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# Regular interval Cantor sets of $S^1$ and minimality

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**Abstract.** It is known that not every Cantor set of $S^1$ is $C^1$-minimal. In this work we prove that every member of a subfamily of what we here call *regular interval Cantor set* is not $C^1$-minimal. We also prove that no member of a class of Cantor sets that includes this subfamily is $C^{1+\epsilon}$-minimal, for any $\epsilon > 0$.

**Keywords:** Cantor sets, $C^1$-minimal sets.

**Mathematical subject classification:** 37E10, 37C45.

## 1 Introduction

If $f: S^1 \to S^1$ is a diffeomorphism without periodic points, there exists a unique set $\Omega(f) \subset S^1$ minimal for $f$ (we say that $\Omega(f)$ is $C^1$-minimal for $f$). In this case $\Omega(f)$ is either a Cantor set or $S^1$. The $C^1$-minimal Cantor sets known up to now are the Denjoy examples and its conjugates. On the other hand, we know that some families are not $C^1$-minimal. For example, in [2] Mc Duff shows that the usual middle thirds Cantor set is not $C^1$-minimal and gives some conditions for a Cantor set not to be $C^1$-minimal. In [6] we can find other conditions that also imply they are not $C^1$-minimal. In [5] A. Norton shows that the affine Cantor sets are not $C^1$-minimal. In this work we construct new families of Cantor sets that are not $C^1$-minimal and other families of Cantor sets that are not $C^{1+\epsilon}$-minimal, for any $\epsilon > 0$.

### 1.1 Regular interval Cantor sets

The regular interval Cantor set construction imitates the procedure used to obtain the usual middle thirds Cantor set. Given two sequences $\{m_i\}$ and $\{\theta_i\}$ with $m_i$ a positive integer and $0 < \theta_i < 1$, we proceed as follows. First we remove from

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the circle $m_1$ open intervals with the same measure, distributed in the same way, obtaining the closed set $K_1 = \cup\Delta_{i_1}$ ($i_1 = 1, \ldots, m_1$) with Lebesgue measure $|K_1| = \theta_1$, where $\Delta_{i_1}$ are the connected components of $K_1$. In the second step, we remove from each connected component $\Delta_{i_1}$, $m_2$ open intervals of the same measure, distributed in the same way, obtaining the closed set $K_2 = \cup\Delta_{i_1 i_2}$ ($i_2 = 1, \ldots, m_2 + 1$) with measure $|K_2| = \theta_2|K_1|$, where $\Delta_{i_1 i_2}$ are the connected components of $K_2$. Proceeding like this, we obtain, for each $n$, a closed set $K_n \subset S^1$, contained in $K_{n-1}$, with measure $|K_n| = \theta_n|K_{n-1}|$, and $K_n = \cup\Delta_{i_1\ldots i_n}$ ($i_n = 1, \ldots, m_n + 1$ ), where $\Delta_{i_1\ldots i_n}$ are connected components of $K_n$. We define $K = \bigcap K_n$. This set is a Cantor set, and we will call **regular interval Cantor set** every set $K$ constructed in this way.

### 1.2 Quasi regular interval Cantor sets

Now we are going to construct a family of Cantor sets that contains the regular interval Cantor sets. Given a sequence $\{n_i\}$ of positive integers with $\sum_{i<j} n_i \leq n_j$, we proceed as follows. First we remove from $S^1$, $n_1$ open intervals of the same measure, obtaining a closed set $K_1 = \bigcup \Delta_{1i_1}$ ($i_1 = 1, \ldots, n_1$), where $\Delta_{1i_1}$ are the connected components of $K_1$. In the second step, we remove from $K_1$, $n_2$ open intervals of the same measure, removing at least an interval of each connected component of $K_1$, obtaining the closed set $K_2 = \bigcup \Delta_{2i_2}$ ($i_2 = 1, \ldots, n_1 + n_2$), where $\Delta_{2i_2}$ are the connected components of $K_2$. We do not require the intervals removed to be likewise distributed. Inductively, for each $m$ we obtain a closed set $K_m \subset S^1$ contained in $K_{m-1}$ and we write $K_m = \bigcup \Delta_{mi_m}$ ($i_m = 1, \ldots, n_1 + \cdots + n_m$) where $\Delta_{mi_m}$ are the connected components of $K_m$. Then, we define $K = \bigcap K_m$. The set $K$ is a Cantor set if, and only if,

$$\nu_m = \max\left\{|\Delta_{mi_m}| \colon i_m = 1, \ldots, n_1 + \cdots + n_m\right\} \to 0$$

when $m \to \infty$. We will call **quasi regular interval Cantor set** every Cantor set $K$ constructed in this way. Note that with this procedure we do not obtain all Cantor sets of $S^1$. If

$$\mu_m = \min\left\{|\Delta_{mi_m}| \colon i_m = 1, \ldots, n_1 + \cdots + n_m\right\},$$

the number $\delta = \inf\{\mu_m/\nu_m \colon m \in \mathbf{N}\}$ gives an idea of the irregularity of the Cantor set $K$. This number depends on the set $K$ as well as on the procedure to obtain $K$. Then, we define the regularity of $K$ as the supreme of the set of $\delta's$, taken over all the possible procedures to obtain $K$. Note that if the Cantor set $K$ is a regular interval Cantor set, its regularity is 1.

## 2 Main results

**Theorem 1.** *If the Cantor set $K$ is $C^1$-minimal for a diffeomorphism $f$, and $K^c$ has only one orbit of wandering intervals, then $K$ is not a quasi regular interval Cantor set.*

**Theorem 2.** *If $K$ is a quasi regular interval Cantor set of regularity different from 0, then $K$ is not $C^{1+\epsilon}$-minimal for any $\epsilon > 0$.*

As all regular interval Cantor sets have regularity 1 then, from the previous theorem, we have the following result.

**Corollary 1.** *If $K$ is a regular interval Cantor set, then $K$ is not $C^{1+\epsilon}$-minimal for any $\epsilon > 0$.*

If the regular interval Cantor set $K$ has positive measure and we suppose that it is $C^1$-minimal for $f$ we obtain several conditions for $f'$. Let $m_i$ be the number of intervals removed in the step $i$ of the construction of $K$.

**Theorem 3.** *If $K$ is a regular interval Cantor set of positive measure and the sequence $\{m_i\}$ is not bounded, then $K$ is not $C^1$-minimal.*

**Definition 2.1.** *If $K$ is a regular interval Cantor set, for each prime integer we define $A_q = \{i \in \mathbf{N} \colon m_i + 1 = 0 (\operatorname{mod} q)\}$.*

For the case that $A_q$ is an infinite set we denote its elements by $t_n$ ($n \in \mathbf{N}$), with $t_n < t_{n+1}$. Now we can state the following result.

**Theorem 4.** *If $K$ is a regular interval Cantor set of positive measure and there exists a prime integer $q$ such that $A_q$ is infinite and $t_{n+1} - t_n \to \infty$, then $K$ is not $C^1$-minimal.*

## 3 Generalities

The following lemmas are going to be very useful in the proofs of the main results.

