BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 40(1), 77-84

A characterization of isometries on an open convex set, II*

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Abstract. Let E^n be an n-dimensional Euclidean space with $n \geq 2$. In this paper, we generalize a classical theorem of Beckman and Quarles by proving that if a mapping, from an open convex subset C_0 of E^n into E^n, preserves a distance ρ, then the restriction of f to an open convex subset C_∞ of C_0 is an isometry.

Keywords: Aleksandrov problem, isometry, distance preserving mapping, restricted domain.

Mathematical subject classification: Primary: 51K05; Secondary: 51F20, 51M25.

1 Introduction

Let X and Y be normed spaces. A mapping $f : X \to Y$ is called an isometry (or a congruence) if f satisfies

$$\|f(x) - f(y)\| = \|x - y\|$$

for all $x, y \in X$. A distance $\rho > 0$ is said to be contractive (or non-expanding) by $f : X \to Y$ if $\|x - y\| = \rho$ always implies $\|f(x) - f(y)\| \leq \rho$. Similarly, a distance ρ is said to be extensive (or non-shrinking) by f if the inequality $\|f(x) - f(y)\| \geq \rho$ is true for all $x, y \in X$ with $\|x - y\| = \rho$. We say that ρ is conservative (or preserved) by f if ρ is contractive and extensive by f simultaneously.

If f is an isometry, then every distance $\rho > 0$ is conservative by f, and conversely. At this point, we can raise a question:

Is a mapping that preserves certain distances an isometry?

Received 22 January 2007.

*This work was supported by 2007 Hongik University Research Fund.

In 1970, A.D. Aleksandrov [1] had raised a question whether a mapping $f : X \to X$ preserving a distance $\rho > 0$ is an isometry, which is now known to us as the Aleksandrov problem. Without loss of generality, we may assume $\rho = 1$ when X is a normed space (see [16]).

Indeed, earlier than Aleksandrov, F.S. Beckman and D.A. Quarles [2] solved the Aleksandrov problem for finite-dimensional real Euclidean spaces $X = E^n$:

> *If a mapping* $f : E^n \to E^n$ $(2 \le n < \infty)$ *preserves distance* 1, *then* f *is a linear isometry up to translation.*

For $n = 1$, they suggested the mapping $f : E^1 \to E^1$ defined by

$$f(x) = \begin{cases} x + 1 & \text{for } x \in \mathbb{Z}, \\ x & \text{otherwise} \end{cases}$$

as an example for a non-isometric mapping that preserves distance 1. For $X = E^\infty$, Beckman and Quarles also presented an example for a unit distance preserving mapping that is not an isometry (*cf.* [13]).

We may find a number of papers on a variety of subjects in the Aleksandrov problem (see [3, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and also the references cited therein).

In [10], the author proved the theorem of Benz [3] when the relevant domains are open convex sets. In this connection, we will prove the theorem of Beckman and Quarles when the relevant domains are open convex sets.

2 Preliminaries

Throughout this section, let E^n denote an n-dimensional Euclidean space, where $n \ge 2$ is a fixed integer, and assume that j is a fixed integer satisfying $0 \le j \le n$. Furthermore, suppose there exist unit vectors $w_1, \ldots, w_j \in E^n$ and a subspace E_s of E^n such that $E^n = E_s \oplus Sp(w_1) \oplus \cdots \oplus Sp(w_j)$ and $E_s, Sp(w_1), \ldots, Sp(w_j)$ are pairwise orthogonal, where $Sp(w_i)$ is the subspace of E^n which is spanned by w_i.

