

Schottky groups can not act on P_C^{2n} as subgroups of $PSL_{2n+1}(C)$

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Abstract. In this paper we look at a special type of discrete subgroups of $PSL_{n+1}(C)$ called Schottky groups. We develop some basic properties of these groups and their limit set when $n > 1$, and we prove that Schottky groups only occur in odd dimensions, *i.e.*, they cannot be realized as subgroups of $PSL_{2n+1}(C)$.

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1 Introduction

Schottky groups play a significant role in the theory of classical Kleinian groups and Riemann surfaces (see for instance [3, 4, 5]). Their analogues in higher dimensions were introduced by Nori [8] and Seade-Verjovsky [9], though these groups were also known to N. Hitchin (see the commentary of Nori in in [8]). These are a special type of discrete groups of automorphisms of complex projective spaces having non-empty region of discontinuity, where the action is “free” with compact quotient. Hence they are a rich source for complex compact manifolds equipped canonically with a projective structure. Schottky groups also have very interesting dynamics in their limit set, the complement of the region of discontinuity. Moreover, these groups are neither Fuchsian (*i.e.*, subgroups of $PU(n, 1)$) nor affine in general. Thus, if we want to study Kleinian actions on higher dimensional complex projective spaces, Schottky groups provide a very nice starting point.

So far Schottky groups have been studied only for odd-dimensional projective spaces (in [8, 9]). It is thus natural to ask whether Schottky groups exists in even

dimensions. In this paper we prove they do not: Schottky groups cannot act by complex automorphism on P_C^{2n} . Hence, in order to construct discrete groups of automorphisms of P_C^{2n} with a rich underlying geometry and dynamics one must follow different methods. This is done for P_C^2 in [1, 6, 7].

This paper is divided into four sections. In section 1 we define what Schottky groups are and we state the main result of this article. In section 2 we develop some basic dynamical and algebraic facts about Schottky groups. In section 3 we look at the limit set of infinite cyclic groups; and in section 4 we use the previous information to show that Schottky groups cannot be realized in even dimensions.

2 Notations and the Main Result

We recall that the complex projective space P_C^n is defined as:

$$P_C^n = (C^{n+1} - \{0\}) / \sim,$$

where “ $\sim$ ” denotes the equivalence relation given by $x \sim y$ if and only if $x = \alpha y$ for some non-zero complex scalar α . We know that P_C^n is a compact connected complex n -dimensional manifold, which is naturally equipped with the Fubini-Study metric (see for instance [7]).

If $[\]_n : C^{n+1} - \{0\} \rightarrow P_C^n$ represents the quotient map, then a non-empty set $H \subset P_C^n$ is said to be a projective subspace of dimension k (in symbols $\dim_C(H) = k$) if there is a C -linear subspace $\tilde{H}$ of dimension $k+1$ (in symbols $\dim_C(\tilde{H}) = k+1$), such that $[\tilde{H}]_n = H$. Given a set of points P in P_C^n , we define

$$\langle P \rangle = \bigcap \{l \subset P_C^n \mid l \text{ is a projective subspace and } P \subset l\}.$$

So that $\langle P \rangle$ is a projective subspace of P_C^n , see [2].

From now on, the symbols $e_1, \dots, e_{n+1}$ will either denote the elements of the standard basis in C^{n+1} or their images under $[\]_n$.

Consider the action of Z_n (regarded as the n -roots of unity) on $SL(n, C)$ given by $\alpha(a_{i,j}) = (\alpha a_{i,j})$. The quotient $PSL_n(C) = SL_n(C)/Z_n$ is a Lie Group whose elements are called projective transformations. Every representative $\tilde{\gamma}$ of the coset $\gamma = Z_n \tilde{\gamma} = \gamma \in PSL_n(C)$ will be called a lifting of γ . Observe that $\gamma \in PSL_{n+1}(C)$ acts on P_C^n as a biholomorphic map by $\gamma([w]_n) = [\tilde{\gamma}(w)]_n$, where $[w]_n \in P_C^n$ and $\tilde{\gamma}$ is a lifting of γ .

Definition 1. A subgroup $\Gamma \leq PSL_{n+1}(C)$ is called a Schottky group if:

1. There are $2g$, $g \geq 2$, opens sets $R_1, \dots, R_g, S_1, \dots, S_g$ in P_C^n with the property that:
 - (a) each of these open sets is the interior of its closure; and
 - (b) the closures of the $2g$ open sets are pairwise disjoint.
2. Γ has a generating set $\text{Gen}(\Gamma) = \{\gamma_1, \dots, \gamma_g\}$ such that $\gamma(R_j) = P_C^n - \overline{S_j}$ for all $1 \leq j \leq g$, here the bar means topological closure.

From now on $\text{Int}(A)$ will denote the topological interior and $\partial(A)$ the topological boundary of the set A and for each $1 \leq j \leq g$, R_j and S_j will be denoted by $R_{\gamma_j}^*$ and $S_{\gamma_j}^*$ respectively.

Example 1.