**Definition 3.1.** *If $f \colon S^1 \to S^1$ is a diffeomorphism, then for each $x \in S^1$ and for each positive integer $n$ we define*

$$F(x, n) = \sum_{i=0}^{n-1} \log f'(f^i(x)) = \log(f^n)'(x).$$

**Lemma 3.1.** *If the Cantor set $K$ is $C^1$-minimal for $f$, then there exists $x \in K$ such that $F(x, n) \geq 0$, for all positive integer $n$.*

**Proof.** Assume that for all $x \in K$ there exists $m_x$ such that $F(x, m_x) < 0$. By the continuity of $f'$, for each $x \in K$ there exists $\delta_x > 0$ such that for every point $y$ in the interval $(x - \delta_x, x + \delta_x)$, $F(y, m_x) < 0$. As the family of intervals $(x - \delta_x, x + \delta_x)$ with $x \in K$ is a covering of $K$, and $K$ is a Cantor set, then there exists a finite refinement $\{I_i,\ i = 1, \ldots, p\}$ of this covering of open intervals, pairwise disjoint, that is a covering of $K$. So, for each $I_i$ there exists $m_i \in \mathbf{N}$ such that for all $y \in I_i$ we have $F(y, m_i) < 0$. Besides, $S^1 \setminus \bigcup_{i=1}^p I_i$ is a finite union of closed intervals, each of which is contained in a connected component of $K^c$ that we call $J_i$, with $i = 1, \ldots, p$. We consider $m = \max\{m_i : i = 1, \ldots, p\}$ and $M \geq 1$ the maximum of $f'$. We consider a wandering interval $T$ of the past of $J_1$ such that $|T| M^m < \min\{|J_1|, \ldots, |J_p|\}$. Now we will show that if $j$ is a positive integer then $|f^j(T)| < |J_1|$, getting a contradiction. By the choice of $T$, we know that $T$ is contained in $I_i$ for some $i$. By the Mean Value Theorem, there exists $\theta \in I_i$ such that

$$|f^{m_i}(T)| = |T|(f^{m_i})'(\theta).$$

As $F(\theta, m_i) < 0$, we have $(f^{m_i})'(\theta) < 1$ and so

$$|f^{m_i}(T)| < |T|.$$

We can repeat this process with $f^{m_i}(T)$ instead of $T$. Inductively we conclude that there exists a sequence $\nu_1, \nu_2, \ldots, \nu_k, \ldots$ with $\nu_k \in \{m_1, \ldots, m_p\}$ such that for all positive integer $r$

$$|f^{\sum_{k=1}^r \nu_k}(T)| < |T|.$$

As for all $j$ there exists $r_0 \geq 0$ such that

$$\sum_{k=1}^{r_0} \nu_k \leq j < \sum_{k=1}^{r_0+1} \nu_k,$$

we have

$$|f^j(T)| = |f^{j - \sum_{k=1}^{r_0} \nu_k}(f^{\sum_{k=1}^{r_0} \nu_k}(T))| \leq M^m |T| < |J_1|. \qquad \square$$

Let $K$ be a Cantor set of the circle and let $K^c = \bigcup I_j$, where $I_j$ are the connected components of $K^c$. We define the spectrum of $K$ $(E_K)$ as the ordered set $\{\lambda_i\}$ $(\lambda_{i+1} < \lambda_i)$, with $\lambda_i$ the length of $I_j$, for some $j$.

**Lemma 3.2.** *If the Cantor set $K$ is $C^1$-minimal for $f$ and $\lambda_n/\lambda_{n+1} \not\to 1$, there exists $\eta > 0$ and $x \in K$ such that $F(x, m) \leq -\eta$, for all positive integer $m$.*

**Proof.** As $\lambda_n/\lambda_{n+1} \not\to 1$, there exist $\epsilon_0 > 0$ and a sequence $\{n_k\}$ such that $1 + \epsilon_0 \leq \frac{\lambda_{n_k}}{\lambda_{n_k+1}}$. Let $I_{n_k}$ be a connected component of $K^c$ such that $|I_{n_k}| \geq \lambda_{n_k}$ and for all $j > 1$, $|f^j(I_{n_k})| \leq \lambda_{n_k+1}$. By the choice of $I_{n_k}$ we have that $|I_{n_k}| \to 0$ when $k \to \infty$. Let $x$ be an accumulation point of the set of the intervals $I_{n_k}$ $(x \in K)$ and $\{k_i\}$ a sequence such that $d(x, I_{n_{k_i}}) \to 0$ when $i \to \infty$. Therefore, for every $m \geq 1$, there exists $i$ sufficiently large such that

$$1 + \epsilon_0 \leq \frac{\lambda_{n_{k_i}}}{\lambda_{n_{k_i}+1}} \leq \frac{|I_{n_{k_i}}|}{|f^m(I_{n_{k_i}})|}.$$

Then

$$F(x, m) = \log(f^m)'(x) = \log\left(\lim_{i\to\infty} \frac{|f^m(I_{n_{k_i}})|}{|(I_{n_{k_i}})|}\right) \leq -\log(1 + \epsilon_0). \qquad \square$$

**Lemma 3.3.** *If the Cantor set $K$ is $C^1$-minimal for $f$ and $\lambda_n/\lambda_{n+1} \not\to 1$ then for every point $x \in K$, $F(x, m)$ is not bounded.*

**Proof.** By the transitivity of $K$ (for $f$), it is enough to prove the property for any point of $K$. Let $x$ and the number $\eta$ be as in lemma 3.2 and suppose by contradiction that $F(x, m)$ is bounded. Therefore if $y = \inf\{F(x, m) \colon m \in \mathbf{N}\}$, there exists a positive integer $p$ such that $|F(x, p) - y| < \eta/2$. So

$$\begin{aligned} F(f^p(x), m) &= F(x, m+p) - F(x, p) \\ &= F(x, m+p) - y - (F(x, p) - y) > \frac{-\eta}{2} \end{aligned} \tag{1}$$

for all positive integer $m$. We consider $\{n_k\}$ such that $f^{p+n_k}(x)$ has limit $x$ when $k \to \infty$. From the uniform continuity of $f'$ we have that

$$\begin{aligned} &|F(f^p(x), p + n_k) - F(x, p + n_k)| \\ &\quad \leq \sum_{i=0}^{p-1} |\log f'(f^{p+n_k+i}(x)) - \log f'(f^i(x))| \\ &\quad = \delta(n_k) \to 0 \end{aligned}$$

when $k \to \infty$. Then

$$F(f^p(x), p + n_k) < F(x, p + n_k) + \delta(n_k) < -\eta + \delta(n_k),$$

so using (1) we have a contradiction. $\square$

## 4 Geometric rigidity

In this section we are going to prove two geometric properties for the quasi regular interval Cantor sets and that if, we suppose that a Cantor set $K$ of this family is $C^1$-minimal for $f$, we obtain rigidity conditions for $f'$.

**Lemma 4.1.** *If $K$ is a quasi regular interval Cantor set, $\mu_n < \frac{2\pi}{2^{n-1}}$, for all integer $n > 1$.*

**Proof.** We are going to prove that if

$$\mu_n < \frac{2\pi}{2^{n-1}}, \quad \mu_{n+1} < \frac{2\pi}{2^n}.$$

As a result of this, $\mu_1 < 2\pi$ and we would have proved the lemma. From the construction of $K$ we know that there exist integers $j_1$, $j_2$ and $j_3$ such that $\Delta_{nj_1} < \frac{2\pi}{2^{n-1}}$ and such that $\Delta_{n+1,j_2}$ and $\Delta_{n+1,j_3}$ are contained in $\Delta_{nj_1}$. Therefore

$$\min\left\{|\Delta_{n+1,j_2}|, |\Delta_{n+1,j_3}|\right\} \leq \frac{|\Delta_{n,j_1}|}{2} < \frac{2\pi}{2^n},$$

and from here the lemma follows. □

**Lemma 4.2.** *If $K$ is a quasi regular interval Cantor set then $\lambda_n/\lambda_{n+1} \not\to 1$.*

**Proof.** Let $\{l_i\}$ be the sequence where $l_i$ is the length of the open intervals removed in the step $i$ of the construction of $K$. From the construction of $K$ we have that the open intervals removed in the step $n$ are contained in $K_{n-1}$ , so from the previous lemma we have that $l_n < 2\pi/2^{n-2}$ for $n > 2$. Then, for $n > 2$ we have

$$\#\left(\{\log \lambda_i\} \cap [-(n-2)\log 2 + \log 2\pi, 0]\right) < n. \tag{2}$$

Suppose that $\lambda_n/\lambda_{n+1} \to 1$. Then for all $\epsilon > 0$ there exists $n_0 > 0$ such that for all $n \in \mathbf{N}$

$$0 < \log \lambda_{n_0+n-i} - \log \lambda_{n_0+n+1-i} < \log(1+\varepsilon)$$

with $i = 0, \ldots, n$, so

$$0 > \log \lambda_{n_0+n} > \log \lambda_{n_0} - n\log(1+\varepsilon).$$

Then

$$\#(\{\log \lambda_i\} \cap [\log \lambda_{n_0} - n\log(1+\varepsilon), 0]) \geq n_0 + n. \tag{3}$$

