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Islands at infinity on manifolds of asymptotically nonnegative curvature

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The first author dedicates this paper to his parents José Martiniano and Zoraide

Abstract. We introduce an invariant which measures the R-eccentricity of a point in a complete Riemannian manifold M and show that it goes to zero when the point goes to infinity, if M has asymptotically nonnegative curvature. As a consequence we show that the isometry group is compact if M has asymptotically nonnegative curvature and a point with positive sectional curvature.

Keywords: nonnegative curvature, convexity, isometry group.

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0 Introduction

In this paper, we will study the "islands" (geodesic balls with all sectional curvatures bounded from below by a positive constant) at infinity on complete Riemannian manifolds with asymptotically nonnegative curvature. This paper was motivated by our intuition that a complete manifold with asymptotically nonnegative curvature should have, in some sense, sectional curvatures close to zero at the infinity. For example, it should not admit sequences of uniformly large balls going to infinity with uniform positive lower bound. For a better presentation, we introduce the following definition, which relates the radius of an island and the positive lower bound of its curvature.

Definition 1. *Let $p \in M$. We call the number*

$$h_R(p) = \sup_{r \in (0,R]} \{r^2 \inf\{K(q) : q \in B_p(r)\}\}$$

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as the R-eccentricity *of* p. *Here* $K(q)$ *is the infimum of the sectional curvatures at* p, R *is a positive constant and* $B_p(r)$ *is the ball of radius* r *centered at* p.

It is easy to see that $r^2 \inf\{K(q) \colon q \in B_p(r)\}$ is invariant under a positive constant scaling of metric. We consider the asymptotic behavior of $h_R(p)$ on complete noncompact manifolds. Recall (see [Ab1]) that the curvature of a complete manifold M is *asymptotically nonnegative* if there exists a nonincreasing function $\kappa \colon [0, +\infty) \to [0, +\infty)$ such that $\int_0^{+\infty} t\kappa(t)dt < +\infty$, and $K(x) \geq -\kappa\big(d(o, x)\big)$, for a fixed point o. Under this assumption we obtain:

Theorem 1. *Let* M *be a complete manifold with asymptotically nonnegative curvature. Then for any constant* $R > 0$ *we have*

$$\lim_{p\to\infty} h_R(p) = 0.$$

In other words, the theorem says that on a complete manifold with asymptotically nonnegative curvature any sequence $q_k \to \infty$, with $K \geq \delta_k > 0$ in $B_{q_k}(r_k)$, and $r_k \leq R$, for a fixed number R, satisfies $r_k^2 \delta_k \to 0$.

It should be noticed that on the universal covering of a torus with a nonflat metric there exists a sequence of points $q_k \to \infty$ such that $h_R(q_k) = \text{constant} > 0$ for some $R > 0$. So some suitable integrability conditions about curvatures are reasonable and the asymptotically nonnegative curvature condition has been studied extensively by Abresch ([Ab1], [Ab2]).

Remark 0.1. We present an example (see Example 3.4 in the third section) which shows that there exists a complete manifold M with nonnegative curvature and a sequence of points $p_k \to \infty$ and $\{r_k\} \subset \mathbb{R}^+$ such that $h_{r_k}(p_k) \geq 1/128$ for all k. So the condition on the finiteness of R is essential in Theorem 1, even if $K \geq 0$.

Remark 0.2. A trivial consequence of conditions in Theorem 1 is that $\liminf_{p\to\infty} K(p) = 0$. In general we don't have $\lim_{p\to\infty} K(p) = 0$. Proposition 3.2 in section 3 shows that there exists a surface with nonnegative curvature and $\limsup_{p\to\infty} K(p) = +\infty$.

Remark 0.3. Let M be a complete two-dimensional Riemannian manifold with finite integral of the negative part of Gaussian curvature. The theorems of Cohn-Vossen ([CV]) and Huber ([Hu]) assert that $\int_M K dV \leq 2\pi \mathcal{X}(M)$, where $\mathcal{X}(M)$ is the Euler characteristic of M. Let $\{D_k, k = 1, 2, \cdots\}$ be a sequence of disjoint domains in M with area $A(D_k) \geq A_0$ and μ_k the infimum of Gaussian curvature in D_k. Then $\liminf_{k\to+\infty} \mu_k \leq 0$. Otherwise there exists a constant $\mu_0 > 0$ and a subsequence of $\{D_k\}$ which we still denote by $\{D_k\}$ such that $\mu_k \geq \mu_0$. Then the positive part of Gaussian curvature satisfies

$$\int_M K_+ dV \geq \int_{\bigcup_{k=1}^{+\infty} D_k} K_+ dV \geq \sum_{k=1}^{+\infty} \int_{D_k} \mu_0 dV = +\infty.$$

Since the integral of the negative part of the curvature is finite and the integral of the curvature is finite, the positive part of K must have finite integral, which leads to a contradiction. This provides in dimension 2 a phenomenon similar to Theorem 1.

We can use Theorem 1 to study the isometry group of a noncompact Riemannian manifold. Let $\mathrm{Isom}(M)$ be the isometry group of M with the topology of uniform convergence on compact subsets. We have

Corollary 2. *Let M be a complete and noncompact manifold with asymptotically nonnegative curvature. Assume that M contains a point with positive sectional curvature. Then* $\mathrm{Isom}(M)$ *is compact. Moreover, if we take a sequence $\{f^k\}$ for some isometry f, then, for any point p of positive curvature, there exists a convergent subsequence $f^{k_i} \to g$ such that $g(p) = p$.*

We would like to remark that the hypothesis of existence of a point with positive curvature in Corollary 2 cannot be removed. For example, $\mathrm{Isom}(\mathbb{R}^2)$ is not compact.

The rest of this paper is organized as follows. Since the proof of Theorem 1 is quite long we divide it into two sections: in Section 1 we give some estimate for distance function in manifolds of positive curvature which is needed in the proof of Theorem 1. Then in the second section we prove Theorem 1 and Corollary 2. In the third section we present some examples mentioned in the introduction.

1 The Distance Function in Manifolds of Positive Curvature

Let M be a complete Riemanian manifold. All geodesics unless otherwise stated are assumed to be normalized. Recall that a connected set C is convex if, given $p \in C$, there exists a ball $B_p(\varepsilon)$ such that the set $U = B_p(\varepsilon) \cap C$ is strongly convex (this means that given any two points $x, y \in U$, there exists a unique minimizing geodesic γ in M joining x and y, and γ is contained in C). If C is also closed, Theorem 1.6 in [CG] says that C is a k-dimensional submanifold with smooth and totally geodesic interior $\mathbf{int}(\mathbf{C})$ and a boundary $\boldsymbol{\partial}\mathbf{C}$ of C^0 class. We assume that $\partial C \neq \emptyset$. Given $p \in \mathrm{int}(C)$ we say that a normal geodesic σ is a