1. Every classical Schottky group of Möbius transformations (see [3, 4, 5]) is Schottky in the sense of Definition 1. Moreover by the characterization of Schottky groups acting on the Riemann sphere given by Maskit [4], it is not hard to prove that every group of Möbius transformations which is Schottky in the sense of Definition 1, is a Schottky group in $PSL_2(C)$.
2. In [8] Nori gave the following construction of the higher-dimensional analogues of the classical Schottky groups: let $n = 2k + 1$, $k > 1$ and $g \geq 1$. Choose $2g$ mutually disjoint projective subspaces $L_1, \dots, L_{2g}$ of dimension k in P_C^n and $0 < \alpha < \frac{1}{2}$. For every integer $1 \leq j \leq g$ choose a basis of C^{n+1} so that $L_j = [\{z_0, \dots, z_k = 0\} - \{0\}]_n$ and $L_{g+j} = [\{z_{k+1}, \dots, z_n = 0\} - \{0\}]_n$. Define $\phi_j : P_C^n \rightarrow R$ by the formula

$$\phi_j[z_0, \dots, z_n] = \frac{|z_0|^2 + \dots + |z_k|^2}{|z_0|^2 + \dots + |z_n|^2}$$

and consider the open neighborhoods $V_j = \{x \in P_C^n : \phi_j(x) < \alpha\}$ and $V_{g+j} = \{x \in P_C^n : \phi_j(x) > \alpha\}$ of L_j and L_{g+j} respectively. Consider the automorphism γ_j of P_C^n given by

$$\gamma_j[z_0, \dots, z_n] = [\lambda z_0, \dots, \lambda z_k, z_{k+1}, \dots, z_n]$$

where $\lambda \in C$ and $|\lambda| = \frac{1}{\alpha} - 1$. Then $\gamma_j(V_j) = P_C^n - \overline{V_{g+j}}$. Moreover for all α small the group Γ generated by $\gamma_1, \dots, \gamma_g$ is a Schottky group.

3. Let $L = \{L_1, \dots, L_g\}$, $g > 1$, be a set of g projective subspaces of dimension n of P_C^{2n+1} , all of them pairwise disjoint. In [9] it is shown that:
- (a) There exists a set $\{V_1, \dots, V_g\}$ of pairwise disjoint open sets of P_C^{2n+1} such that $P_C^{2n+1} - \partial(V_i)$ has 2 connected components for each $1 \leq j \leq g$, L_j is contained in V_j and the closures of the g open sets are pairwise disjoint.
 - (b) There are involutions $T_1, \dots, T_g$ of P_C^{2n+1} , such that each T_j , $j = 1, \dots, g$, interchanges the two connected components of $P_C^{2n+1} - \partial(V_j)$ and the boundary $\partial(V_j)$ is an invariant set.
 - (c) Let Γ be the group generated by $T_1, \dots, T_g$ and let $\tilde{\Gamma} \cup \{id\}$ be the subgroup consisting of elements of Γ which can be written as reduced words of even length in the generators (recall that $w = z_n^{\varepsilon_n} \dots z_2^{\varepsilon_2} z_1^{\varepsilon_1} \in \Gamma$ is a reduced word of length n if $z_\ell \in \{T_1, \dots, T_g\}$; $\varepsilon_\ell \in \{-1, +1\}$ and if $z_j = z_{j+1}$ then $\varepsilon_j = \varepsilon_{j+1}$). For $g > 2$ it is verified that $\tilde{\Gamma}$ is a Schottky group in the sense of Definition 1.

We prove:

Theorem 1. *If $\Gamma \leq PSL_{2n+1}(C)$ is a discrete subgroup, then Γ cannot be a Schottky group acting on P_C^{2n} .*

2.1 Basic properties of Schottky groups

Definition 2. *For a subgroup $\Gamma \leq PSL_n(C)$ satisfying Definition 1 we define:*

1. $F(\Gamma) = P_C^n - \left(\bigcup_{\gamma \in \text{Gen}(\Gamma)} R_\gamma^* \cup S_\gamma^* \right).$
2. $\Omega(\Gamma) = \bigcup_{\gamma \in \Gamma} \gamma(F(\Gamma)).$

Example 2. If $\Gamma \leq PSL_{2n}(C)$ is any of the groups of the Example 1, then $P_C^{2n+1} - \Omega(\Gamma)$ is homeomorphic to $P_C^n \times C$, where C is a Cantor set, see [5, 8, 9].

Proposition 1. *If Γ is a Schottky group, then:*

1. Γ is a free group generated by $\text{Gen}(\Gamma)$.
2. $\Omega(\Gamma)/\Gamma$ is a compact complex n -manifold and $\text{Int}(F(\Gamma))$ is a fundamental domain for the action of Γ .

Before we prove this result we state a definition and prove a technical lemma.

Definition 3. Let $\Gamma \leq PSL_n(C)$ be a subgroup. For an infinite subset $H \subset \Gamma$ and a non-empty, Γ -invariant open set $\Omega \subset P_C^n$, we define $Ac(H, \Omega)$ to be the closure of the set of cluster points of HK , where K runs over all the compact subsets of Ω . Recall that p is a cluster point of HK if there is a sequence $(g_n)_{n \in \mathbb{N}} \subset H$ of different elements and $(x_n)_{n \in \mathbb{N}} \subset K$ such that $g_n(x_n) \xrightarrow{n \rightarrow \infty} p$.