Applying the inequalities (2) and (3) we have

$$\begin{aligned} &\#(\{\log \lambda_i\} \cap [-(n-2)\log 2 + \log 2\pi, 0]) < n < n_0 + n \\ \leq\ &\#(\{\log \lambda_i\} \cap [\log \lambda_{n_0} - n \log(1+\epsilon), 0]). \end{aligned}$$

Therefore

$$-(n-2)\log 2 + \log 2\pi \geq \log \lambda_{n_0} - n \log(1+\epsilon).$$

As this inequality is true for all $n \in \mathbf{N}$ and for all $\epsilon > 0$, taking $\epsilon$ such that $\log(1+\epsilon) < \log 2$ we get contradiction. □

**Lemma 4.3.** *If a quasi regular interval Cantor set $K$ is $C^1$-minimal for $f$, there exists $x \in K$ such that $f'(x) > 1$.*

**Proof.** From the previous lemma, we know that there exists $\epsilon_0 > 0$ and an increasing sequence of positive integers $\{n_j\}$ such that $\lambda_{n_j}/\lambda_{n_j+1} > 1 + \epsilon_0$, for all $n_j$. Let $I$ be a connected component of $K^c$. Then, the family $\{f^{-n}(I)\}$ with $i \in \mathbf{N}$ is a family of open intervals, pairwise disjoint, so $|f^{-n}(I)| \to 0$ when $n \to \infty$. Therefore, if $j$ is sufficiently large there exists $p(j) \in \mathbf{N}$ such that $|f^{-p(j)}(I)| \leq \lambda_{n_j+1}$ and $|f^{-p(j)+1}(I)| \geq \lambda_{n_j}$. Then, we have

$$\frac{|f^{-p(j)+1}(I)|}{|f^{-p(j)}(I)|} \geq \frac{\lambda_{n_j}}{\lambda_{n_j+1}} > 1 + \epsilon_0. \tag{4}$$

Applying the Mean Value Theorem, there exists a point $\theta_{p(j)} \in f^{-p(j)}(I)$ such that

$$|f^{-p(j)+1}(I)| = f'(\theta_{p(j)})|f^{-p(j)}(I)|$$

so

$$\frac{|f^{-p(j)+1}(I)|}{|f^{-p(j)}(I)|} = f'(\theta_{p(j)}). \tag{5}$$

From (4) and (5) we have

$$f'(\theta_p) > 1 + \varepsilon_0. \tag{6}$$

If $x$ is an accumulation point of the set $\{f^{-p(j)}(I)\}$, it is an accumulation point of the set $\{\theta_{p(j)}\}$ too and, as $f \in C^1$, we have that $f'(\theta_p) \to f'(x)$ when $j \to \infty$, so from (6) we obtain that $f'(x) > 1$. □

If $K$ is a quasi regular interval Cantor set and $y \in K$ we denote by $K_n^y$ the connected component of $K_n$ that contains $y$. The following remarks will be useful for the proofs of the next lemmas.

1. If $K$ is a quasi regular interval Cantor set, $C^1$-minimal for $f$, for all $\epsilon > 0$ there exists a positive integer $n(\epsilon)$ such that if $n > n(\epsilon)$ and $x_1, x_2$ belong to the same connected component of $K_n$,
$$\frac{1}{1+\varepsilon} < \frac{f'(x_1)}{f'(x_2)} < 1+\varepsilon.$$
2. For all positive integer $n$ and all point $x \in K$ there exists a positive number $\upsilon$ such that if $\lambda$ is an element of the spectrum of $K$, smaller than $\upsilon$, there exists a connected component of $K^c$, of length $\lambda$, contained in $K_n^{f(x)}$ such that its preimage is contained in $K_n^x$.

**Lemma 4.4.** *Let $K$ be a quasi regular interval Cantor set that is $C^1$-minimal for $f$ and $x$ any point in $K$. For any $\epsilon > 0$, any integer $n$ and any small enough $I$, connected component of $K^c$, there exists a connected component $I^*$ of $K^c$ such that*
$$\frac{(f'(x))^m}{1+\varepsilon} < \frac{|I^*|}{|I|} < (f'(x))^m(1+\varepsilon).$$

**Proof.** First we suppose that $m \geq 0$. We consider $\epsilon_1 > 0$ sufficiently small and $n = n(\epsilon_1)$ as in remark 1. Let $K_n$ be as in the construction of $K$. If $I$ is a short enough connected component of $K^c$, there exists another connected component $I_1$ of $K^c$, contained in $K_n^x$ such that its length is $|I|$. From the Mean Value Theorem we have that there exists $\theta \in I_1$ such that
$$|f(I_1)| = f'(\theta)|I_1| = f'(\theta)|I|.$$
As $\theta \in K_n^x$, by remark 1 we have
$$\frac{f'(x)}{1+\epsilon_1} < \frac{|f(I_1)|}{|I|} < f'(x)(1+\epsilon_1).$$
If $I$ is sufficiently small we can repeat this procedure with $f(I_1)$ instead of $I$. Then there exists $I_2$, connected component of $K^c$, such that
$$\frac{f'(x)}{1+\epsilon_1} < \frac{|f(I_2)|}{|f(I_1)|} < f'(x)(1+\epsilon_1).$$
By induction we conclude that there exist $I_3, \ldots, I_m$, connected components of $K^c$, such that
$$\frac{f'(x)}{1+\epsilon_1} < \frac{|f(I_{i+1})|}{|f(I_i)|} < f'(x)(1+\epsilon_1),$$

with $i = 1, \ldots, m-1$. So

$$\frac{(f'(x))^m}{(1+\epsilon_1)^m} < \frac{|f(I_m)|}{|I|} < (f'(x))^m(1+\epsilon_1)^m. \tag{7}$$

Given $\epsilon > 0$ we choose $\epsilon_1 > 0$ such that $(1+\epsilon_1)^m < 1+\epsilon$. Then, from (7) the lemma follows. In the case $m < 0$ we proceed as follows. If $I$ is a connected component of $K^c$, sufficiently small, there exists $I_1$, connected component of $K^c$ too, of length $|I|$, contained in $K_n^{f(x)}$ such that $f^{-1}(I_1)$ is contained in $K_n^x$. Therefore, there exists $\theta \in I_1$ such that

$$|f^{-1}(I_1)| = (f^{-1})'(\theta)|I_1| = \frac{|I_1|}{f'(f^{-1}(\theta))}.$$

As $f^{-1}(\theta) \in K_n^x$, from remark 1 we have

$$\frac{1}{(1+\epsilon_1)f'(x)} < \frac{|f^{-1}(I_1)|}{|I_1|} = \frac{1}{f'(f^{-1}(\theta))} < \frac{1+\epsilon_1}{f'(x)}.$$

So, as in the first case, we obtain the desired result. □

**Lemma 4.5.** *If the quasi regular interval Cantor set $K$ is $C^1$-minimal for $f$, then $f'$ takes finitely many values. Moreover, if the set of values of $f'$ restricted to $K$ is $\{a_1, \ldots, a_n\}$, then $\log a_i / \log a_j \in \mathbb{Q}$ ( $a_j \neq 1$).*

**Proof.** Let $\epsilon_0$ and $\{n_j\}$ be as in the proof of lemma 4.3. We need to prove that $A = \{f'(x) \colon x \in K\}$ is a finite set. Assume that $A$ is infinite. As $f'$ is continuous in $S^1$, the set $A$ has an accumulation point. We conclude that there exist $a, b \in K$, $a \neq b$, such that

$$\frac{1}{1+\epsilon_0} < \frac{f'(a)}{f'(b)} < 1. \tag{8}$$

Let $\epsilon_1$ be a positive number such that

$$1+\epsilon_1 < \min\left\{\sqrt{\frac{f'(b)}{f'(a)}}, \sqrt{(1+\epsilon_0)\frac{f'(a)}{f'(b)}}\right\}.$$