C-minimal connection to ∂C if the distance $d_C(p, \partial C) = L(\sigma)$, where $L(\sigma)$ is the length of σ and d_C is the intrinsic distance of C. Let $\boldsymbol{\gamma}\colon [0, a] \to \operatorname{int}(C)$ be a geodesic. Let $\boldsymbol{\theta}(\mathbf{s})$ be the angle between $\gamma'(s)$ and the C-minimal connection to ∂C. Set $\boldsymbol{\varphi} = d_C \circ \gamma$. When the sectional curvature of M is nonnegative, by Theorem 1.10 in [CG] φ is concave (The original statement of Theorem 1.10 in [CG] refers to the distance d instead of d_C. It is shown in Example 3.5 in the third section that this modification is necessary.) and, given $s_0 \in [0, a]$, it holds that

$$\varphi(s) \leq \varphi(s_0) - (s - s_0)\cos\theta(s_0)$$

for sufficiently small $|s - s_0|$. In the following result we show that, if $K \geq \delta > 0$, then the graph of φ stays below a parabola.We say that C is *γ-convex* if, for any C-minimal connection $\sigma_s\colon [0, \varphi(s)] \to C$ between $\gamma(s)$ and ∂C, it holds that any geodesic $\tau\colon [0, +\infty) \to M$ with $\tau'(0) \perp \sigma_s'(\varphi(s))$ has $\tau(u) \notin \operatorname{int}(C)$ for small $u \geq 0$.

Proposition 1.1. *Assume that C is γ-convex. Suppose that there exists a constant $\delta > 0$ such that the sectional curvatures $\geq \delta$ along all C-minimal connections between $\gamma(s)$ and ∂C. Then*

$$\varphi(s) \leq \varphi(0) - s\cos\theta(0) - \frac{\bar{d}\lambda\delta}{2}s^2$$

for all s, where $\lambda = \min\{\sin^2\theta(0), \sin^2\theta(a)\}$ and $\bar{d} = \min\{\varphi(0), \varphi(a)\}$.

To prove the proposition we need some notations and prove several technical lemmas. For a geodesic $\tau\colon [0, b] \to M$, set $\mathbf{f}[\boldsymbol{\nu}, \boldsymbol{\tau}, \mathbf{L}](\mathbf{t}, \mathbf{s}) = \exp_{\tau(t)} sP_t\nu$, $(t, s) \in L$ where $P_t\nu$ is the parallel transport of ν along τ and $L \subset [0, b] \times \mathbb{R}$. Let $\mathbf{D}$ be a compact neighborhood of γ and $\boldsymbol{\epsilon}_{\mathbf{D}} > 0$ be such that for any $p \in D$ the ball $B_p(\epsilon_D)$ of center p and radius ϵ_D is a normal geodesic ball. By decreasing ϵ_D if necessary, we can assume that $B_{\gamma(s)}(\epsilon_D) \subset D$, for all $s \in [0, a]$. Let $\boldsymbol{\sigma}_{\mathbf{s}}$ be some C-minimal connection between $\gamma(s)$ and ∂C.

Lemma 1.2 below follows easily from the continuous differentiability of the exponential map, the continuity of the function $K(x)$, and from Lemmas 1.4 and 1.7 in [CG]. Note that by Lemma 1.4 in [CG], given any point $p \in \partial C$ there exists a small ball $U = B_p(\varepsilon)$ such that any point $x \in \operatorname{int}(C) \cap U$ is joined to p by a unique minimal geodesic contained in $\operatorname{int}(C)$.

Lemma 1.2. *Given $\varepsilon > 0$, there exists $\mu > 0$ such that, for any $s_0 \in [0, a]$ and any $\nu \perp \sigma_{s_0}'(0)$, $|\nu| = 1$, if we set $f = f[\nu, \sigma_{s_0}, L]$, where $L = [0, \varphi(s_0)] \times [0, \mu]$, then the geodesic $s \longmapsto f(t, s)$ is free of focal points to $\sigma_{s_0}(t)$, for all*

$t \in [0, \varphi(s_0)]$, the geodesic $s \longmapsto f\big(\varphi(s_0), s\big)$ is entirely away from $\operatorname{int}(C)$*, and it holds that $K \geq \delta - \varepsilon$ on the image of f.*

The following result is an elementary fact about the geometry of spheres.

Lemma 1.3. *Consider the sphere $S^2(\delta) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = R^2\}$, $R = 1/\sqrt{\delta}$. Let $\sigma : [0, d] \to S^2(\delta)$ be a geodesic. Take a unitary tangent vector v orthogonal to $\sigma'(0)$. Set $f = f[v, \sigma, L]$, where $L = [0, d] \times R\pi/2$. Consider the curve $f_s(t) = f(t, s)$. Then the length $L(f_s) = d \cdot \cos(s\sqrt{\delta})$.*

Lemma 1.4. *Take s and s_0 so that $s > s_0$ and $\theta(s_0) < \pi/2$. Let τ be a C-minimal connection between $\gamma(s)$ and σ_{s_0}, with $\tau(0) = \sigma_{s_0}(t)$ and $\tau(u) = \gamma(s)$. Take in the plane the triangle $(s - s_0, t, u)$ with corresponding angles $(\tilde{\alpha}, \tilde{\beta}, \tilde{\theta})$. Given $\varepsilon > 0$, there exists $\mu = \mu(\gamma, \varepsilon) > 0$ so that if $s - s_0 < \mu$ then $\sin\tilde{\theta} > \sin\theta(s_0)/\sqrt{1 + \varepsilon}$.*

Proof of Lemma 1.4. Note that by Toponogov Theorem (see for example [S]) we have $\tilde{\theta} \leq \theta(s_0)$. Let K_0 be an upper bound of the sectional curvatures in D. By construction of τ, it holds that $u \leq (s - s_0)$. Then the greatest side of the triangle $\big(\gamma_{|[s_0,s]}, (\sigma_{s_0})_{|[0,t]}, \tau\big)$ does not exceed $2(s - s_0)$. Suppose that

$$0 < s - s_0 < \min\left\{\frac{\epsilon_D}{2}, \frac{\pi}{2\sqrt{K_0}}\right\}.$$

The geodesics of the triangle are minimizing and contained in normal balls centered at each of the three vertices. Since $0 < s - s_0 < \pi/(2\sqrt{K_0})$, its perimeter does not exceed $2\pi/\sqrt{K_0}$. As it is observed in [Gv], p. 197, the Rauch Comparison Theorem implies that there exists a small triangle $\triangle = (s - s_0, t, u)$ in $S^2(K_0)$ with corresponding angles $(\alpha', \beta', \theta')$, that satisfy $\tilde{\alpha} \leq \alpha'$, $\tilde{\beta} \leq \beta'$ and $\tilde{\theta} \leq \theta'$.

