form of degree $d \geq 3$ and k is a non-zero rational integer; the unknown x and y being rational integers. The name is given in honor of the Norwegian mathematician A. Thue [11] who proved that it has only finitely many solutions. Upper bounds for the solutions have been given using A. Baker's [1] theory on linear forms in logarithms of algebraic numbers. So we consider the following Thue equation

$$\begin{aligned}\Phi_n(x, y) \ = \ & x^3 - n(n^2+n+3)(n^2+2)x^2y \\ & - (n^3+2n^2+3n+3)xy^2 - y^3 = \pm 1.\end{aligned} \tag{1.1}$$

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The aim of this paper is to solve this equation using Baker's method. In fact, since E. Thomas [10] has solved the first parameterized family of Thue equations of positive discriminant, several families of parameterized Thue equations have been studied. Many authors are able to solve cubic, quartic, quintic, sextic Thue equations. In 2004, Heuberger, Togbé, and Ziegler ([7]) solved the first octic family of Thue equations. In Thomas' example, the family of cubic fields have some special properties particularly they are Galois fields. It is also the case for the cubic fields related with the above equation. Recently, for the first time, we used Baker's method to solve a sextic family of Thue equations (see [12]). Our main result is the following:

Theorem 1.1. *For $n \geq 0$, the family of parameterized Thue equations*

$$\begin{aligned} \Phi_n(x,y) \;=\; & x^3 - n(n^2+n+3)(n^2+2)x^2y \\ & - (n^3+2n^2+3n+3)xy^2 - y^3 = \pm 1 \end{aligned} \tag{1.2}$$

has only the integral solutions

$$\pm\{(1,0),\ (0,1)\}, \tag{1.3}$$

except for $n = 0$, where we have:

$$\pm\{(-3,2),\ (-1,1),\ (-1,3),\ (0,1),\ (1,0),\ (2,1)\}. \tag{1.4}$$

Kishi [8] studied the family

$$\begin{aligned} \Phi_n(x,1) \;=\; & \phi_n(x) = x^3 - n(n^2+n+3)(n^2+2)x^2 \\ & - (n^3+2n^2+3n+3)x - 1. \end{aligned} \tag{1.5}$$

His result is useful to prove Theorem 1.1. With $n \in \mathbb{Z}$, the polynomial $\phi_n(x)$ is irreducible. Let $\mathbb{K}_n$ be the number field related with $\phi_n(x)$. There are three real roots $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$ of $\phi_n(x)$. For a solution (x,y) of (1.1), we have

$$\Phi_n(x,y) = \prod_{j=1}^{3}\left(x - \theta^{(j)}y\right) = N_{\mathbb{Q}(\theta^{(1)})/\mathbb{Q}}\left(x - \theta^{(j)}y\right) = 1. \tag{1.6}$$

This means that $x - \theta^{(j)}y$ is a unit in the order $\mathcal{O}_{\mathbb{K}_n} := \mathbb{Z}[\theta^{(1)}, \theta^{(2)}]$.

In Section 2, we will give some elementary properties of the polynomials, recall the result obtained by Kishi [8], and we consider the index $I = [E : \langle -1, \theta^{(1)}, \theta^{(2)}\rangle] \leq 2$, for $n \geq 2$, to work. In Section 3, we will study approximation properties of solutions to (1.1) and we obtain a lower bound for $\log(y)$

very useful for the proof of Theorem 1.1. We use bounds on linear forms in logarithms of algebraic numbers to prove that this equation has only the trivial solutions for large n in Section 4. Solutions for medium sized n are discussed in Section 5 using heavy computational verifications. Finally, the case of small n is covered in Section 6. In fact, we use Kash [5] to solve (1.1) for $0 \leq n \leq 75$. Most of the computations involve manipulations with asymptotic approximations done using Maple.

2 Elementary properties of the polynomials

We have the following properties:

1. $\Phi_n(-x, -y) = -\Phi_n(x, y)$; hence if (x, y) is a solution to (1.1), so is $(-x, -y)$. Without loss of generality, we will consider only the solutions (x, y) to (1.1) with y positive.
2. The couples in (1.3) are solutions to (1.1). In fact
   - if $y = 0$, then $x = \pm 1$;
   - if $y = 1$, then

     $$\phi_n(x) = x^3 - n(n^2 + n + 3)(n^2 + 2)x^2 - (n^3 + 2n^2 + 3n + 3)x - 1$$

     has no integral solutions for $n \geq 1$. So we suppose $y \geq 2$.

Let us recall some properties obtained by Kishi. In fact, in a very nice paper published in 2003 (see [8]), Kishi proved the following result:

Theorem 2.1. *Let θ and θ' be two distinct roots of $f_n(X)$. Then $\{\theta, \theta'\}$ is a system of fundamental units of the order $\mathbb{Z}[\theta, \theta']$. Let E denote the unit group of $\mathcal{O}_{\mathbb{K}_n}$ and put*

$$N := \frac{(n^2 + 3)(n^4 + n^3 + 4n^2 + 3)}{4^{\delta_1} \cdot 9^{\delta_2}}$$

where

$$\delta_1 = \begin{cases} 0 & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd}; \end{cases} \qquad \delta_2 = \begin{cases} 0 & \text{if } n \equiv 2 \pmod 3, \\ 1 & \text{if } n \not\equiv 2 \pmod 3. \end{cases}$$

Suppose N is squarefree. Then the index $[E\colon \langle -1, \theta, \theta' \rangle]$ is equal to 1, that is, $\{\theta, \theta'\}$ is a system of fundamental units of $\mathbb{K}_n$, except for $n = \pm 1, -2$. For $n = \pm 1$, we have $[E\colon \langle -1, \theta, \theta' \rangle] = 7$, and for $n = -2$, $[E\colon \langle -1, \theta, \theta' \rangle] = 3$.