From remark 1 we have that there exists $n(\varepsilon_1)$ such that if $x_1$ and $x_2$ are in the same connected component of $K_{n(\epsilon_1)}$, then

$$\frac{1}{1+\epsilon_1} < \frac{f'(x_1)}{f'(x_2)} < 1+\epsilon_1. \tag{9}$$

Let $I_1$ be a connected component of $K^c$ contained in the connected component of $K_{n(\epsilon_1)}$ that contains the point $a$. From the construction of $K$ we have that $K^c_{n(\epsilon_1)}$ only contains a finite number of connected components of $K^c$. By the Mean Value Theorem, there exists $\theta_1 \in I_1$ such that

$$|f(I_1)| = |I_1| f'(\theta_1).$$

By (9), as $\theta_1$ and $a$ are in the same connected component of $K_{n(\epsilon_1)}$, we have

$$\frac{|I_1| f'(a)}{1+\epsilon_1} < |f(I_1)| < |I_1|(1+\epsilon_1) f'(a). \tag{10}$$

If $|I_1|$ is sufficiently small there exists $I_2$, connected component of $S^1 \setminus K$, of length $|f(I_1)|$, such that $f^{-1}(I_2)$ is in the connected component of $K_{n(\varepsilon_1)}$ that contains $b$ (remark 2). By the Mean Value Theorem there exists $\theta_2 \in I_2$ such that

$$|f^{-1}(I_2)| = |I_2|(f^{-1})'(\theta_2) = \frac{|I_2|}{f'(f^{-1}(\theta_2))}.$$

From the choice of $I_2$ we have that $f^{-1}(\theta_2)$ and $b$ are in the same connected component of $K_{n(\varepsilon_1)}$; so applying (9) we obtain

$$\frac{|f(I_1)|}{f'(b)} \frac{1}{1+\epsilon_1} \leq |f^{-1}(I_2)| \leq \frac{|f(I_1)|}{f'(b)}(1+\epsilon_1).$$

From this last inequality and (10) we have

$$\frac{|I_1|}{(1+\epsilon_1)^2} \frac{f'(a)}{f'(b)} \leq |f^{-1}(I_2)| \leq |I_1|(1+\epsilon_1)^2 \frac{f'(a)}{f'(b)},$$

and therefore, by the choice of $\epsilon_1$, we have

$$1 < \frac{|I_1|}{|f^{-1}(I_2)|} < 1+\epsilon_0.$$

Summarizing, we have proved that if $I$ is a connected component of $S^1 \setminus K$ with sufficiently small length, there exists another connected component $I^*$ of $K^c$ such that

$$1 < |I|/|I^*| < 1+\epsilon_0.$$

Taking $I$ of length $\lambda_{n_j}$ sufficiently small we have

$$1+\epsilon_0 > \frac{|I|}{|I^*|} \geq \frac{\lambda_{n_j}}{\lambda_{n_j+1}} > 1+\epsilon_0$$

and this is a contradiction. Then, $A$ is finite.

Now, assume that there exist $i$ and $j$ such that $\log a_i / \log a_j \notin \mathbb{Q}$. We are going to prove (as in the previous case) that if $I$ is a short enough connected component of $K^c$, there exists another connected component $I^*$ of $K^c$ such that

$$1 < |I|/|I^*| < 1 + \varepsilon_0$$

and we get a contradiction. As $\log a_i / \log a_j \notin \mathbb{Q}$ then for all $\epsilon_1 > 0$ there exist integers $m$ and $n$ such that

$$-\epsilon_1 < m \log a_i - n \log a_j < 0,$$

so there exist $x, y \in K$ such that

$$e^{-\epsilon_1} < (f'(x))^m (f'(y))^{-n} < 1. \tag{11}$$

From lemma 4.4 we have that given $\epsilon_2 > 0$ and $I$ a sufficiently small connected component of $K^c$, there exist $I^*$ and $I^{**}$ such that

$$\frac{(f'(x))^m}{1+\epsilon_2} < \frac{|I^{**}|}{|I|} < (f'(x))^m (1+\epsilon_2) \tag{12}$$

and

$$\frac{(f'(x))^{-n}}{1+\epsilon_2} < \frac{|I^{*}|}{|I^{**}|} < (f'(x))^{-n} (1+\epsilon_2) \tag{13}$$

By equations (11), (12) and (13) we have

$$\frac{(f'(x))^{-m}(f'(y))^n}{(1+\epsilon_2)^2} < \frac{|I|}{|I^*|} < \frac{(1+\epsilon_2)^2}{e^{-\epsilon_1}}. \tag{14}$$

We take $\epsilon_2$ such that

$$\frac{(f'(x))^{-m}(f'(y))^n}{(1+\epsilon_2)^2} > 1,$$

and $\epsilon_1$ such that

$$\frac{(1+\epsilon_2)^2}{e^{-\epsilon_1}} < 1 + \epsilon_0.$$

So, by (14) we have proved what we want. $\square$

## 5 Proof of the theorem 1

**Remark 1.** Let $R_\theta \colon S^1 \to S^1$ be the rotation of angle $\theta$ (irrational in $\pi$). Take $x \in S^1$ and $m$ a positive integer. There exist $n > m$ such that the set $A_n = \{R_\theta^i(x) \colon i = 0, \dots, n\}$ determines the intervals $T_1, \dots, T_p$ (with the same length) and $J_1, \dots, J_q$ (with the same length) such that $f(T_i) = T_{i+1}$ and $f(J_j) = J_{j+1}$. This can be easily deduced from basic facts about the combinatorics of rotations.

**Remark 2.** If $f: S^1 \to S^1$ is a continuous function and $R_\theta$ is the rotation of irrational angle $\theta$, for all point $x \in S^1$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{0\le i\le n} f(R_\theta^i(x)) = \int_{S^1} f\,dx.$$

This follows from the fact that $f$ is continuous in $S^1$ applying Birkhoff theorem (see [4]).

Now we prove theorem 1.

**Proof.** Assume, that there exists a quasi regular interval Cantor set $K$, $C^1$-minimal for $f$, and that $K^c$ has only one orbit of wandering intervals. Let $h: S^1 \to S^1$ be the semiconjugate such that $h\circ f = R_\theta \circ h$, with $R_\theta: S^1 \to S^1$ the rotation of angle $\theta$ (irrational in $\pi$). From lemma 4.5 we have that there exists a covering of $K$ formed by pairwise disjoints closed intervals $H_1, \dots, H_r$, such that $f'/H_i \bigcap K = a_i$. It is possible to choose the intervals $H_i$ so that each connected component of the complement of $\bigcup_{i=1}^r H_i$ is a connected component of $K^c$. If $L_1, \dots, L_r$ are the connected components of the complement of $\bigcup_{i=1}^r H_i$, then the image of each $L_i$ by $h$ is a point $y_i$. As $f$ has only one orbit of wandering intervals, then the points $y_i$ are in the same orbit in the rotation $R_\theta$. Let $A_m$, $T_1, \dots, T_p$, $J_1, \dots, J_q$ be as in remark 1 such that $\{y_1, \dots, y_r\} \subset A_m$. Now, we define

$$g: \bigcup_1^p T_i \cup \bigcup_1^q J_j \to \mathbb{R}$$

such that $g(x) = f'(h^{-1}(x))$ (note that $g$ is well defined even in the case that $h^{-1}(x)$ is an interval). By the choice of the intervals $T_i$ and $J_j$ we have that $g$ is constant in each of them. Even more, if $y$ is a point of $S^1$ such that $h(y)$ does not belong to $\bigcup_{j\in\mathbf{N}} R_\theta^{-j}(A_m)$ (preorbit of the end points of the intervals $T_i$ and $J_j$) then

$$F(y,n) = \sum_{i=0}^{n-1} \log(g(R_\theta^i(h(y)))).$$

**Claim.** $\displaystyle\int_{(\bigcup T_i)\cup(\bigcup J_j)} \log g\,dx = 0.$

Assume that $\int_{(\bigcup T_i)\cup(\bigcup J_j)} \log g\,dx \neq 0$. Supposing that

$$\int_{(\bigcup T_i)\cup(\bigcup J_j)} \log g\,dx > 0,$$

we have that there exists a continuous function $g_1 : S^1 \to S^1$ such that $g_1 < g$ and $\int_{S^1} \log g_1 dx > 0$. So, by remark 2 we have that given $x \in S^1$ and $k > 0$ there exists $n = n(x, k)$ such that

$$\sum_{i=0}^{n-1} \log \left( g_1(R_\theta^i(x)) \right) > k.$$

Therefore, if $x \in K$ and $h(x) \notin \bigcup_{j \in \mathbf{N}} R_\theta^{-j}(A_m)$ we have that for each $k > 0$ there exists a positive integer $n$ such that

$$F(x, n) = \sum_{i=0}^{n-1} \log \left( g(f^i(x)) \right) \geq \sum_{i=0}^{n-1} \log \left( g_1(R_\theta^i(h(x))) \right) > k. \tag{15}$$