Let $E \subset S^2(K_0)$ be the set bounded by $\triangle$ and that has the smallest area. By using Gauss-Bonnet Theorem in E we have that

$$\pi - \tilde{\theta} = \tilde{\alpha} + \tilde{\beta} \leq \alpha' + \beta' = \pi - \theta' + \int_E K_0 \, dS,$$

where dS represents the element of area of $S^2(K_0)$. Let $\psi(\ell)$ be the area of the equilateral triangle of side ℓ in $S^2(K_0)$. Therefore $\pi - \tilde{\theta} \leq \pi - \theta' + K_0\, \psi\big(2(s - s_0)\big)$. Then $\theta(s_0) - \tilde{\theta} \leq \theta' - \tilde{\theta} \leq K_0\, \psi\big(2(s - s_0)\big)$. Lemma 1.4 follows easily from this. □

Proof of Proposition 1.1. The proof has a local part, in which we show that for the geodesic γ as in the statement of Proposition 1.3 there exists $\mu > 0$ such that if $|s - s_0| < \mu$, then

$$\varphi(s) \leq \mathbf{h_{s_0}(s)} := \varphi(s_0) - (s - s_0)\cos\theta(s_0) - \frac{\bar{d}\lambda(\delta-\varepsilon)}{2(1+\varepsilon)^3}(s-s_0)^2.$$

We prove later that for all $s \in [0, a]$, it holds that

$$\varphi(s) \leq h_0(s) = \varphi(0) - s\cos\theta(0) - \frac{\bar{d}\lambda(\delta-\varepsilon)}{2(1+\varepsilon)^3}s^2.$$

By making $\varepsilon \to 0$, we obtain the desired inequality.

A. Local part of the proof

Fix an $\varepsilon > 0$ such that $\varepsilon < \delta$. There are three cases: $\theta(s_0) = \pi/2$, $\theta(s_0) > \pi/2$ and $\theta(s_0) < \pi/2$. First we obtain that, for s sufficiently close to s_0 and such that $s \geq s_0$, it holds that

$$\varphi(s) \leq \mathbf{f_{s_0}(s)} = \varphi(s_0)\cos\frac{(s-s_0)\sqrt{\delta-\varepsilon}\ \sin\theta(s_0)}{1+\varepsilon} - (s-s_0)\cos\theta(s_0).$$

The case that $s < s_0$ is reduced to the other one by changing the orientation of γ and replacing $\theta(s_0)$ by $\pi - \theta(s_0)$. We note also that the local part is clearly true if $\theta(s_0) = 0$ or $\theta(s_0) = \pi$. So we always assume that $0 < \theta < \pi$. The local part of the proof will be completed by showing that $f_{s_0}(s) \leq h_{s_0}(s)$ for s sufficiently close to s_0. For simplicity of notation along the proof of the local part we set

$$d = \varphi(s_0),\ \theta = \theta(s_0),\ \sigma = \sigma_{s_0}.$$

Claim 1. *Let $s_0 \in [a, b]$. For s sufficiently close to s_0 it holds that $\varphi(s_0) \leq f_{s_0}(s)$.*

Case 1. $\theta = \pi/2$.

Take $\mu > 0$ given by Lemma 1.2. Set $F = f[\gamma'(s_0), \sigma, L]$, with $L = [0, d] \times [0, \mu]$. Set $F_t(s) = F^s(t) = F(t, s - s_0)$. By hypothesis the geodesic F_d is entirely away from $\mathrm{int}(C)$. Thus we have $\varphi(s) \leq L(F^s)$. Because of Lemmas 1.2 and 1.4, and the Berger's extension of the Rauch Theorem (see for example [Gv], p. 194) we have

$$\varphi(s) \leq L(F^s) \leq d\cos\big((s-s_0)\sqrt{\delta-\varepsilon}\big) \leq d\cos\frac{(s-s_0)\sqrt{\delta-\varepsilon}}{1+\varepsilon} = f_{s_0}(s).$$

Case 2. $\theta > \pi/2$.

In the plane determined by $\gamma'(s_0)$ and $\sigma'(0)$, consider a vector $E \perp \sigma'(0)$ with $|E| = 1$ and such that the angle $\measuredangle\big(E, \gamma'(0)\big) < \pi/2$. Take μ given by Lemma 1.2. Set $F = F[(\sin\theta)E, \sigma, L]$, with $L = [0, d] \times [0, \mu]$. Set $F_t(s) = F^s(t) = F(t, s - s_0)$. As in Case 1, the geodesic F_d is entirely away from $\mathrm{int}(C)$ and so we have

$$d\big(F_0(s), \partial C\big) \leq L(F^s) \leq d\cos\big((s - s_0)\sqrt{\delta - \varepsilon}\ \sin\theta\big).$$

By the triangle inequality we have

$$\varphi(s) \leq d\big(F_0(s), \partial C\big) + d\big(\gamma(s), F_0(s)\big).$$

Set $x = d\big(\gamma(s), F_0(s)\big)$. We only need to prove that $x \leq -(s - s_0)\cos\theta$. Assume that $s - s_0 < \epsilon_D$. Consider in the plane the triangle $\triangle$ of sides $s - s_0$, $(s - s_0)\sin\theta$ and angle $\theta - \pi/2$ between them. By Toponogov Theorem (see [S]) the value x does not exceed the third side of $\triangle$. Furthermore we have $\sin\theta = \cos(\theta - \pi/2)$, hence $\triangle$ is in fact a right triangle. So it's third side is equal to $(s - s_0)\sin(\theta - \pi/2) = -(s - s_0)\cos\theta$, as we wanted to prove.

Case 3. $\theta < \pi/2$.

First we prove that

$$\varphi(s) \leq \mathbf{g_{s_0}(s)} = \big(d - (s - s_0)\cos\theta\big)\cos\frac{(s - s_0)\sqrt{\delta - \varepsilon}\ \sin\theta}{\sqrt{1 + \varepsilon}}.$$

After this we obtain that $g_{s_0}(s) \leq f_{s_0}(s)$.

Step 1. *For small $s - s_0$ we have $\varphi(s) \leq g_{s_0}(s)$.*

Let μ be as in Lemma 1.2. Assume further that $\mu < \epsilon_D$. Let $\tau : [0, u_s] \to V$ be a normal geodesic that realizes the distance between the geodesic σ and the point $\gamma(s)$. Set $t_s > 0$ such that $\tau(0) = \sigma(t_s)$. We have $\tau(u_s) = \gamma(s)$. By taking $f[\tau'(0), \sigma, L]$ with $L = [0, d] \times [0, \mu]$, we obtain as above

$$\varphi(s) \leq (d - t_s)\cos\left(u_s\sqrt{\delta - \varepsilon}\right). \tag{1.1}$$

Set $\alpha = \pi/2 = \measuredangle\big(-\sigma'(t_s), \tau'(0)\big)$ and $\beta = \measuredangle\big(\tau'(u_s), \gamma'(s)\big)$. Consider in the plane the triangle $(s - s_0, t_s, u_s)$ with corresponding angles $(\tilde{\alpha}, \tilde{\beta}, \tilde{\theta})$. By

Toponogov Theorem (see [S]) it holds that $\alpha \geq \tilde{\alpha}$, $\beta \geq \tilde{\beta}$ and $\theta \geq \tilde{\theta}$. We have $t_s = (s - s_0)\cos\tilde{\theta} + u_s \cos\tilde{\alpha}$. Since $\tilde{\alpha} \leq \alpha = \pi/2$ we obtain

$$t_s \geq (s - s_0)\cos\tilde{\theta} \geq (s - s_0)\cos\theta. \tag{1.2}$$

Let H be the height relative to the side t_s. Then

$$u_s \geq H = (s - s_0)\sin\tilde{\theta}. \tag{1.3}$$

By (1.1), (1.2) and (1.3) we obtain

$$\varphi(s) \leq \big(d - (s - s_0)\cos\theta\big)\cos\big((s - s_0)\sqrt{\delta - \varepsilon}\ \sin\tilde{\theta}\big).$$

By Lemma 1.4, for sufficiently small μ we have $\varphi(s) \leq g_{s_0}(s)$ and we conclude the proof of Step 1.