Lemma 1. For a subgroup $\Gamma \leq PSL_n(C)$ satisfying Definition 1 one has:

1. For each reduced word $w = z_n^{\varepsilon_n} \cdots z_2^{\varepsilon_2} z_1^{\varepsilon_1} \in \Gamma$ one has:

(a) If $\varepsilon_n = 1$ then $w(\text{Int}(F(\Gamma))) \subset S_{z_n}^*$.

(b) If $\varepsilon_n = -1$ then $w(\text{Int}(F(\Gamma))) \subset R_{z_n}^*$.

2. Let $\gamma \in \text{Gen}(\Gamma)$. Then

$$R(\gamma) = \bigcap_{k \in \mathbb{N} \cup \{0\}} \gamma^{-k}(R_\gamma^*) \quad \text{and} \quad S(\gamma) = \bigcap_{k \in \mathbb{N} \cup \{0\}} \gamma^k(S_\gamma^*)$$

are closed disjoint sets contained in $P_C^n - \Omega(\Gamma)$.

3. Let $F_k = \{\gamma(f) : f \in F(\Gamma) \text{ and } \gamma \in \Gamma \text{ is a reduced word of Length at most } k\}$. Then $F(\Gamma) \subset F_1(\Gamma) \subset \dots \subset F_k(\Gamma) \subset \dots$ and

$$\Omega(\Gamma) = \bigcup_{k \in \mathbb{N} \cup \{0\}} \text{Int}(F_k(\Gamma)).$$

4. For each $\gamma \in \text{Gen}(\Gamma)$ one has that $\emptyset \neq Ac(\{\gamma^n\}_{n \in \mathbb{N}}, \Omega(\Gamma)) \subset S(\gamma)$ and $\emptyset \neq Ac(\{\gamma^{-n}\}_{n \in \mathbb{N}}, \Omega(\Gamma)) \subset R(\gamma)$.

Proof.

(1) Let us proceed by induction on the length of the reduced words. Clearly the case $k = 1$ is done by the definition of Schottky group. Now assume we have proven the statement for $j = k$. Let $w = z_{k+1}^{\varepsilon_{k+1}} \cdots z_1^{\varepsilon_1}$ be a reduced word and $x \in \text{Int}(F(\Gamma))$. By the inductive hypothesis we deduce that

$$\begin{aligned} z_{k+1}^{-\varepsilon_{k+1}} w(x) &\in P_C^n - \overline{R_{z_{k+1}}^*} && \text{if } \varepsilon_{k+1} = 1 \quad \text{and} \\ z_{k+1}^{-\varepsilon_{k+1}} w(x) &\in P_C^n - \overline{S_{z_{k+1}}^*} && \text{if } \varepsilon_{k+1} = -1. \end{aligned}$$

Now the proof follows by the definition of Schottky group.

- (2) Let $\gamma \in \text{Gen}(\Gamma)$. Since $\gamma^m(\overline{S_\gamma^*}) \subset \gamma^{m-1}(S_\gamma^*)$ we deduce that

$$\bigcap_{m \in N} \gamma^m(\overline{S_\gamma^*}) \subset \bigcap_{m \in N} \gamma^{m-1}(S_\gamma^*) = S(\gamma).$$

To conclude observe that:

$$S(\gamma) = \bigcap_{m \in N} \gamma^{m-1}(S_\gamma^*) \subset \bigcap_{m \in N} \gamma^{m-1}(\overline{S_\gamma^*}) \subset \bigcap_{m \in N} \gamma^m(\overline{S_\gamma^*}).$$

- (3) We will prove that $F(\Gamma) \subset \text{Int}(F_1(\Gamma))$. Let $x \in \partial(F(\Gamma))$, then there is $\gamma_0 \in \text{Gen}(\Gamma)$ such that $x \in \partial S_{\gamma_0}^* \cup \partial R_{\gamma_0}^*$, for simplicity we will assume that $x \in \partial S_{\gamma_0}^*$. Define $r_1 = \min\{d(x, \gamma_0(\overline{S_\gamma^*})) : \gamma \in \text{Gen}(\Gamma)\}$, $r_2 = \min\{d(x, \overline{R_\gamma^*}) : \gamma \in \text{Gen}(\Gamma)\}$, $r_3 = \min\{d(x, \gamma_0(\overline{R_\gamma^*})) : \gamma \in \text{Gen}(\Gamma) - \{\gamma_0\}\}$, $r_4 = \min\{d(x, \overline{S_\gamma^*}) : \gamma \in \text{Gen}(\Gamma) - \{\gamma_0\}\}$ and $r = \min\{r_1, r_2, r_3, r_4\}$ (here d denotes the Fubini-Study metric). Clearly $r > 0$. Now, let $y \in B_{r/4}(x) \cap \overline{S_{\gamma_0}^*}$ then by the definition of r we have that $y \in F(\Gamma) \cup \gamma(F(\Gamma))$. If $y \in B_{r/2}(x) \cap P_C^n - \overline{S_{\gamma_0}^*}$ then by definition of r we deduce $y \in F(\Gamma)$. In other words, we have shown $F(\Gamma) \subset \text{Int}(F_1(\Gamma))$. Therefore:

$$F_k(\Gamma) \subset \{\gamma(f) : \gamma \text{ is a reduced word of length at most } k \text{ and } \text{Int}(F_1(\Gamma))\} \\ \subset F_{k+1}(\Gamma)$$

i.e., $F_k(\Gamma) \subset \text{Int}(F_{k+1}(\Gamma))$. To conclude the proof observe that:

$$\Omega(\Gamma) = \bigcup_{k \in N \cup \{0\}} F_k(\Gamma) \subset \bigcup_{k \in N \cup \{0\}} \text{Int}(F_{k+1}(\Gamma)) \subset \bigcup_{k \in N \cup \{0\}} \text{Int}(F_k(\Gamma)).$$

- (4) Let $K \subset \Omega(\Gamma)$ be a compact set and x a cluster point of $\{\gamma^m(K)\}_{m \in N}$. Then there is a subsequence $(n_m)_{m \in N} \subset (m)_{m \in N}$ and a sequence $(x_m)_{m \in N} \subset K$ such that $\gamma^{n_m}(x_m) \xrightarrow{m \rightarrow \infty} x$. In case $x \notin S(\gamma)$ it is deduced that there is $k_0 \in N$ such that $x \notin \gamma^{k_0}(\overline{S_\gamma^*})$. Taking $r = d(x, \gamma^{k_0}(\overline{S_\gamma^*}))$ we have that:

$$B_{r/2}(x) \cap \gamma^{k_0}(\overline{S_\gamma^*}) = \emptyset. \quad (1)$$

On the other hand, observe that since K is compact, by part (3) of the present lemma there is $l_0 \in N$ such that $K \subset F_{l_0}(\Gamma)$; also observe that since $(n_m)_{m \in N}$ is an strictly increasing sequence, there is $k_1 \in N$ such that $n_m > l_0 + 1 + k_0$ for $m > k_1$. With these facts in mind we deduce $\gamma^{l_0+1}(K) \subset \overline{S_\gamma^*}$ and therefore:

$$\gamma^{n_m}(x_m) \in \gamma^{n_m-l_0-1}(\overline{S_\gamma^*}) \subset \gamma^{k_0}(\overline{S_\gamma^*}) \text{ for } m > k_1.$$

Hence $x \in \gamma^{k_0}(\overline{S_\gamma^*})$, which contradicts 1. Thus $\emptyset \neq Ac(\{\gamma^n\}_{n \in N}, \Omega(\Gamma)) \subset S(\gamma)$. Observe that similar arguments prove also $\emptyset \neq Ac(\{\gamma^{-n}\}_{n \in N}, \Omega(\Gamma)) \subset R(\gamma)$. $\square$

3 Proof of Proposition 1.

- (1) Assume there is a reduced word h with length > 0 such that $h = Id$. Now, let $x \in \text{Int}(F(\Gamma))$, then by part (1) of Lemma 1,

$$x = h(x) \in \bigcup_{\gamma \in \text{Gen}(\Gamma)} (R_\gamma^* \cup S_\gamma^*),$$

which contradicts the choice of x . Therefore Γ is free.

- (2) Let $K \subset \Omega(\Gamma)$ be a compact set, then by part (3) of Lemma 1, there is $k \in N$ such that $K \subset F_k(\Gamma)$. Assume there is a word w with length $\geq 2k + 2$ such that $w(F_k(\Gamma)) \cap F_k(\Gamma) \neq \emptyset$. So there are $x_1, x_2 \in F(\Gamma)$ and words w_1, w_2 of length at most k such that $x_1 = w_1^{-1}w^{-1}w_2x_2$. On the other hand $w_1^{-1}w^{-1}w_2$ is a word with length ≥ 2 . By (1) of Lemma 1,

$$x_1 = w_1^{-1}w^{-1}(w_2(x_2)) \in \bigcup_{g \in \text{Gen}(\Gamma)} S_j^* \cup R_j^*,$$

but this contradicts the choice of x_1 . Therefore Γ acts properly discontinuously and freely on $\Omega(\Gamma)$. $\square$

Remark 1. All the results in this section remain valid if we change P_C^n for P_R^n .

4 Dynamics of projective transformations

Lemma 2. Let V be a C -linear space with $\dim_C(V) = n$, $T : V \rightarrow V$ an invertible linear transformation and $\lambda \in C$ such that $|\alpha| < |\lambda|$ for every eigenvalue α of T . For every $l \in N$ we have uniform convergence

$$\lambda^{-m} \binom{m}{l} T^m \xrightarrow{m \rightarrow \infty} 0$$

on compact subsets of V .

Here $\binom{m}{l}$ denotes the number of sets with l elements from a set with m elements.