The following remark will be important later.

Remark 2.2. In fact, Kishi proved that for all $n \geq 2$ and $n \leq -2$, we have $[E\colon \langle -1, \theta, \theta' \rangle] \leq 2$, (see [8], page 98). So in our computations, we will consider $I = [E\colon \langle -1, \theta, \theta' \rangle] \leq 2$.

We adopt here the L-notation defined in [7], pages 1151-1152, that we recall here. Let c be a real number, assume $f(x)$, $g(x)$, and $h(x)$ are real functions and $h(x) > 0$ for $x > c$. We will write

$$f(x) = g(x) + L_c(h(x)) \quad \text{if} \quad g(x) - h(x) \leq f(x) \leq g(x) + h(x), \text{ for } x > c.$$

So some asymptotic expressions of $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$ are given by

$$\begin{aligned}
\theta^{(1)} &= n^5 + n^4 + 5n^3 + 2n^2 + 6n \\
&\quad + \tfrac{1}{n^2} + \tfrac{1}{n^3} - \tfrac{3}{n^4} - \tfrac{1}{n^5} + \tfrac{8}{n^6} - \tfrac{3}{n^7} - \tfrac{17}{n^8} + L_{40}\left(\tfrac{0.2}{n^8}\right), \\
\theta^{(2)} &= -\tfrac{1}{n^3} + \tfrac{1}{n^4} + \tfrac{2}{n^5} - \tfrac{4}{n^6} - \tfrac{2}{n^7} + \tfrac{11}{n^8} + L_{40}\left(\tfrac{0.2}{n^8}\right), \\
\theta^{(3)} &= -\tfrac{1}{n^2} + \tfrac{2}{n^4} - \tfrac{1}{n^5} - \tfrac{4}{n^6} + \tfrac{5}{n^7} + \tfrac{6}{n^8} + L_{40}\left(\tfrac{0.2}{n^8}\right).
\end{aligned} \tag{2.1}$$

We use the asymptotic expressions of $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$ to determine those of $\log\left|\theta^{(i)}\right|$ and $\log\left|\theta^{(i)} - \theta^{(k)}\right|$. In fact, we know that for the function

$$f(x) = \log(x) = \log(1 + u),$$

we have

$$f^{(n+1)}(x) = (-1)^n \frac{n!}{x^{n+1}}, \; n \geq 0.$$

The error associated with the approximation of $f(x)$ by the 3rd Taylor polynomial is:

$$|R_3(1+u)| = \left|\frac{f^{(4)}(z)}{4!}\right| u^4 = \frac{u^4}{4z^4}$$

where z is between 1 and $1 + u$. In this interval, $|R_3(1+u)|$ is maximal when $z = 1$ if $u > 0$ and $z = 1 - |u|$ when $u < 0$. So we have

$$|R_3(1+u)| \leq \begin{cases} u^4/4, & \text{if } u > 0, \\ \dfrac{u^4}{4(1-|u|)^4}, & \text{if } u < 0. \end{cases}$$

By applying Lemma 3 in [7], we obtain:

$$\begin{aligned}
\log|\theta^{(1)}| &= 5\log(n) + \tfrac{1}{n} + \tfrac{9}{2n^2} - \tfrac{8}{3n^3} + L_{75}\left(\tfrac{197}{50n^4}\right), \\
\log|\theta^{(2)}| &= -3\log(n) - \tfrac{1}{n} - \tfrac{5}{2n^2} + \tfrac{5}{3n^3} + L_{75}\left(\tfrac{229}{100n^4}\right), \\
\log|\theta^{(3)}| &= -2\log(n) - \tfrac{2}{n^2} + \tfrac{1}{n^3} + L_{75}\left(\tfrac{58}{25n^4}\right),
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
\log|\theta^{(1)}-\theta^{(2)}| &= 5\log(n)+\tfrac{1}{n}+\tfrac{9}{2n^2}-\tfrac{8}{3n^3}+L_{75}\left(\tfrac{197}{50n^4}\right),\\
\log|\theta^{(1)}-\theta^{(3)}| &= 5\log(n)+\tfrac{1}{n}+\tfrac{9}{2n^2}-\tfrac{8}{3n^3}+L_{75}\left(\tfrac{197}{50n^4}\right),\\
\log|\theta^{(2)}-\theta^{(3)}| &= -2\log(n)-\tfrac{1}{n}-\tfrac{3}{2n^2}+\tfrac{5}{3n^3}+L_{75}\left(\tfrac{482}{125n^4}\right).
\end{aligned} \tag{2.3}$$

3 Approximation properties of solutions

Let $(x, y) \in \mathbb{Z}^2$ be a solution to (1.1). We define $\beta := x - \theta y$. Let us consider an integer j such that $j \in \{1, 2, 3\}$. Now we define the type j of a solution (x, y) of (1.1) such that

$$\left|\beta^{(j)}\right| := \min_{i=1,2,3} \left|\beta^{(i)}\right|. \tag{3.1}$$

The following lemma will be very useful for the proof of Theorem 1.1:

Lemma 3.1. *Let $n \geq 75$ and (x, y) be a solution to (1.1) of type j such that $y \geq 2$. Then we have*

$$\left|\beta^{(j)}\right| \leq \begin{cases} \dfrac{4}{y^2 n^{10}} & \text{if } j = 1,\\ 4/(n^9) & \text{if } j = 2,\\ 4/(n^7) & \text{if } j = 3, \end{cases} \tag{3.2}$$

$$\log\left|\beta^{(i)}\right| = \log(y) + \log|\theta^{(i)} - \theta^{(j)}| + \begin{cases} L_{75}\left(\frac{1}{n^{15}}\right) & \text{if } j = 1,\\ L_{75}\left(\frac{8}{n^{10}}\right) & \text{if } j = 2,\\ L_{75}\left(\frac{8}{n^7}\right) & \text{if } j = 3. \end{cases} \tag{3.3}$$