Step 2. *There exists $\eta = \eta(\delta, \varepsilon, \bar{d}) > 0$, such that if $0 < s - s_0 < \eta$ then $g_{s_0}(s) \leq f_{s_0}(s)$, where $\bar{d} = \min\{\varphi(0), \varphi(a)\}$.*

Note that by the concavity of φ we have $d \geq \bar{d}$. Set

$$u = s - s_0,\ \ A = \frac{\sqrt{\delta - \varepsilon}\ \sin\theta}{1 + \varepsilon}.$$

Then we have

$$\begin{aligned} f_{s_0}(s) - g_{s_0}(s) &= d\big(\cos uA - \cos uA\sqrt{1 + \varepsilon}\big) - u\cos\theta\big(1 - \cos uA\sqrt{1 + \varepsilon}\big) \\ &= 2\sin^2\frac{uA\sqrt{1+\varepsilon}}{2}\left(d\left(1 - \frac{\sin^2\frac{uA}{2}}{\sin^2\frac{uA\sqrt{1+\varepsilon}}{2}}\right) - u\cos\theta\right) \\ &\geq 2\sin^2\frac{uA\sqrt{1+\varepsilon}}{2}\left(\bar{d}\left(1 - \frac{\sin^2\frac{uA}{2}}{\sin^2\frac{uA\sqrt{1+\varepsilon}}{2}}\right) - u\right). \end{aligned}$$

We have $|uA| \leq \dfrac{u\sqrt{\delta - \varepsilon}}{1 + \varepsilon}$. Since

$$\lim_{x \to 0}\left(1 - \frac{\sin^2 x}{\sin^2 x\sqrt{1 + \varepsilon}}\right) = \frac{\varepsilon}{1 + \varepsilon} > 0,$$

it is easy to conclude Step 2. So we have proved that in all cases ($\theta = \pi/2$, $\theta > \pi/2$ and $\theta < \pi/2$), if $|s - s_0|$ is sufficiently small, then

$$\varphi(s) \leq f_{s_0}(s) = d\cos\frac{(s - s_0)\sqrt{\delta - \varepsilon}\ \sin\theta}{1 + \varepsilon} - (s - s_0)\cos\theta.$$

To complete the Local Part of the proof of Proposition 1.1 it suffices to prove

Claim 2. *There exists $\eta = \eta(\delta, \varepsilon, \bar{d}, \lambda) > 0$ such that*

$$f_{s_0}(s) \leq h_{s_0}(s) = d - (s - s_0)\cos\theta - \frac{\bar{d}\lambda(\delta - \varepsilon)}{2(1+\varepsilon)^3}(s - s_0)^2,$$

if $|s - s_0| < \eta$, *where* $\lambda = \min\{\sin^2\theta(0), \sin^2\theta(a)\}$.

It is an easy consequence of the concavity of the distance function φ that $\theta(s_1) \geq \theta(s_2)$, if $s_1 < s_2$. So we conclude that $\sin^2\theta \geq \lambda$. We have:

$$f_{s_0}(s_0) = h_{s_0}(s_0) = d, \quad f'_{s_0}(s_0) = h'_{s_0}(s_0) = -\cos\theta,$$

$$h_{s_0}{}''(s) = -\frac{\bar{d}\lambda(\delta - \varepsilon)}{(1+\varepsilon)^3},$$

$$\begin{aligned} f_{s_0}{}''(s) &= -\frac{d(\delta-\varepsilon)\sin^2\theta}{(1+\varepsilon)^2}\cos\frac{(s-s_0)\sqrt{\delta-\varepsilon}\,\sin\theta}{1+\varepsilon} \\ &\leq -\frac{\bar{d}\lambda(\delta-\varepsilon)}{(1+\varepsilon)^2}\cos\frac{|s-s_0|\sqrt{\delta-\varepsilon}}{1+\varepsilon}. \end{aligned}$$

Since

$$\lim_{x\to 0}\cos\frac{x\sqrt{\delta-\varepsilon}}{1+\varepsilon} = 1,$$

it is easy to conclude that for certain $\eta = \eta(\delta, \varepsilon, \bar{d}, \lambda) > 0$ we have $f''_{s_0}(s) \leq h''_{s_0}(s)$, if $|s - s_0| < \eta$. So Claim 2 is proved and the local part of the proof of Proposition 1.1 is completed.

B. Global part of the proof

By the local part, there exists $\eta > 0$ such that if $|s - s_0| < \eta$ then

$$\varphi(s) \leq h_{s_0}(s) = \varphi(s_0) - (s - s_0)\cos\theta(s_0) - \frac{\bar{d}\lambda(\delta-\varepsilon)}{2(1+\varepsilon)^3}(s - s_0)^2.$$

To conclude the proof of Proposition 1.1 we must prove that for all $s \in [0, a]$,

$$\varphi(s) \leq h_0(s) = \varphi(0) - s\cos\theta(0) - \frac{\bar{d}\lambda(\delta-\varepsilon)}{2(1+\varepsilon)^3}s^2. \tag{1.4}$$

Consider numbers $0 = s_0 < s_1 < s_2 < \ldots < s_m = a$ such that, for all $i \in \{1, 2, \ldots, m\}$, it holds that $s_i - s_{i-1} < \eta$. Then

$$s \in [s_{i-1},\ s_{i+1}] \ \Rightarrow\ \varphi(s) \le h_{s_i}(s), \quad \text{for } i = 1, 2, \ldots, m-1.$$

For $s \in [0,\ s_1]$, it holds that $\varphi(s) \le h_0(s)$.

Claim 1. *Let $i,\ j \in \{0, 1, \ldots, m\}$. It occurs exactly one of the three conditions below:*

(a) *h_{s_i} and h_{s_j} coincide;*

(b) *the graphs of h_{s_i} and h_{s_j} have no intersection;*

(c) *the graphs of h_{s_i} and h_{s_j} have exactly one intersection.*

In fact, the functions h_{s_i} and h_{s_j} are quadratic functions with the same second derivative. The intersection of their graphs is obtained by a linear equation. Claim 1 follows from this.

Claim 2. *Let $1 \le i \le m-1$. It holds that*

$$s \ge s_i \ \Rightarrow\ h_{s_i}(s) \le h_{s_{i-1}}(s).$$

In fact we have

$$h_{s_{i-1}}(s_{i-1}) = \varphi(s_{i-1}) \le h_{s_i}(s_{i-1}), \tag{1.5}$$

and

$$h_{s_i}(s_i) = \varphi(s_i) \le h_{s_{i-1}}(s_i). \tag{1.6}$$

By (1.5) and (1.6) there exists a point $s' \in [s_{i-1},\ s_i]$ such that

$$h_{s_{i-1}}(s') = h_{s_i}(s'). \tag{1.7}$$

By (1.6) and (1.7) it follows from Claim 1 that

$$s \ge s_i \ \Rightarrow\ h_{s_i}(s) \le h_{s_{i-1}}(s),$$

and Claim 2 is proved.