Let us define a sequence (s_k) by

$$s_0 = 0, \quad s_1 = \rho, \quad s_2 = \left[1 + \sqrt{\frac{2(n+1)}{n}}\right]\rho,$$

$$s_k = \left[n + 2 + \sqrt{\frac{2(n+1)}{n}} + \sum_{i=4}^{k} (n+1)^{5-i}\right]\rho$$

for $k \in \{3, 4, 5, \ldots\}$, where we assume that ρ is a fixed positive constant throughout this paper. We denote by s_∞ the limit point of s_k for convenience, i.e.,

$$s_\infty = \lim_{k\to\infty} s_k = \left[n + 2 + \sqrt{\frac{2(n+1)}{n}} + \frac{(n+1)^2}{n} \right] \rho.$$

By C_k we denote an open convex subset (cylinder) of E^n defined by

$$C_k = \big\{ x + \lambda_1 w_1 + \cdots + \lambda_j w_j \,:\, x \in E_s; \\ c_i + s_k < \lambda_i < d_i - s_k \text{ for } i = 1, \ldots, j \big\}$$

for every $k \in \{0, 1, 2, \ldots\} \cup \{\infty\}$, where $c_i, d_i \in \mathbb{R} \cup \{\pm\infty\}$ are constants with $d_i - c_i > 2s_\infty$ for all $i \in \{1, \ldots, j\}$. We then remark that $C_\infty \subset \cdots \subset C_{k+1} \subset C_k \subset \cdots \subset C_0 \subset E^n$.

Given a constant $\beta > 0$, a set of n distinct points of C_k which are pairwise of distance β will be called a β-set in C_k. Assume that α is a positive real number satisfying

$$\gamma(\alpha, \beta) = 4\alpha^2 - 2\beta^2 \left(1 - \frac{1}{n}\right) > 0$$

and that P is a β-set in C_k. The α-associated points of P are the uniquely determined two distinct points of E^n, which have distance α from each point of P, and the distance between α-associated points is $\sqrt{\gamma(\alpha, \beta)}$ (*cf.* [4]). We will refer to this fact many times in the proof of Lemma 2.

In view of the definition of C_k and 2), 3) in Section 2 of [4] and referring to Fig. 1 below, we can easily prove the following lemma. It only suffices to remark that $\sqrt{\gamma(\rho, \rho)} = \sqrt{2(n+1)/n}\,\rho$.

Lemma 1.

(a) *Let P be a ρ-set in C_1. There exist exactly two distinct ρ-associated points of P, namely x and y, which are in C_0 and $\|x - y\| = \sqrt{2(n+1)/n}\,\rho$.*

(b) *If $x, y \in C_1$ are given with $\|x - y\| = \sqrt{2(n+1)/n}\,\rho$, then there exists a ρ-set P in C_0 such that x and y are the ρ-associated points of P.*

We will stepwise prove that $f|_{C_1}$ resp. $f|_{C_2}$ preserves some special distance, ε_1 resp. ε_2 in the following lemma. This lemma is indispensable for the proof of our main theorem.

Lemma 2. *Assume that a mapping $f : C_0 \to E^n$ preserves the distance ρ.*

(a) *If x and y are points of C_1 with $\|x - y\| = \varepsilon_1 = \sqrt{\gamma(\rho, \rho)} = \sqrt{2(n+1)/n}\,\rho$, then $\|f(x) - f(y)\| = \varepsilon_1$;*

(b) *If x and y are points of C_2 satisfying $\|x - y\| = \varepsilon_2 = \sqrt{\gamma(\varepsilon_1, \varepsilon_1)} = (n+1)(2\rho/n)$, then $\|f(x) - f(y)\| = \varepsilon_2$;*

(c) *If x and y are points of C_2 with $\|x - y\| = \sqrt{\gamma(\rho, \varepsilon_1)} = 2\rho/n$, it then holds that $\|f(x) - f(y)\| \le 2\rho/n$.*

Proof.