Proof. For $i \neq j$ we have

$$y\left|\theta^{(i)} - \theta^{(j)}\right| \leq 2\left|\beta^{(i)}\right|,$$

then we obtain

$$\left|\beta^{(j)}\right| = \frac{1}{\prod_{i\neq j}\left|\beta^{(i)}\right|} \leq \frac{4}{y^2} \cdot \frac{1}{\prod_{i\neq j}\left|\theta^{(i)} - \theta^{(j)}\right|}. \tag{3.4}$$

Since

$$\prod_{i\neq j}\left|\theta^{(i)}-\theta^{(j)}\right|\geq n^{10},\qquad \text{for } n\geq 75 \text{ and } j=1, \tag{3.5a}$$

$$\prod_{i\neq j}\left|\theta^{(i)}-\theta^{(j)}\right|\geq n^{3},\qquad \text{for } n\geq 75 \text{ and } j=2,\ 3, \tag{3.5b}$$

we have $|\theta^{(j)}-x/y|<1/(2y^2)$, hence $|x|/|y|$ is a convergent to $|\theta^{(j)}|$. Using our asymptotic expansions (2.1), we see that

$$\frac{0.97}{n^3}<-\theta^{(2)}<\frac{1}{n^3}\quad \text{for } n\geq 75 \qquad \text{and} \qquad n^3<-\frac{1}{\theta^{(2)}}<\frac{1}{0.97}n^3.$$

So

$$-\theta^{(2)}=[0,a_1,\ldots] \text{ with } a_1\geq n^3.$$

So if $y\geq 2$, we have $y\geq a_1\geq n^3$. For $\theta^{(3)}$, with the same method, one can show that if $y\geq 2$, we have $y\geq n^2$.

One can see that if (x,y) is a solution not contained in (1.3), then

$$y\geq\begin{cases} n^3 & \text{if } j=2,\\ n^2 & \text{if } j=3.\end{cases} \tag{3.6}$$

Therefore (3.4) yields

$$|\beta^{(j)}|\leq\begin{cases} \dfrac{4}{n^9} & \text{if } j=2,\\[2ex] \dfrac{4}{n^7} & \text{if } j=3.\end{cases} \tag{3.7}$$

So we obtain (3.2). Moreover, we know that

$$\frac{\left|\beta^{(i)}\right|}{y\left|\theta^{(i)}-\theta^{(j)}\right|}=\left|1+\frac{\beta^{(j)}}{y(\theta^{(j)}-\theta^{(i)})}\right|.$$

Then we take the log of the previous expression and we use equations (2.1), (3.2), and (3.6) to get

$$\log\left|\beta^{(i)}\right|=\log y+\log\left|\theta^{(i)}-\theta^{(j)}\right|+\begin{cases} L_{75}\left(\frac{1}{n^{15}}\right) & \text{if } j=1,\\ L_{75}\left(\frac{8}{n^{10}}\right) & \text{if } j=2,\\ L_{75}\left(\frac{8}{n^{7}}\right) & \text{if } j=3.\end{cases} \tag{3.8}$$

This completes the proof. □

Lemma 3.2. *Let* (x, y) *be a solution to* (1.1) *with* $y \geq 2$ *and* $n \geq 75$. *Then*

$$\log y \geq \left(\frac{19n}{6} + \frac{19}{12} + \frac{4}{3n}\right)\log^2(n) + \left(\frac{19}{3n} - \frac{11}{3}\right)\log(n). \tag{3.9}$$

Proof. If (x, y) is a solution to (1.1), then β is a unit in $\mathbb{Z}[\theta]$. By Remark 2.2 there are integers $u_1,\ u_2,\ I$ with $I \leq 2$ such that

$$\beta^I = \pm\left(\theta^{(1)}\right)^{u_1}\left(\theta^{(2)}\right)^{u_2}. \tag{3.10}$$

But a generator σ of the Galois group G of $\mathbb{K}_n$ is defined by

$$\sigma(\theta^{(1)}) = \frac{(-n^2 - n - 1)\theta^{(1)} - 1}{(n^4 + n^3 + 3n^2 + n + 1)\theta^{(1)} + n}. \tag{3.11}$$

One can check that

$$\theta^{(2)} = \sigma^{-1}\left(\theta^{(1)}\right),\ \theta^{(3)} = \sigma^{-1}\left(\theta^{(2)}\right),\ \theta^{(1)} = \sigma^{-1}\left(\theta^{(3)}\right). \tag{3.12}$$

So we have

$$\begin{cases} |(\beta^{(1)})^I| = |\theta^{(1)}|^{u_1}|\theta^{(2)}|^{u_2}, \\ |(\beta^{(2)})^I| = |\theta^{(2)}|^{u_1}|\theta^{(3)}|^{u_2}, \\ |(\beta^{(3)})^I| = |\theta^{(3)}|^{u_1}|\theta^{(1)}|^{u_2}; \end{cases} \tag{3.13}$$

therefore

$$\begin{cases} \log|\beta^{(1)}| = \dfrac{u_1}{I}\log|\theta^{(1)}| + \dfrac{u_2}{I}\log|\theta^{(2)}|, \\ \log|\beta^{(2)}| = \dfrac{u_1}{I}\log|\theta^{(2)}| + \dfrac{u_2}{I}\log|\theta^{(3)}|, \\ \log|\beta^{(3)}| = \dfrac{u_1}{I}\log|\theta^{(3)}| + \dfrac{u_2}{I}\log|\theta^{(1)}|. \end{cases} \tag{3.14}$$