Claim 3. *For $s \in [0, a]$, it holds that $\varphi(s) \le h_0(s)$.*

If $s \in [0,\ s_1]$, the assertion is true. Assume that $s \in [s_i,\ s_{i+1}]$, with $1 \le i \le m-1$. By applying successively Claim 2 we obtain

$$\varphi(s) \le h_{s_i}(s) \le h_{s_{i-1}}(s) \le h_{s_{i-2}}(s) \le \ldots \le h_{s_0}(s) = h_0(s).$$

So the inequality (1.4) is proved. By making $\varepsilon \to 0$ we conclude the proof of Proposition 1.1. $\square$

2 Balls Going to Infinity

In this section we will finish the proofs of Theorem 1 and corollary 2 stated in the introduction. We say that a normal geodesic $\gamma : [0, \infty) \to M$ is a ray if $d\big(\gamma(0), \gamma(t)\big) = t$, for all t. It is easy to see that for all $p \in M$ there exists a ray starting at p. Let $\mathbf{\Gamma_p}$ be the set of all rays which start at p. Set $S_t = \partial B_o(t)$. We define as in [W] the function

$$F_o(x) = \lim_{t\to+\infty} \big(t - d(x, S_t)\big).$$

Set $C_t = \{x \in M | F_o(x) \le t\}$. It follows from [W] that F_o is a well defined Lipschitz function satisfying

$$F_o(x) \le d(o, x), \quad \text{and } d(x, \partial C_t) = t - F_o(x) \text{ if } F_o(x) < t. \tag{2.1}$$

Furthermore, if $\sigma_j(0) \to p$, $\sigma_j'(0) \to v$ and σ_j is a minimal connection between $\sigma_j(0)$ and S_{t_j}, where $t_j \to +\infty$, then $\gamma(t) := \exp_p tv$ is a ray satisfying

$$F_o\big(\gamma(t)\big) = F_o(p) + s. \tag{2.2}$$

Lemma 1.4 in [K] implies that $\lim_{x\to\infty} F_o(x) = +\infty$ and that, given $\varepsilon > 0$, there exists $r > 0$ such that, if $F_o(x) > r$ and $\gamma \in \Gamma_p$ satisfies (2.2), then the angle between $\gamma'(0)$ and any minimal connection between x and o is greater than $\pi - \varepsilon$. This implies by standard arguments of [C] that for $t > r$ the complement $M \backslash \mathrm{int}(C_t)$ is a finite union of ends U of the form $\partial U \times [0, +\infty)$, where ∂U is connected. From now on we assume by contradiction that there exists a sequence q_k going to infinity with $K \ge \delta_k > 0$ in the ball $B_{q_k}(r_k)$ and such that $\delta_k r_k^2 \ge \eta > 0$, and $r_k \le R$. We can assume that q_k is contained in some end U. Take any sequence $p_k \to \infty$, with $p_k \in U$. If s is sufficiently large we have $S_s \cap \partial U = \emptyset$. For each large k take a minimal connection σ_k between p_k and S_s. The geodesic σ_k is contained in a minimal geodesic $\tilde{\sigma}_k$ joining o and p_k. By taking a subsequence we obtain a ray $\gamma : [0, +\infty) \to M$ starting at o such that $\gamma(t) \in U$ for $t \ge s$.

Lemma 2.1. *Let o be the base point of M. Let $\gamma : [0, +\infty) \to M$ be a ray starting at o such that $\gamma(t) \in U$ for $t \ge s$. Then for sufficiently large k we have $d(q_k, \gamma) > r_k/2$.*

Proof. By [Ab1] we know that $\int_0^{+\infty} \kappa(t)dt < +\infty$. Since $K\big(\gamma(t)\big) \ge -\kappa(t)$ we obtain that the improper integral $\int_0^{+\infty} \big(K(\gamma(t)\big)dt$ is well defined. By [Am] or [MZ2] we conclude that $\int_0^{+\infty}\big(K(\gamma(t)\big)dt < +\infty$. So for sufficiently large s we have

$$\max\left\{\int_s^{+\infty}\big(K(\gamma(t)\big)dt, \int_s^{+\infty}\kappa(t)dt\right\} < \frac{\eta}{2R}.$$

Assume by contradiction that there exists some subsequence, which we still denote by q_k, such that $d(q_k, \gamma) \leq r_k/2$. Thus it is easy to obtain an interval $I_k \subset [s, +\infty)$ of length r_k such that $\gamma(I_k) \subset B_{q_k}(r_k)$. Since $\delta_k r_k \geq \eta/R$ we have

$$\frac{\eta}{2R} > \int_s^{+\infty} K\big(\gamma(t)\big)dt \geq \frac{\eta}{R} - \int_{[s,+\infty)\setminus I_k}\kappa(t)dt > \frac{\eta}{2R}.$$

This contradiction proves Lemma 2.1. □

Lemma 2.2. *Take q_k as above. Choose $\tilde{q}_k \in B_{q_k}(r_k/3)$. Assume that $\tilde{q}_k \in \partial C_{t_k}$. Given a geodesic σ of length $\ell \leq r_k/3$, if $\sigma(0) = \tilde{q}_k$ and $\sigma(\ell) \in \partial C_{t_k}$, then for sufficiently large k we have $\sigma \subset C_{t_k}$.*

Proof. Since $t_k = F_o(\tilde{q}_k) \to +\infty$, for sufficiently large k we can assume that

$$\int_{t_k}^{+\infty}\kappa(t)dt < \frac{\eta}{6R}.$$

Assume by contradiction that there exists p_k in the image of σ such that $p_k \notin C_{t_k}$. Then there exists $\varepsilon > 0$ and $s_0 > 0$ such that if $s \geq s_0$ then $s - d(p_k, S_s) \geq t_k + \varepsilon$. So there exists p_{ks} in the image of σ such that $d(p_{ks}, S_s)$ is minimal. We still have $s - d(p_{ks}, S_s) \geq t_k + \varepsilon$. By taking a subsequence, a minimal geodesic joining p_{ks} and S_s converges to a ray τ, where:

$$\tau(0) = p,\ F_o(p) \geq t_k + \varepsilon,\ d\big(\tau(s), \sigma\big) = s,\ F_o\big(\tau(s)\big) = F_o(p) + s.$$

For $0 \leq s \leq r_k/3$ we have $\tau(s) \in B_{q_k}(r_k)$, hence $K\big(\gamma(s)\big) \geq \delta_k$. For all $s \geq 0$ we have $d\big(o, \tau(s)\big) \geq F_o\big(\tau(s)\big) = F_o(p) + s > t_k + s$, hence $K\big(\tau(s)\big) \geq -\kappa(t_k + s)$. So we obtain

$$\int_0^S K \circ \tau \geq \frac{\delta_k r_k}{3} - \int_{r_k/3}^S \kappa(t_k + s)ds \geq \frac{\eta}{3R} - \int_{t_k+r_k/3}^{t_k+S}\kappa(u)du \geq \frac{\eta}{6R}.$$