Proof. Decomposing T into one or more Jordan blocks according to Jordan's Normal Form Theorem we reduce the problem to the case where there is $0 < |\lambda| < 1$ and an ordered basis $\beta = \{v_1, \dots, v_n\}$, $n \geq 2$, such that the matrix of T with respect to β (in symbols $[T]_\beta$) satisfies:

$$[T]_\beta = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

An inductive argument shows that for all $m > n$:

$$[T^m]_\beta = \begin{pmatrix} \lambda^m & \binom{m}{1} \lambda^{m-1} & \binom{m}{2} \lambda^{m-2} & \cdots & \binom{m}{n-1} \alpha^{m+1-n} \\ 0 & \lambda^m & \binom{m}{1} \lambda^{m-1} & \cdots & \binom{m}{n-2} \lambda^{m+2-n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda^m \end{pmatrix}. \quad (2)$$

For a compact subset $K \subset V$ set

$$\sigma(K) = \sup \left\{ \sum_{j=1}^n |\alpha_j| : \sum_{j=1}^n \alpha_j v_j \in K \right\}.$$

Let $z \in K$, $z = \sum_{j=1}^n \alpha_j v_j$, then by equation 2 we deduce:

$$|T^m(z)| \leq \sigma(K) \max \{ |v_j| : 1 \leq j \leq n \} \sum_{j=1}^n \sum_{k=0}^{j-1} \binom{m}{k} |\alpha_j|^{m-k},$$

Hence it is sufficient to observe that:

$$\left| \binom{m}{l} \binom{m}{k} \alpha^{m-k} \right| \leq m^{2 \max\{k, l\}} |\alpha|^{m-k} \xrightarrow{m \rightarrow \infty} 0. \quad \square$$

Definition 4. Let V be a C -linear space with $\dim_C(V) = n$ and let $T: V \rightarrow V$ be a C -linear transformation. We define $\text{Eve}(T) = \langle \{v \in V : v \text{ is an eigenvector of } T\} \rangle$. Where $\langle \{v \in V : v \text{ is an eigenvector of } T\} \rangle$ will denote the linear subspace generated by the eigenvectors of T .

Lemma 3. Let $l, k \in N \cup \{0\}$ with $l < k$. Then $\binom{m}{l} \binom{m}{k}^{-1} \xrightarrow{m \rightarrow \infty} 0$.

Proof. $\binom{m}{l} \binom{m}{k}^{-1} = \prod_{j=l}^{k-1} \frac{j+1}{m-j} \leq \left(\frac{k}{m-l}\right)^{k-l} \xrightarrow{m \rightarrow \infty} 0$. □

Lemma 4. Let V be a C -linear space with $\dim_C(V) = n > 1$ and let $T: V \rightarrow V$ be an invertible linear transformation such that there are $\lambda \in C$, with $|\lambda| = 1$, and an ordered basis $\beta = \{v_1, \dots, v_n\}$ for which:

$$[T]_\beta = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

that is $[T]_\beta$ is a $n \times n$ -Jordan block. Then for every $v \in V - \{0\}$ there is a unique $k(v, T) \in N \cup \{0\}$ such that the set of cluster points of

$$\left\{ \binom{m}{k(v, T)}^{-1} T^m(v) \right\}_{m \in N}$$

lies in $\langle\langle v_1 \rangle\rangle - \{0\}$.

Proof. Let $z = \sum_{j=0}^n \alpha_j v_j$ and $k(z, T) = \max\{1 \leq j \leq n : \alpha_j \neq 0\} - 1$, then we have that:

$$\binom{m}{k(v, T)}^{-1} T^m(z) = \sum_{j=1}^n \left(\sum_{k=0}^{n-j} \binom{m}{k} \binom{m}{k(v, T)}^{-1} \lambda^{m-k} \alpha_{k+j} \right) v_j$$

The result now follows from Lemma 3. □

Corollary 1. Let V be a C -linear space with $\dim_C(V) = n$ and let $T: V \rightarrow V$ be a linear transformation such that there are $\alpha_1, \dots, \alpha_n \in C$, with $|\alpha_j| = 1$ for each $0 \leq j \leq n$, and an ordered basis $\beta = \{v_1, \dots, v_n\}$ for which $T(\sum_{j=0}^n \beta_j v_j) = \sum_{j=0}^n \alpha_j \beta_j v_j$. Then $k(v, T) = 0$ is the unique positive integer for which the set of cluster points of

$$\left\{ \binom{m}{k(v, T)}^{-1} T^m(v) \right\}_{m \in N},$$

where $v \in V - \{0\}$, lies on $V - \{0\}$.

Corollary 2. *Let V be a C -linear space with $\dim_C(V) = n$ and let $T : V \rightarrow V$ be an invertible linear transformation such that each of its eigenvalues is a unitary complex number. Then for every $v \in V - \{0\}$ there is a unique $k(v, T) \in N \cup \{0\}$ for which the set of cluster points of*

$$\left\{ \binom{m}{k(v, T)}^{-1} T^m(v) \right\}_{m \in N}$$

lies in $\text{Eve}(T) - \{0\}$.