We compute the asymptotic expression of the regulator and we obtain:

$$R = 19\log^2(n) + \left(\frac{8}{n} + \frac{34}{n^2} + L_{75}\left(\frac{20.7}{n^3}\right)\right)\log(n) + \frac{1}{n^2} + L_{75}\left(\frac{7.2}{n^3}\right). \tag{3.15}$$

Moreover, we have:

$$R < (19 + 0.05)\log^2(n), \tag{3.16}$$

for $n \geq 75$. For each j, from (3.14), we consider the subsystem not containing $\beta^{(j)}$ that we solve to determine u_1 and u_2 using Cramer's method. Then we use the asymptotic expressions (2.1), (2.2), and (3.3) to obtain

$$\frac{Ru_1}{I} = \begin{cases} \left(7\log(n)+\frac{1}{n}+\frac{13}{2n^2}+L_{75}\left(\frac{3.8}{n^3}\right)\right)\log(y)+35\log^2(n) \\ \quad +\left(\frac{12}{n}+\frac{64}{n^2}+L_{75}\left(\frac{37.5}{n^3}\right)\right)\log(n)+\frac{1}{n^2}+L_{75}\left(\frac{12}{n^3}\right) & \text{if } j=1, \\ \left(8\log(n)+\frac{2}{n}+\frac{7}{n^2}+L_{75}\left(\frac{4.5}{n^3}\right)\right)\log(y)+19\log^2(n) \\ \quad +\left(\frac{5}{n}+\frac{71}{2n^2}+L_{75}\left(\frac{18.6}{n^3}\right)\right)\log(n)+L_{75}\left(\frac{5.2}{n^3}\right) & \text{if } j=2, \\ \left(\log(n)+\frac{1}{n}+\frac{1}{2n^2}+L_{75}\left(\frac{1}{n^3}\right)\right)\log(y)-16\log^2(n) \\ \quad +\left(-\frac{7}{n}-\frac{57}{2n^2}+L_{75}\left(\frac{18.8}{n^3}\right)\right)\log(n)-\frac{1}{n^2}+L_{75}\left(\frac{6.1}{n^3}\right) & \text{if } j=3, \end{cases} \tag{3.17}$$

and

$$\frac{Ru_2}{I} = \begin{cases} \left(-\log(n)-\frac{1}{n}-\frac{1}{2n^2}+L_{75}\left(\frac{0.72}{n^3}\right)\right)\log(y)-5\log^2(n) \\ \quad +\left(-\frac{6}{n}-\frac{7}{n^2}+L_{75}\left(\frac{7}{n^3}\right)\right)\log(n)-\frac{1}{n^2}+L_{75}\left(\frac{5.1}{n^3}\right) & \text{if } j=1, \\ \left(7\log(n)+\frac{1}{n}+\frac{13}{2n^2}+L_{75}\left(\frac{3.7}{n^3}\right)\right)\log(y) \\ \quad +\left(\frac{-5}{n}+\frac{5}{2n^2}+L_{75}\left(\frac{3.6}{n^3}\right)\right)\log(n)-\frac{1}{n^2}+L_{75}\left(\frac{4.1}{n^3}\right) & \text{if } j=2, \\ \left(8\log(n)+\frac{2}{n}+\frac{7}{n^2}+L_{75}\left(\frac{4.4}{n^3}\right)\right)\log(y)+5\log^2(n) \\ \quad +\left(\frac{1}{n}+\frac{19}{2n^2}+L_{75}\left(\frac{2.9}{n^3}\right)\right)\log(n)+L_{75}\left(\frac{1.1}{n^3}\right) & \text{if } j=3. \end{cases} \tag{3.18}$$

For each j, we define the following linear combinations of the $\frac{Ru_k}{I}$:

$$\frac{Rv_j}{I} := \begin{cases} -\frac{Ru_1}{I} - 7\frac{Ru_2}{I} & \text{if } j=1, \\ 7\frac{Ru_1}{I} - 8\frac{Ru_2}{I} - 7R & \text{if } j=2, \\ 8\frac{Ru_1}{I} - \frac{Ru_2}{I} + 7R & \text{if } j=3, \end{cases} \tag{3.19}$$

i.e.

$$v_j := \begin{cases} -u_1 - 7u_2 & \text{if } j=1, \\ 7u_1 - 8u_2 - 7I & \text{if } j=2, \\ 8u_1 - u_2 + 7I & \text{if } j=3, \end{cases} \tag{3.20}$$

where $I \leq 2$ (See Remark 2.2). Then from (3.19), we use the expressions (3.17) and (3.18) to get

$$\frac{Rv_j}{I} = \begin{cases} \left(\frac{6}{n} - \frac{3}{n^2} + L_{75}\left(\frac{1}{n^3}\right)\right)\log(y) \\ \quad +\left(\frac{30}{n} - \frac{15}{n^2} + L_{75}\left(\frac{5}{n^3}\right)\right)\log(n) + \frac{6}{n^2} + L_{75}\left(\frac{126.9}{n^3}\right) & \text{if } j = 1, \\ \left(\frac{6}{n} - \frac{3}{n^2} + L_{75}\left(\frac{1}{n^3}\right)\right)\log(y) \\ \quad +\left(\frac{19}{n} - \frac{19}{2n^2} + L_{75}\left(\frac{38}{3n^3}\right)\right)\log(n) + \frac{1}{n^2} + L_{75}\left(\frac{330.7}{n^3}\right) & \text{if } j = 2, \\ \left(\frac{6}{n} - \frac{3}{n^2} + L_{75}\left(\frac{1}{n^3}\right)\right)\log(y) \\ \quad +\left(-\frac{1}{n} + \frac{1}{2n^2} + L_{75}\left(\frac{29}{3n^3}\right)\right)\log(n) - \frac{1}{n^2} + L_{75}\left(\frac{146.1}{n^3}\right) & \text{if } j = 3. \end{cases} \tag{3.21}$$