Then

$$\liminf_{S\to+\infty}\int_0^S K\big(\tau(s)\big)ds > 0,$$

which contradicts Corollary 1 in [MZ1], since $d\big(\tau(s), \sigma\big) = s$, for all $s \geq 0$. Lemma 2.2 is proved. □

Consider the solution f of the equation

$$\begin{cases} f''(r) - \kappa(r) f(r) = 0 \\ f(0) = 0, \quad f'(0) = 1 \end{cases} \tag{2.3}$$

for $r \geq 0$. Now let $\tilde{M}$ be the plane equipped with the metric

$$ds^2 = dr^2 + f(r)^2 d\theta^2.$$

This metric was studied by Abresch in [Ab1], and he proved a version of Toponogov Theorem comparing triangles with vertex o in M with triangles with vertex $\tilde{o} := (0,0)$ in $\tilde{M}$. The curvature of $\tilde{M}$ is

$$K(r) = -\frac{f''(r)}{f(r)} = -\kappa(r).$$

Of course $\tilde{o}$ is a pole. In [GW] it is proved that there exists

$$F = \lim_{r \to +\infty} f'(r) \geq 1.$$

We prove now a simple result about $\tilde{M}$.

Lemma 2.3. *Let $\tilde{\gamma}, \tilde{\sigma}$ be rays starting at $\tilde{o} \in \tilde{M}$ with an angle $\measuredangle\big(\tilde{\gamma}'(0), \tilde{\sigma}'(0)\big) = \tilde{\theta}$. Let V be the region defined by $\tilde{\gamma}, \tilde{\sigma}$ and $\tilde{\theta}$. Then*

$$\int_V K = \tilde{\theta}(1 - F).$$

Proof. Set $V_t = \{\exp_{\tilde{o}} sv | 0 \leq s \leq t, v \in I\}$ where I is the arc in the unit tangent circle of angle $\tilde{\theta}$ which joins $\tilde{\gamma}'(0)$ and $\tilde{\sigma}'(0)$. For any $t > 0$ we have

$$\int_{V_t} K = \int_0^{\tilde{\theta}} d\theta \int_0^t -\kappa(r) f(r) dr = \tilde{\theta} \int_0^t \big(-f''(r)\big) dr = \tilde{\theta}\big(1 - f'(t)\big).$$

When we let $t \to +\infty$ we obtain the desired equality. □

From now on we denote by τ_{xy} any minimal geodesic joining x and y. Let $q_k \in M$ be a sequence as above and set $t_k = F_o(q_k)$. Again we have that $U \cap \big(M \backslash \operatorname{int}(C_{t_k})\big)$ is homeomorphic to $(U \cap \partial C_{t_k}) \times [0, +\infty)$, where $U \cap \partial C_{t_k}$ is connected. From Lemma 2.1 and the connectedness of $U \cap \partial C_{t_k}$ we can choose a point $\tilde{q}_k \in U \cap \partial C_{t_k}$ with $d(q_k, \tilde{q}_k) = r_k/10$.

Lemma 2.4. *Take $\varepsilon > 0$. For sufficiently large k, any $\tau = \tau_{q_k \tilde{q}_k}$, and any minimal connection σ between q_k and $\partial C_{t'}$ with $0 < t' - t_k < r_k/10$ satisfies*

$$\frac{\pi}{2} \leq \measuredangle\big(\sigma'(0),\ \tau'(0)\big) < \frac{\pi}{2} + \varepsilon. \tag{2.4}$$

Proof. Because of Lemma 2.2, the same proof of Lemma 1.7 in [CG] implies that $C_{t'}$ is τ-convex. Thus the left inequality in (2.1) is an easy consequence of the first variation formula together with (1.1). Let us prove the right inequality. Take $\mu = \tau_{o\tilde{q}_k}$ and set $\ell = L(\mu)$. Let $\gamma = \tau_{oq_k}$ and set $t = L(\gamma)$. By (4.1) we have $\ell, t \geq t_k$. Set: $\alpha_k = \measuredangle\big(-\gamma'(t),\ \tau'(0)\big)$, $\beta_k = \measuredangle\big(\mu'(\ell),\ \tau'(r_k/10)\big)$.

Consider in $\tilde{M}$ the comparison triangle $(\ell, t, r_k/10)$ with corresponding angles $(\tilde{\alpha}_k, \tilde{\beta}_k, \tilde{\theta}_k)$, where $\tilde{o}$ is the vertex opposite to $r_k/10$. By the extension of the Toponogov Theorem due to Abresch (see [Ab1]) we have $\alpha_k \geq \tilde{\alpha}_k,\ \beta_k \geq \tilde{\beta}_k$. Since $\tilde{M}$ satisfies $K \leq 0$ and $r_k/t_k \leq R/t_k \to 0$ we can consider a triangle in the plane with the same lengths and conclude easily that $\tilde{\theta}_k \to 0$ by the extension of the Rauch Comparison Theorem (see [Gv], p. 197). So for sufficiently large k we have $\tilde{\theta} F < \varepsilon/3$.

Claim 1. *For sufficiently large k it holds that $\beta_k < \pi/2 + \varepsilon/3$.*

By Lemma 1.4 in [K], if k is sufficiently large then any minimal connection ρ between $\tilde{q}_k$ and $\partial C_{t'}$ satisfies $\measuredangle\big(\rho'(0),\ \mu'(\ell)\big) < \varepsilon/3$. Then

$$\begin{aligned}
\beta_k &= \pi - \measuredangle\big(-\tau'(r_k/10), \mu'(\ell)\big) \\
&\leq \pi - \big[\measuredangle\big(-\tau'(r_k/10), \rho'(0)\big) - \measuredangle\big(\rho'(0), \mu'(\ell)\big)\big] \\
&= \pi - \measuredangle\big(-\tau'(r_k/10),\ \rho'(0)\big) + \measuredangle\big(\rho'(0),\ \mu'(\ell)\big).
\end{aligned}$$

By the left inequality in (4.4) applied to τ and ρ instead of τ and σ we have $\measuredangle\big(-\tau'(r_k/10), \rho'(0)\big) \geq \pi/2$. So we obtain

$$\beta_k < \pi - \frac{\pi}{2} + \frac{\varepsilon}{3} = \frac{\pi}{2} + \frac{\varepsilon}{3},$$

and Claim 1 is proved.

Claim 2. $\alpha_k > \pi/2 - 2\,\varepsilon/3$.