Proof. By the Jordan's Normal Form Theorem there are $k \in N$; $V_1, \dots, V_k \subset V$ linear subspaces and $T_i : V_i \rightarrow V_i$, $1 \leq i \leq k$ such that:

1. $\bigoplus_{j=1}^k V_j = V$.
2. For each $1 \leq i \leq k$, T_i is a non-zero C -linear map whose eigenvalues are unitary complex numbers.
3. $\bigoplus_{j=1}^k T_j = T$.
4. For each $1 \leq i \leq k$, T_i is either diagonalizable or $n_i = \dim_C V_i > 1$; V_i contains an ordered basis β_i for which $[T]_{\beta_i}$ is a $n_i \times n_i$ -Jordan block.

Let $v \in V - \{0\}$ then there is a non empty finite set $W \subset \bigcup_{j=1}^k V_j - \{0\}$ such that $v = \sum_{w \in W} w$. Now, take $i : W \rightarrow N$ where $i(w)$ is the unique element in $\{1, \dots, k\}$ such that $w \in V_{i(w)}$, $k(v, T) = \max\{k(w, T_{i(w)}) : w \in W\}$, $W_1 = \{w \in W : k(w, T_{i(w)}) < k(v, T)\}$ and $W_2 = W - W_1$ then:

$$\frac{T^m(v)}{\binom{m}{k(v, T)}} = \sum_{w \in W_1} \frac{\binom{m}{k(w, T_{i(w)})}}{\binom{m}{k(v, T)}} \frac{T_{i(w)}^m(w)}{\binom{m}{k(w, T_{i(w)})}} + \sum_{w \in W_2} \frac{T_{i(w)}^m(w)}{\binom{m}{k(w, T_{i(w)})}} \quad (3)$$

The result now follows from equation 3, Lemmas 3, 4 and Corollary 1. $\square$

Definition 5. *Let $\gamma \in PSL_n(C)$ be an element of infinite order and let $\tilde{\gamma}$ be a lifting of γ . Then we define:*

1. $|\text{Eva}(\gamma)| = \{|\lambda| \in R : \lambda \text{ is an eigenvalue of } \tilde{\gamma}\}$
2. $L_r(\gamma) = \langle \{v \in C^n : v \text{ is an eigenvector of } \tilde{\gamma} \text{ and } |\tilde{\gamma}(v)| = r |v|\} \rangle_n$.
3. $L(\gamma)$ as the closure of accumulation points of $\{\gamma^m(z)\}_{m \in Z}$ where $z \in P_C^n$.

Clearly parts 1 and 2 of this definition do not depend on the choice of $\tilde{\gamma}$.

Proposition 2. *Let $\gamma \in PSL_{n+1}(C)$ be an element of infinite order, then:*

$$L(\gamma) = \bigcup_{r \in |Eva(\Gamma)|} L_r(\gamma).$$

Proof. Since $\bigcup_{r \in |Eva(\Gamma)|} L_r(\gamma) \subset L(\Gamma)$ is trivially verified, it is enough to check that $L(\gamma) \subset \bigcup_{r \in |Eva(\Gamma)|} L_r(\gamma)$. Let $\tilde{\gamma}$ be a lifting of γ , then by the Jordan's Normal Form Theorem there are $k \in N$; $V_1, \dots, V_k \subset C^{n+1}$ linear subspaces; $\gamma_i : V_i \rightarrow V_i$, $1 \leq i \leq k$ and $r_1, \dots, r_k \in R$ which satisfy:

1. $\bigoplus_{j=1}^k V_j = C^{n+1}$.
2. For each $1 \leq i \leq k$, γ_i is a non-zero C -linear map whose eigenvalues are unitary complex numbers.
3. $0 < r_1 < r_2 < \dots < r_k$.
4. $\bigoplus_{j=1}^k r_j \gamma_j = \tilde{\gamma}$.

In what follows $(\tilde{\gamma}, k, \{V_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k, \{r_i\}_{i=1}^k)$ will be called a decomposition for γ . Now let $[v]_n \in P_C^n$, thus $v = \sum_{j=1}^k v_j$ where $v_j \in V_j$. Set $j_0 = \max \{1 \leq j \leq k : v_j \neq 0\}$. One has:

$$\left(\begin{matrix} m \\ k(v_{j_0}, T_{j_0}) \end{matrix} \right)^{-1} \frac{\tilde{\gamma}^m(v)}{r_{j_0}^m} = \sum_{j=1}^k \left(\begin{matrix} m \\ k(v_{j_0}, T_{j_0}) \end{matrix} \right)^{-1} \frac{r_j^m \gamma_j^m(v_j)}{r_{j_0}^m}. \quad (4)$$

By equation 4, Lemma 2 and Corollary 2 we conclude that the set of cluster points of $\{\gamma^m(v)\}_{m \in \mathbb{Z}}$ lies in $[Eva(\gamma_{j_0}) - 0]_n = L_{r_{j_0}}(\gamma)$. $\square$

5 Proof of the Main Theorem

Lemma 5. *Let $\Gamma \leq PSL_{2n+1}(C)$ be a group and Ω a non-empty, Γ -invariant open set where Γ acts properly discontinuously and such that whenever l is a projective subspace contained in $P_C^{2n} - \Omega$ then $\dim_C(l) < n$. Then for every $\gamma \in \Gamma$ with infinite order there is a connected set $\mathcal{L}(\gamma) \subset Ac(\{\gamma\}_{m \in \mathbb{Z}}, \Omega) \cup L(\gamma)$ such that $L(\gamma) \subset \mathcal{L}(\gamma)$.*