As $y \geq 2$, we have $\dfrac{Rv_j}{I} \geq R$. Therefore, (3.21) helps to obtain (3.9). $\square$

4 Large solutions

Suppose that $(x, y) \in \mathbb{Z}^2$ is a non trivial solution of type j. We choose indices (i, k) depending on j:

$$(i, k) = \begin{cases} (2, 3) & \text{if } j = 1, \\ (3, 1) & \text{if } j = 2, \\ (1, 2) & \text{if } j = 3. \end{cases}$$

We use the following Siegel identity

$$\frac{\beta^{(k)}(\theta^{(j)} - \theta^{(i)})}{\beta^{(i)}(\theta^{(j)} - \theta^{(k)})} - 1 = \frac{\beta^{(j)}(\theta^{(k)} - \theta^{(i)})}{\beta^{(i)}(\theta^{(j)} - \theta^{(k)})}.$$

We put

$$\lambda_j = \frac{\theta^{(j)} - \theta^{(i)}}{\theta^{(j)} - \theta^{(k)}}, \qquad \tau_j = \frac{\beta^{(j)}}{\beta^{(i)}}\left(\frac{\theta^{(k)} - \theta^{(i)}}{\theta^{(j)} - \theta^{(k)}}\right)$$

and from (3.14) we obtain the following linear form in logarithms

$$\Lambda_j = \frac{u_1}{I}\log\left|\frac{\theta^{(k)}}{\theta^{(i)}}\right| + \frac{u_2}{I}\log\left|\frac{\theta^{(j)}}{\theta^{(k)}}\right| + \log\left|\lambda_j\right| = \log\left|1 + \tau_j\right|. \tag{4.1}$$

Lemma 4.1. *We have $\Lambda_j \neq 0$.*

Proof. Suppose that $\Lambda_j = 0$, then from (4.1) we have $\tau_j = 0$ or $\tau_j = -2$. It is impossible that $\tau_j = 0$ because the polynomial $\phi_n(x)$ has three distinct nonzero roots. The case $\tau_j = -2$, cannot occur since τ_j obviously has norm 1. $\square$

From (4.1), we have

$$\log\left|\Lambda_j\right| = \log\left|\log\left|1+\tau_j\right|\right| \le \log\left|2\tau_j\right| = \log\left|2\frac{\beta^{(j)}}{\beta^{(i)}}\left(\frac{\theta^{(k)}-\theta^{(i)}}{\theta^{(j)}-\theta^{(k)}}\right)\right|.$$

By (2.3), (3.2) and (3.3), we obtain the following upper bounds of Λ_j:

$$\log\left|\Lambda_j\right| \le -3\log y + \log 8 + \begin{cases} -22\log n - \dfrac{3}{n} & \text{if } j = 1, \\ -\log n + \dfrac{2}{n} & \text{if } j = 2, 3. \end{cases} \tag{4.2}$$

Now we will prove the following result.

Proposition 4.2. *Let* $(x, y) \in \mathbb{Z}^2$ *be a solution to* (1.1) *of type* j *which is not listed in* (1.3). *Then* $n \le N_j$, *where*

$$N_j := \begin{cases} 27301619 & \textit{if } j = 1, \\ 58519951 & \textit{if } j = 2, \\ 5044681 & \textit{if } j = 3. \end{cases} \tag{4.3}$$

Proof. Using (3.19) and (4.1), we rewrite Λ_j as

$$7I\Lambda_1 = u_1 \log\left|\left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^7\left(\frac{\theta^{(3)}}{\theta^{(1)}}\right)\right| + \log\left|\lambda_1^{7I}\left(\frac{\theta^{(3)}}{\theta^{(1)}}\right)^{v_1}\right|, \tag{4.4a}$$

$$8I\Lambda_2 = u_1 \log\left|\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^8\left(\frac{\theta^{(2)}}{\theta^{(1)}}\right)^7\right| + \log\left|\lambda_2^{8I}\left(\frac{\theta^{(1)}}{\theta^{(2)}}\right)^{v_2+7I}\right|, \tag{4.4b}$$

$$8I\Lambda_3 = u_2 \log\left|\left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^8\left(\frac{\theta^{(2)}}{\theta^{(1)}}\right)\right| + \log\left|\lambda_3^{8I}\left(\frac{\theta^{(2)}}{\theta^{(1)}}\right)^{v_3-7I}\right|. \tag{4.4c}$$

For this we use the following result due to Laurent, Mignotte, and Nesterenko (see [9], Corollaire 2, page 288) on linear forms in two logarithms to determine lower bounds of Λ_j. For any non-zero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}\left(X-\gamma^{(j)}\right)$, we denote by

$$h(\gamma) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d}\log\max\left(1, \left|\gamma^{(j)}\right|\right)\right)$$

its absolute logarithmic height.