By Claim 1 and Lemma 2.3 we have

$$\begin{aligned}
\alpha_k \geq \tilde{\alpha}_k &\geq \pi - \tilde{\beta}_k - \tilde{\theta}_k + \tilde{\theta}_k(1 - F) \\
&\geq \pi - \beta_k - \tilde{\theta}_k F > \pi - \Big(\frac{\pi}{2} + \frac{\varepsilon}{3}\Big) - \frac{\varepsilon}{3} = \frac{\pi}{2} - 2\,\frac{\varepsilon}{3}.
\end{aligned}$$

Conclusion of the proof. By Lemma 1.4 in [K], if k is large enough then $\measuredangle\big(\gamma'(t), \sigma'(0)\big) < \varepsilon/3$. So by Claim 2 we obtain

$$\begin{aligned}\measuredangle\big(\tau'(0), \sigma'(0)\big) &\leq \measuredangle\big(\tau'(0), \gamma'(t)\big) + \measuredangle\big(\gamma'(t), \sigma'(0)\big)\\ &= \pi - \alpha_k + \measuredangle\big(\gamma'(t), \sigma'(0)\big)\\ &< \pi - \Big(\frac{\pi}{2} - 2\,\frac{\varepsilon}{3}\Big) + \frac{\varepsilon}{3} = \frac{\pi}{2} + \varepsilon.\end{aligned}$$

This concludes the proof. □

We are now ready to prove the following equivalent version of Theorem 1.

Theorem 2.5. *Let M be a complete manifold with asymptotically nonnegative curvature. Then for any sequence $q_k \to \infty$, with $K \geq \delta_k > 0$ in $B_{q_k}(r_k)$, and $r_k \leq R$, for a fixed number R, satisfies $r_k^2\delta_k \to 0$.*

Proof of Theorem 1. Take a sequence q_k as above. Set $\ell_k = r_k/10$. Let $\varepsilon > 0$ be a constant such that

$$\frac{\sin\varepsilon}{\cos^2\varepsilon} < \frac{\eta}{200}. \tag{2.5}$$

Let $t_k = F_o(q_k)$. There exists a point $\tilde{q}_k \in \partial C_{t_k}$, such that the distance $d(q_k, \tilde{q}_k) = \ell_k$. Let $\tau_k = \tau_{q_k\tilde{q}_k}$. Consider a minimal connection σ_s between $\tau_k(s)$ and $\partial C_{(t_k+\ell_k)}$, for $s \in [0, \ell_k]$. Set $\theta(s) = \measuredangle(\tau_k'(s), \sigma_s'(0))$ and $\varphi(s) = d(\tau_k(s), \partial C_{(t_k+\ell_k)})$. By Lemma 2.2 it is easy to see that $\tau_k([0, \ell_k]) \subset C_{t_k}$. As above the set $C_{t_k+\ell_k}$ is τ_k-convex. So we can apply Proposition 1.3, obtaining that

$$\varphi(\ell_k) \leq \varphi(0) - \ell_k\cos\theta(0) - \frac{\bar{d}\lambda\delta_k}{2}\,\ell_k^2.$$

By Lemma 2.4, for sufficiently large k we have $\pi/2 \leq \theta(0) < \pi/2 + \varepsilon$, and $\pi/2 \leq \pi - \theta(\ell_k) < \pi/2 + \varepsilon$. So we obtain $\lambda \geq \min\{\sin^2\theta(0), \sin^2\theta(\ell_k)\} \geq \cos^2\varepsilon$, because of the monotonicity of the function $\theta(s)$. We obtain also that $-\cos\theta(0) \leq -\cos(\pi/2+\varepsilon) = \sin\varepsilon$. Note that $\varphi(0) = \ell_k$. Since τ_k is contained in C_{t_k}, by (2.1) we obtain $\varphi(s) \geq \ell_k$ for all $s \in [0, \ell_k]$, hence $\bar{d} = \varphi(0) = \ell_k$ and $\ell_k \leq \varphi(\ell_k)$. Thus we conclude that

$$\ell_k \leq \varphi(\ell_k) \leq \ell_k + \ell_k\sin\varepsilon - \frac{\ell_k^3(\cos^2\varepsilon)\delta_k}{2}.$$

Replacing ℓ_k by its value we obtain

$$\frac{\sin\varepsilon}{\cos^2\varepsilon} \geq \frac{\ell_k^2\,\delta}{2} \geq \frac{\eta}{200},$$

which contradicts (2.5) and proves the theorem. □

Let us prove our corollary stated in the introduction.

Proof of Corollary 2. Assume that $K(p) > 0$. So there exists $r > 0$ such that $K \geq K(p)/2$ in the ball $B = B_p(r)$. Let f_k be a sequence of isometries of M. Because of Theorem 1 there exists $S > 0$ such that $f_k(B) \subset B_p(S)$, for all k. So by triangle inequality we have

$$\begin{aligned} d\big(x, f_k(x)\big) &\leq d(x, p) + d\big(p, f_k(p)\big) + d\big(f_k(p), f_k(x)\big) \\ &\leq 2d\big(x, p\big) + S \leq 2T + S, \end{aligned}$$

if $x \in B_p(T)$. So the family of equicontinuous maps f_k sends the compact ball $B_p(T)$ into the compact ball $B_p(3T + S)$. By the Ascoli-Arzela Theorem there exists a subsequence f_{k_i} which converges uniformly on $B_p(T)$. Now considering balls $B_p(s)$, $s \in \mathbb{N}$, and using the classical diagonal argument as in the proof of the Ascoli-Arzela Theorem, we obtain the existence of a subsequence of $\{f_k\}$ which is convergent along each ball $B_p(s)$. Since $\mathrm{Isom}(M)$ is closed, so it is compact.

Now assume that $f_k = f^k$, for some isometry f. We want to find some subsequence converging to g with $g(p) = p$. First we assert that there exists a subsequence f^{k_i} such that $f^{k_i}(p) \to p$. Otherwise there exists $\varepsilon > 0$ such that $d\big(f^k(p), p\big) \geq \varepsilon$, for all $k \geq 1$. Decreasing ε if necessary, we can assume that $K \geq K(p)/2$ in $B = B_p(\varepsilon/4)$. Then it is easy to see that all balls $f^k(B)$ are mutually disjoint, so they cannot be contained in a compact set. This contradicts Theorem 1. So we know that there exists a subsequence f^{s_i} such that $f^{s_i}(p) \to p$. Passing again to a subsequence we can assume that $f^{s_i} \to g$ uniformly on compact subsets. Trivially we have $g(p) = p$. Corollary 2 is proved. $\square$

3 Examples

We give here some examples stated in the introduction. To prove Proposition 3.2 below, we need the following Lemma.

Lemma 3.1. *Take $\varepsilon > 0$. Fix $a, b \in \mathbb{R}$ with $a < b$. There exists a C^∞ function $f \colon \mathbb{R} \to \mathbb{R}$ which satisfies: $f(x) = ax$ if $x \leq -\varepsilon$; $f(x) = bx$ if $x \geq \varepsilon$; $f''(x) \geq 0$ for all $x \in \mathbb{R}$; given $A > 0$, there exist suitable choices of ε and $c \in (-\varepsilon, \varepsilon)$ so that $f''(c) > A$.*

Proof. Let $h \colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function so that $h(x) = a$ if $x \leq -\varepsilon$, $h(x) = b$ if $x \geq \varepsilon$, and $h - (a + b)/2$ is a nondecreasing odd function. In particular we have $\int_{-\varepsilon}^{\varepsilon} h(x)dx = \varepsilon(a+b)$. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = -a\varepsilon + \int_{-\varepsilon}^{x} h(t)dt$. Then $f(x) = ax$ if $x \le -\varepsilon$, and $f(x) = bx$ if $x \ge \varepsilon$. Since $f'' = h'$ we have $f'' \ge 0$. By the Mean Value Theorem there exists $c \in (-\varepsilon, \varepsilon)$, such that

$$f''(c) = h'(c) = \frac{h(\varepsilon) - h(-\varepsilon)}{2\varepsilon} = \frac{b-a}{2\varepsilon}.$$

Thus if we choose $\varepsilon < \dfrac{b-a}{2A}$ Lemma 3.1 follows. □

Proposition 3.2. *There exist a complete surface of revolution M^2 with $K \ge 0$ and a sequence $(p_k) \subset M,\ p_k \to \infty$ such that* $\lim K(p_k) = +\infty$.