Proof. Let $\gamma \in \Gamma$ be an element with infinite order, and choose a decomposition $(\tilde{\gamma}, k, \{V_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k, \{r_i\}_{i=1}^k)$ for γ . Take $j_0 = \min \{1 \leq j \leq k : \sum_{i=1}^j \dim_C(V_i) \geq n+1\}$. From Proposition 2 we can assume that $k \geq 2$. For the

moment let us assume that $j_0 \neq 1$, k . Observe that since $\sum_{i=1}^{j_0} \dim_C(V_i) \geq n+1$ we conclude that there is $w = \sum_{i=1}^{j_0} w_i \in \bigoplus_{j=1}^{j_0} V_j$ non-zero, where $w_i \in V_i$, such that $[w]_{2n} \in \Omega$ and since Ω is open we can assume that w_{j_0} is non-zero. Now, let $z \in \bigoplus_{j>j_0} V_j - \{0\}$ then by Lemma 2

$$w_m(z) = \left[w + \binom{m}{k(w, \gamma_{j_0})} \sum_{j>j_0} \left(\frac{r_{j_0}}{r_j} \right)^m \gamma_j^{-m}(z_j) \right]_{2n} \xrightarrow{m \rightarrow \infty} [w]_{2n}$$

thus for $m(z)$ large $(w_m(z))_{m \geq m(z)} \subset \Omega$. On the other hand, by Corollary 2 there is an strictly increasing sequence $(n_m)_{m \in \mathbb{N}} \subset \mathbb{N}$ and $w_0 \in \text{Eve}(\gamma_{j_0}) - \{0\}$ such that:

$$\left(\binom{n_m}{k(w_{j_0}, \gamma_{j_0})} \right)^{-1} \gamma_{j_0}^{n_m}(w_0) \xrightarrow{m \rightarrow \infty} w_0.$$

From here and Lemma 2 we deduce that:

$$\gamma^{n_m}(w_{n_m}) = \left[\left(\binom{n_m}{k(w_{j_0}, \gamma_{j_0})} \right)^{-1} \sum_{j \leq j_0} \left(\frac{r_j}{r_{j_0}} \right)^{n_m} \gamma_j^{n_m}(w_j) + z \right]_{2n} \xrightarrow{m \rightarrow \infty} [w_0 + z]_{2n}.$$

From here it follows that:

$$\bigcup_{j>j_0} L_{r_j}(\gamma) \subset \left\langle [w_0]_{2n}, \left[\bigoplus_{j>j_0} V_j - \{0\} \right]_{2n} \right\rangle \subset \text{Ac}(\{\gamma^m\}_{m \in \mathbb{Z}}, \Omega) \cup L(\gamma).$$

To conclude consider the following observations:

Obs. 1. Observe that in the previous argument, the assumption $j \neq k$ is not crucial, so for the case $j = 1$ it is verified that there is $w_1 \in L(r_1)$ such that

$$\bigcup_{j>1} L(r_j) \subset \left\langle w_1, \left[\bigoplus_{j>1} V_i - \{0\} \right]_{2n} \right\rangle \subset \text{Ac}(\{\gamma^m\}_{m \in \mathbb{Z}}, \Omega) \cup L(\gamma)$$

thus in case $j = 1$ it is enough to take

$$\mathcal{L}(\gamma) = \left\langle w_1, \left[\bigoplus_{j>1} V_j - \{0\} \right]_{2n} \right\rangle \cup L_{r_1}(\gamma).$$

Obs. 2. Applying the same argument to γ^{-1} in the case $j_0 \neq 1$, k , it is deduced that there is $v \in L(r_{j_0})$ such that:

$$\bigcup_{j<j_0} L(r_j) \subset \left\langle v, \left[\bigoplus_{j<j_0} V_i - \{0\} \right]_{2n} \right\rangle \subset \text{Ac}(\{\gamma^m\}_{m \in \mathbb{Z}}, \Omega).$$

Therefore in this case it is enough to take

$$\mathcal{L}(\gamma) = \left\langle v, \left[\bigoplus_{j < j_0} V_j - \{0\} \right]_{2n} \right\rangle \cup \left\langle [w_0]_{2n}, \left[\bigoplus_{j > j_0} V_j - \{0\} \right]_{2n} \right\rangle \cup L_{r_{j_0}}(\gamma).$$

Obs. 3. To obtain the result in the case $j = k$ it is enough to apply the same argument used in Obs. 1 to γ^{-1} . $\square$

Lemma 6. If $\Gamma \leq PSL_{2n+1}(C)$ is a Schottky group then $P_C^{2n} - \Omega(\Gamma)$ does not contain a projective subspace $\mathcal{V}$ with $\dim_C(\mathcal{V}) \geq n$.