Theorem 4.3. *Let γ_1 and γ_2 be multiplicatively independent and positive algebraic numbers, b_1 and $b_2 \in \mathbb{Z}$ and*

$$\Lambda = b_1 \log \gamma_1 + b_2 \log \gamma_2.$$

Let $D := [\mathbb{Q}(\gamma_1, \gamma_2)\colon \mathbb{Q}]$, for $i = 1, 2$ let

$$h_i \geq \max \left\{ h(\gamma_i),\ \frac{|\log \gamma_i|}{D},\ \frac{1}{D} \right\}$$

and

$$b' \geq \frac{|b_1|}{D\, h_2} + \frac{|b_2|}{D\, h_1}.$$

If $|\Lambda| \neq 0$, then we have

$$\log|\Lambda| \geq -24.34 \cdot D^4 \left(\max \left\{ \log b' + 0.14, \frac{21}{D}, \frac{1}{2} \right\} \right)^2 h_1 h_2.$$

Remark 4.4. We tried to use Corollary 2.2 of [4], but unfortunately we didn't have a better result. So we decide to use the above theorem.

We take $D = 3$.

- For $j = 1$, we take

$$\gamma_1 = \left(\frac{\theta^{(3)}}{\theta^{(2)}} \right)^7 \left(\frac{\theta^{(3)}}{\theta^{(1)}} \right); \quad \gamma_2 = \lambda_1^{7I} \left(\frac{\theta^{(3)}}{\theta^{(1)}} \right)^{v_1}.$$

The algebraic numbers γ_1 and γ_2 are multiplicatively independent because

$$\begin{vmatrix} \log|\gamma_1| & \log|\gamma_2| \\ \log|\sigma(\gamma_1)| & \log|\sigma(\gamma_2)| \end{vmatrix} < -790 \log^2 n.$$

We determine the three conjugates of γ_i, for $i = 1, 2$. Then we use their asymptotic expressions to compare their absolute values to 1. Therefore, we use the above definition of $h(\gamma)$ to obtain

$$h(\gamma_1) \leq \frac{1}{3} \log \left| \left(\frac{\theta^{(1)}}{\theta^{(2)}} \right)^7 \left(\frac{\theta^{(3)}}{\theta^{(2)}} \right) \right|;$$

$$h(\gamma_2) \leq \frac{1}{3} \log \left| \left(\frac{\theta^{(3)} - \theta^{(1)}}{\theta^{(3)} - \theta^{(2)}} \right)^{7I} \left(\frac{\theta^{(1)}}{\theta^{(2)}} \right)^{v_1} \right|.$$

• For $j = 2$, we take

$$\gamma_1 = \left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^8 \left(\frac{\theta^{(2)}}{\theta^{(1)}}\right)^7; \quad \gamma_2 = \lambda_2^{8I} \left(\frac{\theta^{(1)}}{\theta^{(2)}}\right)^{v_2+7I}.$$

The algebraic numbers γ_1 and γ_2 are multiplicatively independent because

$$\begin{vmatrix} \log|\gamma_1| & \log|\gamma_2| \\ \log|\sigma(\gamma_1)| & \log|\sigma(\gamma_2)| \end{vmatrix} < -430 \log^2 n$$

The use a similar method done for $j = 1$ leads to

$$h(\gamma_1) \leq \frac{1}{3} \log \left| \left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^8 \left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^7 \right|;$$

$$h(\gamma_2) \leq \frac{1}{3} \log \left| \left(\frac{\theta^{(3)} - \theta^{(1)}}{\theta^{(3)} - \theta^{(2)}}\right)^{8I} \left(\frac{\theta^{(1)}}{\theta^{(2)}}\right)^{v_2+7I} \right|.$$

• For $j = 3$, we take

$$\gamma_1 = \left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^8 \left(\frac{\theta^{(2)}}{\theta^{(1)}}\right); \quad \gamma_2 = \lambda_3^{8I} \left(\frac{\theta^{(2)}}{\theta^{(1)}}\right)^{v_3-7I}.$$

The algebraic numbers γ_1 and γ_2 are multiplicatively independent because

$$\begin{vmatrix} \log|\gamma_1| & \log|\gamma_2| \\ \log|\sigma(\gamma_1)| & \log|\sigma(\gamma_2)| \end{vmatrix} > 118230 \log^2 n.$$

We apply a similar method done for $j = 1, 2$ to have

$$h(\gamma_1) \leq \frac{1}{3} \log \left| \left(\frac{\theta^{(2)}}{\theta^{(3)}}\right)^8 \left(\frac{\theta^{(1)}}{\theta^{(3)}}\right) \right|;$$

$$h(\gamma_2) \leq \frac{1}{3} \log \left| \left(\frac{\theta^{(3)} - \theta^{(1)}}{\theta^{(3)} - \theta^{(2)}}\right)^{8I} \left(\frac{\theta^{(1)}}{\theta^{(2)}}\right)^{v_3-7I} \right|.$$

Thus the choice of h_1, h_2, and b' depending on j is given in Table 1 below.

Case	h_1	h_2	b'
$j=1$	$21\log n+\frac{6}{n}$	$\left(\frac{32}{19n\log n}-\frac{0.95}{n^2\log n}\right)\log y+\frac{98}{3}\log n+\frac{1012}{57n}+\frac{23}{n^2}$	$\frac{3n}{250}$
$j=2$	$19\log n+\frac{8}{n}$	$\left(\frac{32}{19n\log n}-\frac{16}{19n^2\log n}\right)\log y+\frac{224}{3}\log n+\frac{76}{3n}+\frac{60}{n^2}$	$\frac{11n}{900}$
$j=3$	$\frac{71}{3}\log n+\frac{25}{3n}$	$\left(\frac{32}{19n\log n}-\frac{25}{19n^2\log n}\right)\log y+\frac{20}{19n}-\frac{3}{n^2}$	$\frac{11n}{60}$

Table 1: Choice of h_1, h_2, and b' depending on j.