Proof. Take $g\colon [0,+\infty) \to \mathbb{R}$ satisfying $g(x) = a_k x + b_k,$ if $x \in [k, k+1]$, where the sequences (a_k) and (b_k) satisfy $a_0 = b_0 = 0$, (a_k) is increasing and $a_k \to 1$, b_{k+1} is chosen in such a way that g is continuous at $x = k+1$, that is, $a_k(k+1) + b_k = a_{k+1}(k+1) + b_{k+1}$.

From Lemma 3.1 we can obtain sequences $(\varepsilon_k)_{k\ge0}$, $(c_k)_{k\ge1}$ with $\varepsilon_0 = 0,\ 0 < \varepsilon_k < 1/2$, if $k \ge 1,\ c_k \in (k-\varepsilon_k, k+\varepsilon_k)$, and a C^∞ function $f\colon [0,+\infty) \to [0,+\infty)$, so that $f(x) = g(x)$ if $x \in [k+\varepsilon_k, k+1-\varepsilon_{k+1}]$, $f'' \ge 0$ and $f''(c_k) > k(k-1)$. The surface M given by the rotation of the graph of f around the y-axis furnishes the desired example, as we see next. It is straightforward to verify that

$$K\big(x, f(x), 0\big) = \frac{f''(x)}{x}\,\frac{f'(x)}{\big(1+\big(f'(x)\big)^2\big)^2}.$$

Since the function f' is nondecreasing we have $a_{k-1} \le f'(c_k) \le a_k$, hence

$$\frac{f'(c_k)}{\big(1+\big(f'(c_k)\big)^2\big)^2} \to \frac{1}{4}.$$

Since $f''(c_k) \ge k(k-1)$ it is easy to see that $K\big(c_k, f(c_k), 0\big) \to +\infty$, thus concluding the proof. □

Lemma 3.3. *Let $f(r)$ be a smooth function on* $[0,+\infty)$ *and r_0 be a positive constant such that $f''(r) \le 0$ for any $r \in [r_0, +\infty)$ and $0 < f'(r_0) < 1$. Then there exists a smooth function $g(r)$ on* $[0,+\infty)$ *such that $g''(r) \le 0$ and*

$$g(r) = \begin{cases} r, & r \le r_0 \\ f(r) + c, & r \ge 2r_0, \end{cases} \tag{3.1}$$

where c is a constant.

Proof. Define a function $h(r)$ on $[0, +\infty)$ as $h(r) := \alpha(r) + (1 - \alpha(r)) f'(r)$, with

$$\alpha(r) = \begin{cases} 1, & r \leq r_0 \\ 1 - \beta \displaystyle\int_{r_0}^{r} e^{-\frac{1}{(t-r_0)(t-2r_0)}} dt, & r \in (r_0, 2r_0) \\ 0, & r \geq 2r_0, \end{cases}$$

where

$$\beta = \left(\int_{r_0}^{2r_0} e^{-\frac{1}{(t-r_0)(t-2r_0)}} dt \right)^{-1}.$$

Note that $f'(r) \leq h(r) \leq 1$.

So we have for all $r \geq 0$

$$h'(r) = \alpha'(r)(1 - f'(r)) + (1 - \alpha(r)) f''(r) \leq 0,$$

here we have used the condition $0 < f'(r_0) < 1$ which implies that $f'(r) < 1$ for all $r \geq r_0$. Let $g(r) := \int_0^r h(t)\, dt$. We have $g'(r) = f'(r)$ when $r \geq 2r_0$ and satisfies the desired condition. It is easy to see that the constant c in (5.1) is determined by

$$c = g(2r_0) - f(2r_0) = r_0 + \int_{r_0}^{2r_0} h(t)dt - f(2r_0) \leq 2r_0 - f(2r_0). \qquad \square$$

Here we are ready to give an example to show that the boundness of $\{r_k\}$ in Theorem 1 cannot be deleted in general.

Example 3.4. There exist a Riemannian manifold (M, g) with nonnegative curvature, a sequence of points $q_k \to \infty$, and positive numbers r_k such that

$$r_k^2 \inf \left\{ K(x) \colon x \in B_{q_k}(r_k) \right\} > \frac{1}{128} > 0. \tag{3.2}$$

Proof. Let $f(r) = r^{\frac{1}{2}}$ and $r_0 = 1$. Let g be the function obtained as in Lemma 5.1. Since $g''(r) \leq 0$ on $[0, +\infty)$, and $g'(r) > 0$ for $r \in [0, 1] \cup [2, +\infty)$ we know that $g(r)$ is always positive on $(0, +\infty)$. We note also that here $0 \leq c \leq 1$.

Define $M = (R^n, ds^2)$ with the metric $ds^2 = dr^2 + g^2(r) d\theta^2$ for the spherical coordinates. It is well-known and straightforward to verify that

$$K(r) = -\frac{g''(r)}{g(r)} = -\frac{h'(r)}{g(r)} \geq 0,$$

where $r = r(x)$ is the distance function from origin. So we get

$$K(x) = \frac{1}{4(r^2 + cr^{\frac{3}{2}})} \geq \frac{1}{8r^2}, \qquad \text{for } r \in [2, +\infty).$$

So we can choose $q_k = (k+2, 0, \cdots, 0)$ and $r_k = k$. It follows that

$$k^2 K \geq k^2 K(2k+2) \geq \frac{k^2}{8(2k+2)^2} \geq \frac{k^2}{8(2k+2k)^2} = \frac{1}{128} \qquad \text{in } B_{q_k}(r_k),$$

and this shows (3.2). The proof is complete. □

Now we present a very simple example, which shows that the distance function from the boundary of closed convex sets on nonnegatively curved manifolds may not be concave.

Example 3.5. Consider a smooth curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ satisfying

$$\alpha(r) = \begin{cases} (0, -1, -t), & t \leq -1 \\ (0,\ 1,\ t), & t \geq 1 \\ \big(0, t, f(t)\big), & -1 \leq t \leq 1, \end{cases}$$

where f is a convex smooth even function with a minimum at $f(0) = 0$. If we rotate α around the z-axis we obtain a complete surface S with $K \geq 0$. Let $C = \alpha([-2, 20])$. Clearly C is a closed convex set with $\partial C = \{(0, -1, 2), (0, 1, 20)\}$. Set $\varphi(t) = d\big(\alpha(t), \partial C\big)$, $t \in [-2, 20]$. Clearly φ admits a local strict minimum at $t = 2$, hence φ cannot be concave. Because of this we needed to consider d_C instead of d in Proposition 1.1 above.

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