As for each point $x \in K$ there exists a positive integer $s$ such that $h(f^s(x))$ does not belong to $\bigcup_{j \in \mathbf{N}} R_\theta^{-j}(A_m)$, taking $k$ sufficiently large and applying (15) for the point $h(f^s(x))$, we have that there exists a positive integer $n$ such that

$$F(x, n) > 0.$$

Therefore, the result obtained contradicts lemma 3.2. If

$$\int_{S^1} \log g \, dx < 0,$$

working in analogous form we have that for every $x \in K$ there exists a positive integer $n$ such that $F(x, n) < 0$. This result contradicts lemma 3.1. Then we have proved the claim. Now, we are going to prove that

$$\int_{\bigcup T_i} \log g \, dx = \int_{\bigcup J_j} \log g \, dx = 0. \tag{16}$$

We denote $a_i = g/T_i$ e $b_j = g/J_j$. Then

$$\begin{aligned} \int_{(\bigcup T_i) \cup (\bigcup J_j)} \log g \, dx &= \sum |T_i| \log a_i + \sum |J_j| \log b_j \\ &= |T_1| \sum \log a_i + |J_1| \sum \log b_j = 0. \end{aligned} \tag{17}$$

If $\sum \log a_i \neq 0$, from lemma 4.5 we have $\sum \log b_j / \sum \log a_i \in \mathbb{Q}$. So, by (17) we have that $|T_1|/|J_1| \in \mathbb{Q}$ and this is a contradiction because the end points of the intervals $T_i$ and $J_j$ share an orbit of the irrational rotation $R_\theta$. Then

$$\sum \log b_j = \sum \log a_i = 0.$$

Now, let $y \in K$ be such that $x = h(y) \in T_1$. From the construction of the intervals $T_i$ and $J_j$ we have that $R_\theta^{p+1}(x)$ belongs to $T_1$ or $J_1$. If $R_\theta^{p+1}(x)$ belongs to $T_1$, then $R_\theta^{2p+1}(x)$ belongs to $T_1$ or $J_1$. If $R_\theta^{p+1}(x)$ belongs to $J_1$, then $R_\theta^{p+q+1}(x)$ belongs to $T_1$ or $J_1$. Inductively, we have that there exists an increasing sequence $n_k$ such that $n_{k+1} - n_k$ only takes values $p$ and $q$ and $R_\theta^{n_k+1}(x)$ belongs to $T_1$ or $J_1$. Therefore, from (16) we have that $F(y, n_k) = 0$, for all $k$. Finally, given a positive integer $n$ there exists $k_0$ such that $n_{k_0} \leq n < n_{k_0+1}$ and therefore,

$$F(y,n) = F(y, n_{k_0}) + F(f^{n_{k_0}}(y), n - n_{k_0}) = F(f^{n_{k_0}}(y), n - n_{k_0}).$$

As $n - n_{k_0}$is bounded, $F(y, n)$ is also bounded and this contradicts lemma 3.3, so the proof is finished. □

## 6 Covering and levels

Note that if the quasi regular interval Cantor set $K$ is $C^1$-minimal for $f$, for each positive integer $n$ we have that if $I$ is a connected component of $K^c$, as small as necessary, $I$ and $f(I)$ are contained in $K_n$.

**Definition 6.1.** *The positive integer $s$ is the level of an interval $I \subset S^1$, if $I$ was removed from the construction of $K$ in step $s$ (we denote $s = \mathcal{L}(I)$).*

**Lemma 6.1.** *If $\{\mathcal{T}_{ij}\}$, with $j \in \mathbf{N}$ and $i = 1, \dots, n$, is a family of closed intervals contained in $S^1$ such that $\nu_j = \max\{|\mathcal{T}_{ij}|; i = 1, \dots, n\}$ has limit 0 when $j \to \infty$, there exist a positive integer $k$ and a finite set of pairwise disjoint intervals $\{\mathcal{J}_t\}$, contained in $S^1$, such that $\mathcal{A} = \bigcup \mathcal{J}_t \supset \bigcup_{i=1}^{n} \mathcal{T}_{ik}$ and every interval of $\mathcal{A}^c$ has a greater measure than the measure of $\mathcal{A}$.*

**Proof.** For the proof we will work by induction in $n$. If $n = 1$ it is immediate. Assume the property is true for $n \geq 1$. We are going to prove that the property is true for $n + 1$. For each $j \in \mathbf{N}$, we denote by $\mathcal{B}_j = \bigcup_{i=1}^{n+1} \mathcal{T}_{ij}$ and by $\mathcal{Y}_{sj}$ ($s = 1, \dots, n_j$, with $n_j \leq n + 1$) the connected components of the complement of $\mathcal{B}_j$. Consider two cases. First, suppose that $a_j = \min\{|\mathcal{Y}_{kj}|; k = 1, \dots, n_j\}$ does not have limit 0 when $j \to \infty$. Then, there exist $\epsilon > 0$ and an increasing sequence $\{j_t\}$ such that $a_{j_t}, > \epsilon$ for all $t$. By assumption we know that $\nu_j \to 0$ when $j \to \infty$, then there exists $r \in \mathbf{N}$ such that $\nu_{j_r} < \epsilon/(n + 1)$, so

$$|\mathcal{B}_{j_r}| \leq \sum_{i=1}^{n+1} |\mathcal{T}_{ij_r}| < (n+1)\frac{\epsilon}{n+1} = \epsilon.$$

As $a_{j_r} > \epsilon$, we have that every interval of the complement of $\mathcal{B}_{j_r}$ has length greater than $|\mathcal{B}_{j_r}|$. If we define the intervals $\mathcal{J}_t$ as the connected components of $\mathcal{B}_{j_r}$, we have proved the inductive step in this case. Now, we suppose that $a_j \to 0$ when $j \to \infty$. We denote by $\mathcal{Y}_j^*$ one of the connected components of the complement of $\mathcal{B}_j$ such that its length is $a_j$. We can suppose, without loss of generality, that $\mathcal{Y}_j^*$ is the interval $\mathrm{Arc}(\mathcal{T}_{1j}, \mathcal{T}_{2j}) \setminus (\mathcal{T}_{1j} \cup \mathcal{T}_{2j})$ (considering $j$ sufficiently large and reordering the intervals $\mathcal{T}_{ij}$ as necessary). Now we consider the family of intervals $\mathcal{T}_{ij}^*$ defined as follows. We take

$$\mathcal{T}_{1j}^* = \mathcal{T}_{1j} \cup \mathcal{Y}_j^* \cup \mathcal{T}_{2j}$$

and for $i = 2, \dots, n$

$$\mathcal{T}_{i,j}^* = \mathcal{T}_{i+1,j}.$$

Then by induction there exist a number $k$ and a family of intervals $\mathcal{J}_t$ that satisfy the lemma for the intervals $\mathcal{T}_{ij}^*$. The number $k$ and the family of intervals $\mathcal{J}_t$ obtained for the family of intervals $\mathcal{T}_{ij}^*$ satisfy the thesis of the lemma for the family of intervals $\mathcal{T}_{ij}$. The proof is therefore finished. □

If the point $x$ is the end point of a connected component of $K^c$ of level $s_0$, for each integer $s > s_0$ we denote by $I_s$ the connected component of $K^c$ closest to $x$. Note that if $s$ is sufficiently large then $I_s$ is unique.