(a) Assume that x and y are points of C_1 with $\|x - y\| = \varepsilon_1$. According to Lemma 1(b), there exists a ρ-set P in C_0 such that x and y are the ρ-associated points of P (see also Fig. 1 below). Since f preserves ρ, $P' = f(P)$ is also a ρ-set in E^n. In view of 2) of [4], there are exactly two distinct ρ-associated points of P'. Let us denote these ρ-associated points by x' and y' and we note that $\|x' - y'\| = \varepsilon_1$ (see Lemma 1(a)). Then, since $f(x)$ resp. $f(y)$ is separated by a distance ρ from each point of P' and there exist only two distinct ρ-associated points of P', we get $\{f(x), f(y)\} \subset \{x', y'\}$. Hence, $\|f(x) - f(y)\| = 0$ or ε_1.

Assume that $f(x) = f(y)$. Since $x, y \in C_1$ and $\|x - y\| = \varepsilon_1$, we can choose a $z \in C_0$ with $\|x - z\| = \varepsilon_1$ and $\|y - z\| = \rho$ (see Fig. 1). Repeat the argument in the last paragraph with $x, z \in C_0$ instead of $x, y \in C_1$: By [4], there exists a ρ-set Q in C_0 such that x and z are the ρ-associated points of Q. (As we see in Fig. 1 below, $x \in C_1$ and $\|x - q\| = \rho$ for all $q \in Q$ imply $Q \subset C_0$.)

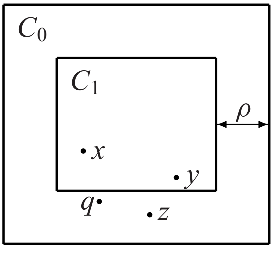

Figure 1

Hence, $Q' = f(Q)$ is also a ρ-set in E^n. According to [4] again, there exist only two distinct ρ-associated points, x'' and z'', of Q' with $\|x'' - z''\| = \varepsilon_1$.

Since f preserves the distance ρ, $f(x)$ resp. $f(z)$ has distance ρ from each point of Q'. Hence, $\{f(x), f(z)\} \subset \{x'', z''\}$. Thus, we obtain $\|f(x) - f(z)\| = 0$ or ε_1, i.e., $\|f(y) - f(z)\| = 0$ or ε_1. However, since $y, z \in C_0$ and f preserves ρ, we would obtain $\rho = \|y - z\| = \|f(y) - f(z)\| = 0$ or ε_1, a contradiction. This implies that $\|f(x) - f(y)\| = \varepsilon_1$.

(b) We note that for any $x, y \in C_2$ with $\|x - y\| = \varepsilon_2$, there exists an ε_1-set P in C_1 such that x and y are the ε_1-associated points of P (see Fig. 2 below). According to (a), $f|_{C_1}$ preserves the distance ε_1. Hence, $P' = f(P)$ is also an ε_1-set in E^n. By [4], there are exactly two distinct ε_1-associated points, x' and y', of P' and $\|x' - y'\| = \varepsilon_2$. Since $f|_{C_1}$ preserves ε_1, $f(x)$ resp. $f(y)$ is separated by a distance ε_1 from each point of P'. Thus, we get $\{f(x), f(y)\} \subset \{x', y'\}$, and hence, $\|f(x) - f(y)\| = 0$ or ε_2.

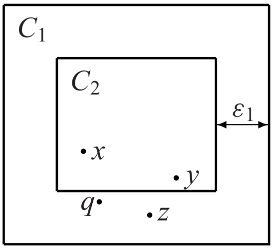

Figure 2

Assume $f(x) = f(y)$. We can obviously choose a $z \in C_1$ with $\|x - z\| = \varepsilon_2$ and $\|y - z\| = \varepsilon_1$. According to [4], there exists an ε_1-set Q in C_1 such that x and z are the ε_1-associated points of Q. (As we see in Fig. 2, $x \in C_2$ and $\|x - q\| = \varepsilon_1$ for all $q \in Q$ imply $Q \subset C_1$.)