Therefore we get

$$\begin{aligned}\log|I\Lambda_1| \geq\ & -1971.54\left(\log\left(\frac{3n}{250}\right)+.14\right)^2\left(21\log n+\frac{6}{n}\right)\\ & \left[\left(\frac{32}{19n\log n}-\frac{0.95}{n^2\log n}\right)\log y+\frac{98}{3}\log n+\frac{1012}{57n}+\frac{23}{n^2}\right]-\log 7I \end{aligned}\tag{4.5a}$$

$$\begin{aligned}\log|I\Lambda_2| \geq\ & -1971.54\left(\log\left(\frac{11n}{900}\right)+.14\right)^2\left(19\log n+\frac{8}{n}\right)\\ & \left[\left(\frac{32}{19n\log n}-\frac{16}{19n^2\log n}\right)\log y+\frac{224}{3}\log n+\frac{76}{3n}+\frac{60}{n^2}\right]-\log 8I, \end{aligned}\tag{4.5b}$$

$$\begin{aligned}\log|I\Lambda_3| \geq\ & -1971.54\left(\log\left(\frac{11n}{60}\right)+.14\right)^2\left(\frac{71}{3}\log n+\frac{25}{3n}\right)\\ & \left[\left(\frac{32}{19n\log n}-\frac{25}{19n^2\log n}\right)\log y+\frac{20}{19n}-\frac{3}{n^2}\right]-\log 8I. \end{aligned}\tag{4.5c}$$

By combining (4.2), (4.5) and Lemma 3.2, we obtain the result. □

5 Solutions for $75 < n \leq N_j$

The aim of this section is to verify that for $75 < n \leq N_j$ the only solutions to (1.1) are those listed in (1.3). As a first step, we use linear forms in logarithms once again in order to obtain an upper bound for $\log y$:

Lemma 5.1. *For $75 < n \leq N_j$, we have*

$$\log y \leq \begin{cases} 7.72\cdot 10^{21}\log n & \text{if } j=1,\\ 7.74\cdot 10^{21}\log n & \text{if } j=2,\\ 6.26\cdot 10^{21}\log n & \text{if } j=3.\end{cases}\tag{5.1}$$

Proof. We use an estimate due to Baker and Wüstholz [3]:

Theorem 5.2 (Baker-Wüstholz). *Let $\gamma_1, \ldots, \gamma_n$ be algebraic numbers, not 0 or 1, $K = \mathbb{Q}(\gamma_1, \ldots, \gamma_n)$ and d the degree $[K : \mathbb{Q}]$. For $i = 1, \ldots, n$ let*

$$h_i \geq \max\left(h(\gamma_i),\ \frac{|\log(\gamma_i)|}{d},\ \frac{1}{d}\right).$$

Let $u_1, \ldots, u_n \in \mathbb{Z}$, $\Lambda = u_1 \log \gamma_1 + \ldots + u_n \log \gamma_n \neq 0$ and $U \geq \max |u_i|$. Then we have

$$\log|\Lambda| > -C(n,d) h_1 \cdots h_n \log U, \tag{5.2}$$

where

$$C(n,d) = 18(n+1)!n^{n+1}(32d)^{n+2}\log(2nd).$$

We note that (3.21) and Lemma 3.2 yield

$$v_1 \leq \frac{12.02}{19n\log^2 n}\log y, \quad v_2 \leq \frac{12.02}{19n\log^2 n}\log y, \quad v_3 \leq \frac{12}{19n\log^2 n}\log y. \tag{5.3}$$

From the asymptotic expansions of u_1 and u_2 for all j, see (3.17) and (3.18), we observe that for $1 \leq j \leq 3$

$$U := \max\{|u_1|, |u_2|\} = \begin{cases} u_1, & \text{if } j = 1, 2, \\ u_2, & \text{if } j = 3, \end{cases} \tag{5.4}$$

then we have

$$U \leq \begin{cases} \dfrac{0.74}{\log n}\log y, & \text{if } j = 1, \\[2ex] \dfrac{0.85}{\log n}\log y, & \text{if } j = 2, 3. \end{cases} \tag{5.5}$$

Now we apply Theorem 5.2 to Λ_j as it is defined in (4.1). So we take $n = 3$, $d = 6$, and for $j = 1, 2, 3$

$$h_1 = h_2 = 3\log(n),\ h_3 = \frac{8I}{3}\log(n).$$

We use the estimate of U given by (5.5) and combine the lower bound with (4.2) to get

$$-2.99\log y \geq \log\left|I\Lambda_j\right| \geq -C(3,6)h_1h_2h_3\log\left(\frac{0.74}{\log n}\log y\right), \quad \text{if } j = 2;$$

and

$$-2.99 \log y \geq \log \left| I\Lambda_j \right| \geq -C(3,6) h_1 h_2 h_3 \log \left(\frac{0.85}{\log n} \log y \right), \quad \text{if } j = 2, 3.$$

Consequently, considering that $75 \leq n \leq N_j$ we obtain

$$\frac{\frac{0.74}{\log n} \log y}{\log \left(\frac{0.74}{\log n} \log y \right)} \leq 1.14 \cdot 10^{20} \quad \text{if } j = 1$$

and

$$\frac{\frac{0.85}{\log n} \log y}{\log \left(\frac{0.85}{\log n} \log y \right)} \leq \begin{cases} 1.31 \cdot 10^{20} & \text{if } j = 2, \\ 1.07 \cdot 10^{20} & \text{if } j = 3. \end{cases}$$

This yields (5.1). □

We write (4.4) as

$$m_j I \Lambda_j = \log \left| \gamma_{j1} \right| + v_j \log \left| \gamma_{j2} \right| + v_j' \log \left| \gamma_{j3} \right|, \tag{5.6}$$

where the notations are defined in Table 2.