**Definition 6.2.** *Let $x$ be the end point of a connected component of $K^c$ of level $s_0$. For each integer $s > s_0$ we define*

$$\varphi_x(s) = s - \mathcal{L}(f(I_s)).$$

**Lemma 6.2.** *If the quasi regular interval Cantor set $K$, has regularity different from 0, is $C^1$-minimal for $f$ and $x$ is the end point of a connected component of $K^c$ of level $s_0$, then $\varphi_x$ is upper bounded.*

**Proof.** As the regularity of $K$ is not 0, there exists a procedure that determines $K$ such that $\delta = \inf\{\mu_m/\nu_m : m \in \mathbf{N}\} > 0$. We suppose that for each $k > 0$ there exists a positive integer $s_k$, such that $\varphi(s_k) = s_k - \mathcal{L}(f(I_{s_k})) > k$. We denote $r_k = \mathcal{L}(f(I_{s_k}))$. By the construction of $K$ we have that $\mu_{s_k} \le 2^{-k}\mu_{r_k}$. If $I_{s_k} = (a_k, b_k)$, with $a_k$ between $x$ and $b_k$, we have that there exists $\theta_k \in [x, a_k]$ such that $d(f(x), f(a_k)) = f'(\theta_k)d(x, a_k)$. So

$$\begin{aligned} d(f(x), f(a_k)) &\le f'(\theta_k)\nu_{s_k} \le f'(\theta_k)\frac{\mu_{s_k}}{\delta} \le \frac{f'(\theta_k)}{\delta}2^{-k}\mu_{r_k} \\ &\le \frac{f'(\theta_k)}{\delta}2^{-k}d(f(x), f(a_k)). \end{aligned}$$

From here it follows that $f'(\theta_k) \to \infty$ when $k \to +\infty$, and this is a contradiction. $\square$

## 7 Proof of the theorem 2

**Proof.** Assume that there exists $\epsilon > 0$ and a diffeomorphism $f$, of class $C^{1+\epsilon}$ such that $K$ is minimal for $f$.

First, we will show that it is possible to choose a set $A = \{\lambda^1, \lambda^2, \dots, \lambda^r\} \subset E_K$, with $r \le q$, and $\nu > 0$ such that the set $\{\lambda^i \nu^m : m \in N^+, i \in \{1, \dots, r\}\}$ is close as necessary from $E_K$. Then, applying lemma 6.1, we will show that $E_K$ satisfies the Mc Duff's condition (see [2]), getting a contradiction.

By lemmas 4.5 and 4.3 we have that there exist a positive integer $n_0$ and an end point $x$ of a connected component of $K^c$, such that:

1. the restriction of $f'$ to $K$ is constant in each connected component of $K_{n_0}$.
2. $f'(x) = \nu > 1$.
3. by the continuity of $f'$ we have that if $n_0$ is sufficiently large, for every connected component $I$ of $K^c$, contained in $K^x_{n_0}$ (connected component of $K_{n_0}$ that contains $x$), we have that $|f(I)| > |I|$, so $f(I)$ and $I$ have different level.

Given a positive integer $n$ we denote by $I_n = (a_n, b_n)$ the interval of level $n + n_0$ contained in $K^x_{n_0}$ nearest to $x$. We fix $m$ and for each integer $n > m$ we consider the family of intervals $\{I^j_n\}_{j\in\mathbf{N}}$ with the following properties:

1. the interval $I^0_n = I_n$.
2. the interval $I^j_n$ is the connected component of $K^c$ with the same level that the level of $f(I^{j-1}_n)$ closest to $x$ (in the proof we are going to work only with a finite number of the $I^j_n$).

Let $q = \max\{\mathcal{L}(I) - \mathcal{L}(f(I))\}$ be the integer given by lemma 6.2. We define $p_n = \min\{j : \mathcal{L}(I^j_n) \le \mathcal{L}(I_{m+q-1}) = n_0 + m + q - 1\}$. We need to prove that the set $D_n = \{j : \mathcal{L}(I^j_n) \le \mathcal{L}(I_{m+q-1})\}$ is not empty. Assume that $D_n$ is empty. Then, for all $j$ we have that $|I^{j-1}_n| < |I^j_n|$ and that $I^j_n$ is between $x$ and $I_{m+q-1}$ and this is a contradiction. So $D_n$ is not empty. Now, we consider the finite family $\{I^j_n\}$ with $j = 1, \dots, p_n$. By lemma 6.2 follows that $n_0 + m + q > \mathcal{L}(I^{p_n}_n) \ge n_0 + m$. So the set $A = \{I^{p_n}_n\}$ has $r$ elements with $r \le q$. By the

Mean Value Theorem we know that there exist $\theta_j \in I_n^j$, $j = 0, \dots, p_n - 1$ such that $|f(I_n^j)| = f'(\theta_j)|I_n^j| = |I_n^{j+1}|$. Therefore,

$$|I_n| = \frac{|I_n^{p_n}|}{f'(\theta_0)\dots f'(\theta_{p_n-1})}. \tag{18}$$

We denote $r_j = \mathcal{L}(I_n^j)$, with $j = 0, \dots, p_n - 1$. Note that as $i \neq j$, $r_i \neq r_j$ and $r_j \geq m + n_0$, for every $j$. For every $j$, we have that $\theta_j$ and $x$ are in the same connected component of $K_{r_j-1}$, so from lemma 4.1 and if $r_j$ is sufficiently large we have

$$|\theta_j - x| < \frac{2}{\delta 2^{r_j-2}}.$$

Therefore, as $f$ is the class $C^{1+\varepsilon}$ (this is $|f'(x) - f'(y)| \leq \widetilde{k}|x-y|^{\epsilon}$) we have

$$1 - \frac{k}{\nu}\frac{1}{2^{(r_j-2)\epsilon}} < \frac{f'(\theta_j)}{\nu} < 1 + \frac{k}{\nu}\frac{1}{2^{(r_j-2)\epsilon}}, \tag{19}$$

where $k = \widetilde{k}(\frac{2}{\delta})^{\epsilon}$. From (18) e (19) we have

$$\frac{|I_n^{p_n}|}{\nu^{p_n}}\prod_{i=0}^{p_n-1}\left\{1 + \frac{k}{\nu}\left(\frac{1}{2^{r_i-2}}\right)^{\epsilon}\right\}^{-1} \leq |I_n| \leq \frac{|I_n^{p_n}|}{\nu^{p_n}}\prod_{i=0}^{p_n-1}\left\{1 - \frac{k}{\nu}\left(\frac{1}{2^{r_i-2}}\right)^{\epsilon}\right\}^{-1}.$$

Therefore,

$$\log|I_n^{p_n}| - p_n\log\nu - P_2(m) \leq \log|I_n| \leq \log|I_n^{p_n}| - p_n\log\nu - P_1(m) \tag{20}$$

where

$$P_1(m) = \sum_{j=m+n_0}^{\infty}\log\left\{1 - \frac{k}{\nu}\left(\frac{1}{2^{j-2}}\right)^{\epsilon}\right\} \leq \log\prod_{i=0}^{p_n-1}\left\{1 - \frac{k}{\nu}\left(\frac{1}{2^{r_i-2}}\right)^{\epsilon}\right\} < 0$$

and

$$P_2(m) = \sum_{j=m+n_0}^{\infty}\log\left\{1 + \frac{k}{\nu}\left(\frac{1}{2^{j-2}}\right)^{\epsilon}\right\} \geq \log\prod_{i=0}^{p_n-1}\left\{1 + \frac{k}{\nu}\left(\frac{1}{2^{r_i-2}}\right)^{\epsilon}\right\} > 0.$$