Proof. If $\mathcal{V} \subset P_C^{2n} - \Omega(\Gamma)$ is a projective subspace with $\dim_C(\mathcal{V}) \geq n$, then:

$$\mathcal{V} \subset P_C^{2n} - \Omega(\Gamma) = P_C^{2n} - \bigcup_{\gamma \in \Gamma} \gamma(F(\Gamma)) \subset P_C^{2n} - F(\Gamma) = \bigcup_{g \in \text{Gen}(\Gamma)} R_g^* \cup S_g^*.$$

Since $\mathcal{V}$ is connected and $(\mathcal{V} \cap \bigcup_{\gamma \in \text{Gen}(\Gamma)} R_\gamma^*, \mathcal{V} \cap \bigcup_{\gamma \in \text{Gen}(\Gamma)} S_\gamma^*)$ is a disconnection for $\mathcal{V}$ we deduce that $\mathcal{V} \subset \bigcup_{\gamma \in \text{Gen}(\Gamma)} R_\gamma^*$ or $\mathcal{V} \subset \bigcup_{\gamma \in \text{Gen}(\Gamma)} S_\gamma^*$. Moreover by an inductive argument we deduce that there is $\gamma_0 \in \text{Gen}(\Gamma)$ such that $\mathcal{V} \subset S_{\gamma_0}^*$ or $\mathcal{V} \subset R_{\gamma_0}^*$. For simplicity let us assume that $\mathcal{V} \subset S_{\gamma_0}^*$. Taking $\sigma \in \text{Gen}(\Gamma) - \{\gamma_0\}$ we have:

$$\sigma^{-1}(\mathcal{V}) \subset \sigma^{-1}(S_{\gamma_0}^*) \subset \sigma^{-1}(P_C^{2n} - \bar{S}_\sigma^*) = R_\sigma^*. \quad (5)$$

Observe that $\mathcal{V}$ and $\sigma^{-1}\mathcal{V}$ are projective subspaces with $\dim_C(\mathcal{V}) + \dim_C(\sigma^{-1}\mathcal{V}) \geq 2n$ then $\mathcal{V} \cap \sigma^{-1}(\mathcal{V}) \neq \emptyset$. However, this is a contradiction since by equation 5 we have that $\mathcal{V} \cap \sigma^{-1}\mathcal{V} \subset R_\sigma^* \cap S_{\gamma_0}^* = \emptyset$. $\square$

Proof of Theorem 1. Assume that there is a group $\Gamma \leq PSL_{2n+1}(C)$ which is a Schottky group and let $\gamma \in \text{Gen}(\Gamma)$. By Lemma 5 there is a connected set $\mathcal{L}(\gamma)$ such that $L(\gamma) \subset \mathcal{L}(\gamma) \subset \text{Ac}(\{\gamma^m\}_{m \in \mathbb{N}}, \Omega(\Gamma))$. On the other hand by (4) of Lemma 1 we have $\text{Ac}(\{\gamma^m\}_{m \in \mathbb{N}}, \Omega(\Gamma)) \subset S(\gamma) \cup R(\gamma)$. Since $(R(\gamma) \cap \mathcal{L}(\gamma), S(\gamma) \cap \mathcal{L}(\gamma))$ is a disconnection for $\mathcal{L}(\gamma)$ we deduce $\mathcal{L}(\gamma) \subset R(\gamma)$ or $\mathcal{L}(\gamma) \subset S(\gamma)$. This implies $L(\gamma) \cap S(\gamma) = \emptyset$ or $L(\gamma) \cap R(\gamma) = \emptyset$. However this contradicts (4) of Lemma 1. Therefore Γ cannot be a Schottky group. $\square$

Remark 2.

1. If in Definition 1 we allow that $R_j^* = S_j^*$ and $\gamma_j^2 = Id$ for $1 \leq j \leq g$, the resulting group is a type of Complex Kleinian Group (see [9]), and by means of Theorem 1 is not hard to see that for $g \geq 3$ this type of groups cannot be realized as subgroups of $PSL_{2n+1}(C)$.
2. Theorem 1 remains valid if we change C by R .

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References

- [1] A. Cano. *On Discrete Subgroups of automorphism of $\mathbb{P}_{\mathbb{C}}^2$* , Preprint (2008).
- [2] A.I. Kostrikin and Yu I. Manin. *Linear Algebra and Geometry*. CRC Press (1989).
- [3] A. Marden. *Schottky groups and circles*. Contributions to Analysis, (1974), 273–278.
- [4] B. Maskit. *A characterization of Schottky groups*. J. Analyse Math., **19** (1967), 227–230.
- [5] B. Maskit. *Kleinian Groups*. Springer-Verlag (1980).
- [6] J.P. Navarrete. *On the Limit Set of Discrete Groups of $PU(2, 1)$* . Geom. Dedicata, **122**(1), 1–13.
- [7] J.P. Navarrete. *The Trace Function and Complex Kleinian Groups in $\mathbb{P}_{\mathbb{C}}^2$* , **19**(7) (2008), 865–890.
- [8] M.V. Nori. *The Schottky Groups in Higher Dimensions*. Cont. Maths, **58**(I), 195–197.
- [9] J. Seade and A. Verjovsky. *Complex Schottky Groups*. Astérisque, **287**, 251–272.

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