j	m_j	γ_{j1}	γ_{j2}	γ_{j3}	v_j'
1	7	λ_1^{7I}	$\frac{\theta^{(3)}}{\theta^{(1)}}$	$\left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^7 \left(\frac{\theta^{(3)}}{\theta^{(1)}}\right)$	$u_1,$
2	8	$\lambda_2^{8I} \left(\frac{\theta^{(1)}}{\theta^{(2)}}\right)^{7I}$	$\frac{\theta^{(1)}}{\theta^{(2)}}$	$\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^8 \left(\frac{\theta^{(2)}}{\theta^{(1)}}\right)$	$u_1,$
3	8	$\lambda_3^{8I} \left(\frac{\theta^{(1)}}{\theta^{(2)}}\right)^{7I}$	$\frac{\theta^{(2)}}{\theta^{(1)}}$	$\left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^8 \left(\frac{\theta^{(2)}}{\theta^{(1)}}\right)$	$u_2.$

Table 2: Notations for (5.6)

We divide by $\log \left| \gamma_{j3} \right|$, use (4.2), (3.9), and $n \geq 75$, and obtain

$$\left| \delta_{j1} + v_j \delta_{j2} + v_j' \right| < 10^{-393\,920}, \tag{5.7}$$

where $\delta_{ji} := \log \left| \gamma_{ji} \right| / \log \left| \gamma_{j3} \right|$ for $i = 1, 2$. We use the following lemma of Heuberger, Pethö, and Tichy [6] (which is implicitly used in Baker and Davenport [2]):

Lemma 5.3. *Let δ_1 and δ_2, $M \in \mathbb{R}$, A and B integers and*

$$|A + B\delta_2 + \delta_1| < M.$$

Furthermore, let $Q \in \mathbb{N}$, $\tilde{\delta_1}, \tilde{\delta_2} \in \mathbb{Q}$ *with* $|\delta_i - \tilde{\delta_i}| < Q^{-2}$ *for* $i = 1, 2$ *and* p/q *a principal convergent of* $\tilde{\delta_2}$ *with* $q < Q$. *Then we have*

$$q\,\|q\tilde{\delta_1}\| \leq Q^2 M + 1 + 2|B|$$

where $\|\cdot\|$ *denotes the distance to the nearest integer.*

One can see that (5.1) and (5.3) imply

$$2 + 2\,|v_1| \leq \frac{4.89 \cdot 10^{21}}{n \log n}, \quad 2 + 2\,|v_2| \leq \frac{4.90 \cdot 10^{21}}{n \log n}, \quad 2 + 2\,|v_3| \leq \frac{3.96 \cdot 10^{21}}{n \log n}. \quad (5.8)$$

For all pairs (j, n) with $1 \leq j \leq 3$ and $75 < n \leq N_j$, we calculate approximations $\tilde{\delta}_{j1}$ and $\tilde{\delta}_{j2}$ such that $|\delta_{j1} - \tilde{\delta}_{j1}| < Q^{-2}$ and $|\delta_{j2} - \tilde{\delta}_{j2}| < Q^{-2}$. For all pairs of (j, n), we compute the successive convergents of $\tilde{\delta}_{j2}$ until we find a convergent p/q of $\tilde{\delta}_{j2}$ with $q < Q$ such that

$$q\,\|q\tilde{\delta}_{j1}\| > \frac{1}{n \log n} \cdot \begin{cases} 4.89 \cdot 10^{21} & \text{if } j = 1, \\ 4.90 \cdot 10^{21} & \text{if } j = 2, \\ 3.96 \cdot 10^{21} & \text{if } j = 3. \end{cases}$$

This is a contradiction to Lemma 5.3 and proves the following proposition:

Proposition 5.4. *For* $75 < n \leq N_j$ *there are no solutions* $(x, y) \in \mathbb{Z}^2$ *to* (1.1) *of type* j *which are not listed in* (1.3).

Here are a few remarks about the computations. The program was developed in MAPLE 11 and executed on a 2.6 GHz Pentium-4 Dell of a lab of Purdue University North Central. We have to use a high precision for the computations because of the large values of the parameter. In general, we did the computations with the accuracy about 100 digits.

- For $j = 1, 2, 3$ (together) and $75 \leq n \leq 5044681$, we ran the program. It took in average 2.74 seconds for each value of n. We start with $Q = 10^{19}$, if it is not successful we try successively 10^{20}, 10^{21}, 10^{22}, $10^{25} \ldots$ until we obtain the desired results.

- For $j = 1, 2$ and $5044681 \leq n \leq 27301619$, we ran the program. It took in average 3.83 seconds for each value of n. For this range, we start with $Q = 10^{40}$, if it is not successful we try successively 10^{41}, 10^{42}, 10^{43}, $10^{44} \ldots$ until we obtain the desired results.

- For $j = 2$ and $27301619 \leq n \leq 58519951$, we did the same thing. It took in average 3.54 seconds for each value of n. Here, we start with $Q = 10^{44}$, if it is not successful we try successively 10^{45}, 10^{46}, 10^{47}, $10^{48} \ldots$ until we obtain the desired results.

6 Solutions with $0 \leq n \leq 75$

Since some of our asymptotic expansions are not valid for $n \leq 75$, we use Kant [5] (Kash Version 2.4) to solve (1.1) for $0 \leq n \leq 75$. We exactly get the solutions listed in (1.3) and (1.4). This completes the proof of Theorem 1.1.

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Alain Togbé
Mathematics Department
Purdue University North Central
1401 S, U.S. 421, Westville IN 46391
USA

E-mail: atogbe@pnc.edu