For each $m$ we define the set $A_m = \{\log|I_r|; r > m\}$ (the difference between this set and the set $\{\log\lambda_i\}$ is a finite number of elements). Now, we consider the quotient $A_m/\log\nu\cdot\mathbb{R} = \mathcal{A}_m$ as a subset of the affine manifold $S = \mathbb{R}/\log\nu\cdot\mathbb{R}$ that is isomorphic to $S^1$. From the inequality (20) we have that for each $m$ there exists a finite number of closed intervals $\mathcal{T}_{mj}$, $j = 1, \dots, q$, contained in $S$ such that $\bigcup_{j=1}^{q} \mathcal{T}_{mj} \supset \mathcal{A}_m$ and $a_m = \max\{|\mathcal{T}_{mj}|;\ j = 1, \dots, q\} = P_2(m) - P_1(m)$. By the definitions of $P_1(m)$ and $P_2(m)$, $a_m$ has limit 0 when $m \to \infty$. By lemma 6.1 we know that there exist $m_0$ and a family of intervals $\mathcal{J}_k$ contained in $S$, with $k = 1, \dots, h$, such that

$$\mathcal{A}_{m_0} \subset \bigcup_{j=1}^{q} \mathcal{T}_{m_0 j} \subset \bigcup \mathcal{J}_k = \mathcal{M}$$

and every connected component of the complement of $\mathcal{M}$ has length greater than $|\mathcal{M}|$. If we consider the lifting of the previous sets we have that there exist a number $\delta > 0$ and a family of intervals $[\alpha_s, \beta_s]$, with $\alpha_s \leq \beta_s$ e $\beta_{s+1} < \alpha_s$, $s = 1, \dots, \infty$ (they are the lifting of the intervals $\mathcal{J}_t$) such that $A_{m_0} \subset \bigcup_{s=1}^{\infty} [\alpha_s, \beta_s]$ and $\alpha_s - \beta_{s+1} < \beta_s - \alpha_s + \delta$. It is easy to see that this condition implies the Mc Duff condition and this is a contradiction (see Proposition 4.2 in [2]). $\square$

## 8 Proof of the theorems 3 and 4

We will start by proving certain lemmas that will be useful in the proofs of theorems 3 and 4. If $I$ and $J$ are sets contained in $S^1 \setminus K$, we denote by $\mathrm{Arc}(I, J)$ the smaller arch that contains $I$ and $J$.

**Lemma 8.1.** *Let $K$ be a regular interval Cantor set and let $I_1$, $I_2$, $I_3$ and $I_4$ be connected components of $S^1 \setminus K$, pairwise disjoint, removed in steps $n_1, n_2, n_3$ and $n_4$ of the construction of $K$, respectively. If $n_4 \geq \max\{n_1, n_2, n_3\}$ and $\mathrm{Arc}(I_3, I_4) \setminus (I_3 \cup I_4)$ is a connected component of $K_{n_4}$, there exists a positive integer $m$ such that $|K \cap \mathrm{Arc}(I_1, I_2)| = m|K \cap \mathrm{Arc}(I_3, I_4)|$.*

**Proof.** From the construction of $K$, we know that $I_1, I_2, I_3, I_4 \subset S^1 \setminus K_{n_4}$, so $\mathrm{Arc}(I_1, I_2) \cap K_{n_4}$ is a union of $m$ connected components of $K_{n_4}$, that we denote by $K_{n_4}^1, \dots, K_{n_4}^m$. Then

$$\mathrm{Arc}(I_1, I_2) \cap K = (\mathrm{Arc}(I_1, I_2) \cap K_{n_4}) \cap K = \left( \bigcup_{i=1}^{m} K_{n_4}^i \right) \cap K.$$

Therefore, $|\mathrm{Arc}(I_1, I_2) \cap K| = \sum_{i=1}^{m} |K_{n_4}^i \cap K|$. So, by the construction of $K$, we have

$$|\mathrm{Arc}(I_1, I_2) \cap K| = m|K_{n_4}^1 \cap K|. \tag{21}$$

As $\operatorname{Arc}(I_3, I_4) \setminus (I_3 \cup I_4)$ is a connected component of $K_{n_4}$ then

$$|K^1_{n_4} \cap K| = |(\operatorname{Arc}(I_3, I_4) \setminus (I_3 \cup I_4)) \cap K| = |\operatorname{Arc}(I_3, I_4) \cap K|. \tag{22}$$

Then from (21) e (22) we have

$$|K \cap \operatorname{Arc}(I_1, I_2)| = m|K \cap \operatorname{Arc}(I_3, I_4)|. \qquad \square$$

**Lemma 8.2.** *If the regular interval Cantor set $K$, of positive measure, is $C^1$-minimal for $f$ and $f'(x) > 1$ for $x \in K$, then $f'(x)$ is a positive integer.*

**Proof.** Let $\epsilon_0$, $\{n_j\}$ and $\{\lambda_{n_j}\}$ be as in the proof of lemma 3.2, and we consider $\epsilon_1 = \min\{\epsilon_0, f'(x) - 1\}$. By lemma 4.5 and the construction of $K$ we know that there exists a positive integer $n$ such that $f'$ is constant in the intersection of $K$ with each connected component of $K_n$ and if $n$ is sufficiently large, by the continuity of $f'$ we have

$$\frac{1}{1+\epsilon_1} < \frac{f'(x_1)}{f'(x_2)} < 1 + \epsilon_1$$

with $x_1$ and $x_2$ in the same connected component of $K_n$. Without loss of generality, we can suppose that $x$ is an end point of a connected component $I$ of $K^c$ such that $I$ and $f(I)$ are contained in $S^1 \setminus K_n$. We consider $j_0$ such that $\lambda_{n_{j_0}}$ is smaller than the length of some connected components of $K^c$ contained in $K_n$. For each $j > j_0$ we consider $I_j$ as the connected component $K^c$ contained in $K_n^x$ (connected component of $K_n$ that contains $x$) nearest to $x$ and $|I_j| \geq \lambda_{n_j}$. Then, we have that $|I_j| \to 0$ and $d(x, I_j) \to 0$ when $j \to \infty$. This implies that there exists a positive integer $j_1$ such that if $j \geq j_1$ then $f(I_j)$ is contained in $K_n^{f(x)}$. By the choice of $\epsilon_1$ we have that

$$d(f(x), f(I_j)) > \frac{f'(x)}{1+\epsilon_1} d(x, I_j) \geq d(x, I_j). \tag{23}$$

Now, we will prove that if $j \geq j_1$ there does not exist another connected component of $K^c$ with length $|f(I_j)|$, contained in $K_n^{f(x)}$ and within $f(x)$ and $f(I_j)$. We suppose that there exists $I^*$ in the previous conditions. Then $f^{-1}(I^*)$ is between $x$ and $I_j$. By the Mean Value Theorem we know that there exists $\theta^* \in f^{-1}(I^*)$ and $\theta_j \in I_j$ such that

$$|f^{-1}(I^*)| = \frac{|I^*|}{f'(\theta^*)} \quad \text{and} \quad |f(I_j)| = f'(\theta_j)|I_j|,$$

so $|f^{-1}(I^*)| = \dfrac{f'(\theta_j)}{f'(\theta^*)}|I_j|$. As $\theta^*$ and $\theta_j$ are in the same connected component of $K_n$, we have

$$\frac{|I_j|}{1+\epsilon_1} \;<\; |f^{-1}(I^*)| \;<\; |I_j|(1+\epsilon_1)$$

so

$$|f^{-1}(I^*)| \;>\; \frac{|I_j|}{1+\epsilon_1} \;>\; \frac{|I_j|}{1+\epsilon_0} \;\geq\; \frac{\lambda_{n_j}}{1+\epsilon_0} \;>\; \lambda_{n_j+1}.$$

From here we conclude that $|f^{-1}(I^*)| \geq \lambda_{n_j}$ and this contradicts the definition of $I_j$. Moreover, utilizing (23) we have that if $f(I_j)$ was removed in the step $n_1$ and $I_j$ was removed in the step $n_2$, $n < n_1 < n_2$. This observation allows us to apply lemma 8.1, so there exists $p \in \mathbf{N}$ such that

$$|K \cap \mathrm{Arc}(f(x), f(I_j))| = p|K \cap \mathrm{Arc}(x, I_j)|. \tag{24}$$

As $f'$ restrict to $K \cap \mathrm{Arc}(x, I_j)$ is constant, then

$$|f(K \cap \mathrm{Arc}(x, I_j))| = f'(x)|K \cap \mathrm{Arc}(x, I_j)| = |K \cap \mathrm{Arc}(f(x), f(I_j)|. \tag{25}$$

Therefore, from (24) e (25) and utilizing that $|K| > 0$ we have that $1 < f'(x) = p \in \mathbf{N}$ and this concludes the proof. □

Now we prove theorem 3.