Hence, by (a), $Q' = f(Q)$ is also an ε_1-set in E^n. According to [4] again, there exist only two distinct ε_1-associated points, x'' and z'', of Q' satisfying $\|x'' - z''\| = \varepsilon_2$, and hence $\{f(x), f(z)\} \subset \{x'', z''\}$ by the same reason as it was previously stated. Therefore, it follows that $\|f(x) - f(z)\| = 0$ or ε_2, i.e., $\|f(y) - f(z)\| = 0$ or ε_2. However, since ε_1 is preserved by $f|_{C_1}$ and $y, z \in C_1$, we get $\varepsilon_1 = \|y - z\| = \|f(y) - f(z)\| = 0$ or ε_2, which is a contradiction. Hence, we conclude that $\|f(x) - f(y)\| = \varepsilon_2$.

(c) Assume that $x, y \in C_2$ are given with $\|x - y\| = 2\rho/n$. We note that there exists an ε_1-set P in C_1 such that x and y are the ρ-associated points of P (see Fig. 2 above). Since, by (a), $f|_{C_1}$ preserves the distance ε_1,

$P' = f(P)$ is also an ε_1-set in E^n. In view of [4], there exist exactly two distinct ρ-associated points, x' and y', of P' and they satisfy $\|x' - y'\| = 2\rho/n$. Since f preserves ρ and there exist exactly two distinct ρ-associated points of P', we get $\{f(x), f(y)\} \subset \{x', y'\}$. Hence, $\|f(x) - f(y)\| = 0$ or $2\rho/n$, i.e., $\|f(x) - f(y)\| \leq 2\rho/n$. □

3 On a theorem of Beckman and Quarles

In this section, let n, j, C_k and ρ be the same ones as in the previous section. We are now ready to prove the theorem of Beckman and Quarles for restricted domains.

Theorem 3. *If a mapping $f : C_0 \to E^n$ preserves the distance ρ, then the restriction $f|_{C_\infty}$ is an isometry.*

Proof. According to (c) and (b) of Lemma 2, the distance $2\rho/n$ is contractive and $(n+1)(2\rho/n)$ is preserved (extensive) by the restriction $f|_{C_2}$.

We now define a sequence (d_k) by

$$d_1 = (n+1)\frac{2}{n}\rho, \quad d_k = (n+1)^{3-k}\frac{2}{n}\rho \quad \text{for } k \in \{2, 3, \ldots\}.$$

Moreover, let $X_k = C_{k+2}$ for each $k \in \{0, 1, 2, \ldots\} \cup \{\infty\}$, and we define

$$d\big(X_{k+1}, \partial X_k\big) = \inf\left\{\|x - y\| \,:\, x \in X_{k+1},\ y \in \partial X_k\right\}$$

for $k \in \{0, 1, 2, \ldots\}$, where ∂X_k denotes the boundary of X_k.

If both X_{k+1} and ∂X_k are bounded for some k, then we get

$$\begin{aligned} d(X_{k+1}, \partial X_k) &= d(C_{k+3}, \partial C_{k+2}) = s_{k+3} - s_{k+2} \\ &= \begin{cases} (n+1)\rho & (\text{for } k = 0), \\ (n+1)^{2-k}\rho & (\text{for } k > 0) \end{cases} \\ &\geq \begin{cases} (n+1)\dfrac{2}{n}\rho & (\text{for } k = 0), \\ (n+1)^{2-k}\dfrac{2}{n}\rho & (\text{for } k > 0) \end{cases} \\ &= d_{k+1}. \end{aligned}$$

Otherwise (if one of X_{k+1} and ∂X_k is unbounded), we set $d(X_{k+1}, \partial X_k) = \infty$ so that $d(X_{k+1}, \partial X_k) \geq d_{k+1}$ (see Section 2 of [10]). Furthermore, we see that

$$X_\infty = C_\infty = \left(\bigcap_{k=0}^{\infty} C_{k+2} \right)^\circ \neq \emptyset.$$

Finally, in view of Theorem 3 of [10], we can conclude that $f|_{X_\infty} = f|_{C_\infty}$ is an isometry because $2\rho/n$ is contractive and $(n+1)(2\rho/n)$ is extensive (preserved) by $f|_{X_0} = f|_{C_2}$. □

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