**Proof.** We suppose, that $K$ is $C^1$-minimal for $f$ and $\{m_i\}$ is not bounded. By lemmas 4.3 and 8.2 we know that there exists an end point of a wandering interval $I$, that we call $x$, such that $f'(x) = p \in \mathbf{N}$ with $p > 1$. Therefore, by the uniform continuity of $f'$ and by lemma 4.1 we know that there exists $n_0 \in \mathbf{N}$ such that $f'/(K \cap K^x_{n_0}) = p$, where $K^x_{n_0}$ is the connected component of $K_{n_0}$ that contains $x$. As $\{m_i\}$ is not bounded, there exists $i_0$ sufficiently large such that $m_{i_0} > p+2$. Let $J_{i_0}$ be the interval of level $i_0$ closest to $x$ and $K^x_{i_0} = [x, y_{i_0}]$ (connected component of $K_{i_0}$ that contains $x$). As $f'$ restricted to $K \cap K^x_{n_0}$ is $p$, then

$$|f(K \cap K^x_{i_0})| = |K \cap [f(x), f(y_{i_0})]| = p|K \cap K^x_{i_0}|.$$

As $K$ has positive measure we have that the interval $[f(x), f(y_{i_0})]$ contains exactly $p$ connected components of $K_{i_0}$. As $f(x)$ is an end point of $f(I)$ (its level is greater than $i_0$, if $i_0$ is sufficiently large) and in step $i_0$ we removed more than $p+2$ intervals, the level of $f(J_{i_0})$ is $i_0$. Therefore $|J_{i_0}| = |f(J_{i_0})|$. Besides, we have that $J_{i_0} \subset K_{i_0-1}$ and $|K_{i_0-1}| \to 0$ when $i_0 \to \infty$. But then, by the continuity of $f'$, we know that if $i_0$ is sufficiently large $|J_{i_0}| < |f(J_{i_0})|$, and this is a contradiction. □

The following lemmas will be useful in proving theorem 4.

**Lemma 8.3.** *If the regular interval Cantor set $K$, of positive measure, is $C^1$-minimal for $f$, and there exists $x \in K$ and a positive integer $p$ ($p > 1$) such that $f'(x) = p$, then $p$ is multiple of $m_i + 1$ for large enough $i$.*

**Proof.** From lemma 4.5 we can suppose that $x$ is an end point of a connected component of $K^c$. We denote by $I_i = (a_i, b_i)$ the connected component of $K^c$ of level $i$ nearest to $x$ (if $i$ is large enough, $I_i$ is determined). Then, $f([x, a_i])$ contains exactly $p$ connected components of $K_i$, so the level of $f(I_i)$ is less than or equal to $i$. If $i$ is sufficiently large we have that $|f(I_i)| > |I_i|$, so the level of $f(I_i)$ is less than $i$. Therefore, the quantity of connected components of $K_i$ that contains $f([x, a_i])$ is multiple of $m_i + 1$. □

**Lemma 8.4.** *If $K$ is a regular interval Cantor set of positive measure, then $\frac{l_n}{\sigma_n} \to 0$ when $n \to \infty$, where $\sigma_n$ is the length of the connected components of $K_n$ and $l_n$ is the length of the open intervals removed in step $n$ of the construction of $K$.*

**Proof.** By the construction of $K$ we have that $|K| = \lim_{n\to\infty} \theta_1 \ldots \theta_n > 0$, so $\theta_n \to 1$. If $x$ is an end point of some open interval that was removed in step $j$, then for all $n > j + 1$ we have

$$\theta_n = \frac{|K_n|}{|K_{n-1}|} = \frac{|K_n^x|(m_n + 1)}{|K_{n-1}^x|} = \frac{|K_n^x|(m_n + 1)}{|K_n^x|(m_n + 1) + m_n l_n},$$

so $\frac{l_n}{|K_n^x|} \to 0$ when $n \to +\infty$. □

Now we prove theorem 4.

**Proof.** We suppose by contradiction that $K$ is $C^1$-minimal for $f$. Let $x$, $I$, $p$ and $n_0$ be as in the proof of theorem 3. For each $i > n_0$, we denote by $J_i = (y_i, z_i)$ the wandering interval of level $i$ closest to $f(x)$. By hypothesis, there exists a positive integer $n_0$ such that if $n \geq n_0$, $t_{n+1} - t_n > 3p$.

**Claim 1:** For all $i > t_{n_0}$, if $f^{-1}(J_i)$ is the interval of level $j$ closest to $x$ then $f^{-1}(J_j)$ is not the interval of level $k = \mathcal{L}(f^{-1}(J_j))$ nearest to $x$. We suppose by contradiction that $f^{-1}(J_j)$ is not in the desired conditions. Therefore $[x, f^{-1}(y_i)]$ is a connected component of $K_j$ and $[x, f^{-1}(y_j)]$ is a connected component of $K_k$. Then $(m_{i+1} + 1) \ldots (m_j + 1) = p$ and $(m_{j+1} + 1) \ldots (m_k + 1) = p$. Utilizing lemma 8.2 and that $q$ is a prime number we have that there exist less than two elements of the set $\{(m_{i+1} + 1), \ldots, (m_j + 1), \ldots, (m_k + 1)\}$ that are multiple of $q$. As this set doest not have more than $2p$ elements, if $i$ is large enough we have a contradiction. Then we have demonstrated claim 1.

**Claim 2:** If $i$ is large enough there exists $k > i$ such that

$$\frac{|J_k|}{|K_k^{f(x)}|} > \frac{3}{2}\frac{|J_i|}{|K_i^{f(x)}|}.$$

By the Mean Value Theorem, for all $i$, there exist $\theta_1$ and $\theta_2$ (they depend on $i$) contained in $[x, f^{-1}(z_i)]$ such that

$$|J_i| = |f^{-1}(J_i)|f'(\theta_1) \quad \text{and} \quad |(f(x), y_i)| = |(x, f^{-1}(y_i))|f'(\theta_2).$$

Then

$$\frac{|J_i|}{|K_i^{f(x)}|} = \frac{|J_i|}{|(f(x), y_i)|} = \frac{f'(\theta_1)}{f'(\theta_2)}\frac{|f^{-1}(J_i)|}{|(x, f^{-1}(y_i))|} \to \frac{|f^{-1}(J_i)|}{|(x, f^{-1}(y_i))|}, \tag{26}$$

when $i \to \infty$. We have two possibilities.

1. If $f^{-1}(J_i)$ is the interval closest to $x$ of level $j = \mathcal{L}(f^{-1}(J_i))$, from claim 1, we have that $f^{-1}(J_j)$ is not the interval of level $k = \mathcal{L}(f^{-1}(J_j))$ closest to $x$, therefore $|(x, f^{-1}(y_j))| > 2.|K_k^x|$. Then, by (26),

$$\frac{|J_i|}{|K_i^{f(x)}|} \to \frac{|J_j|}{|K_j^{f(x)}|} \to \frac{|J_k|}{|(x, f^{-1}(y_j))|} < \frac{|J_k|}{2|K_k^{f(x)}|},$$

   when $i \to \infty$. So, claim 2 follows.

2. If $f^{-1}(J_i)$ is not the interval closest to $x$ of level $k = \mathcal{L}(f^{-1}(J_i))$, then $|(x, f^{-1}(y_i))| > 2.|K_k^x|$. So the proof follows in analogous form to the previous item.

From claim 2 we have that $\dfrac{|J_n|}{|K_n^{f(x)}|} \not\to 0$ when $n \to \infty$ and this contradicts lemma 8.4. $\square$

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