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Discrete subgroups of $PU(2,1)$ acting on $P^2_{\mathbb{C}}$ and the Kobayashi metric

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Abstract. In this paper, we prove the following: If $G \subset PU(2,1)$ is an infinite, discrete group, acting on $P^2_{\mathbb{C}}$ without complex invariant lines, then the component containing $\mathbb{H}^2_{\mathbb{C}}$ of the domain of discontinuity $\Omega(G) = P^2_{\mathbb{C}} \setminus \Lambda(G)$, according to Kulkarni, is G-invariant complete Kobayashi hyperbolic.

Keywords: limit set, discrete subgroup, complex hyperbolic plane, Kobayashi hyperbolic, complex projective plane.

Mathematical subject classification: 51M10, 32H.

1 Introduction

The group $PU(2,1)$ is the projectivization of

$$U(2,1) = \left\{ g \in GL(3,\mathbb{C}) \mid \langle g(u), g(v) \rangle = \langle u, v \rangle \quad \forall u, v \in \mathbb{C}^{2,1} \right\},$$

where $\langle \cdot, \cdot \rangle : \mathbb{C}^{2,1} \times \mathbb{C}^{2,1} \to \mathbb{C}^{2,1}$ is the Hermitian form $\langle Z, W \rangle = z_1\bar{w}_1 + z_2\bar{w}_2 - z_3\bar{w}_3$, where $Z = (z_1, z_2, z_3)$ and $W = (w_1, w_2, w_3)$. Moreover, $PU(2,1)$ is the group of isometries of the complex hyperbolic space

$$\mathbb{H}^2_{\mathbb{C}} = \left\{ [z_1 : z_2 : z_3] \in P^2_{\mathbb{C}} \mid |z_1|^2 + |z_2|^2 - |z_3|^2 > 0 \right\},$$

equipped with the Bergman metric.

The limit set, according to Kulkarni [4], of the group $G \subset PU(2,1)$ acting on complex projective space $P^2_{\mathbb{C}}$ is defined as follows: $L_0(G)$ is the closure of the set of points in $P^2_{\mathbb{C}}$ with infinite isotropy group and $L_1(G)$ is the closure of the set of accumulation points of $\{g(z)\}_{g \in G}$, where z runs over $P^2_{\mathbb{C}} \setminus L_0(G)$.

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Finally, $L_2(G)$ is the closure of the accumulation points of $\{g(K)\}_{g\in G}$, where K runs over compact subsets of $P^2_{\mathbb{C}} \setminus (L_0(G) \cup L_1(G))$.

The set $\Lambda(G) = L_0(G) \cup L_1(G) \cup L_2(G)$ is called the limit set according to Kulkarni. The set $\Omega(G) = P^2_{\mathbb{C}} \setminus \Lambda(G)$ is called the *domain of discontinuity of* G. The group G is called a complex Kleinian group when $\Omega(G)$ is not empty (see [6, 7]).

On the other hand, when G is a discrete subgroup of $PU(2,1)$, there is an action on the complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$ by holomorphic isometries. The limit set $L(G)$ of the group G according to Chen-Greenberg [1] is the set of accumulation points of any G-orbit of a point in $\mathbb{H}^2_{\mathbb{C}}$; in a similar way, as it happens for classical Kleinian groups of real hyperbolic geometry, $L(G)$ is contained in the sphere at infinite $S^3 = \partial\mathbb{H}^2_{\mathbb{C}}$.

It is proved in [5] that if G is a discrete subgroup of $PU(2,1)$ acting in $P^2_{\mathbb{C}}$, and $\Lambda(G)$ is the limit set according to Kulkarni [4] of G, then

$$\Lambda(G) = \bigcup_{z\in L(G)} l_z,$$

where l_z is the only complex projective line tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at z; also $L(G)$ is the limit set, according to Chen-Greenberg [1], of the group G when considered as acting on $\mathbf{H}^2_{\mathbb{C}}$. Furthermore, if G is non-elementary, then the orbit of each line l_z, with $z \in L(G)$, is dense in $\Lambda(G)$ (though the action of G on $\Lambda(G)$ is not minimal).

The Kobayashi pseudo-distance is defined as follows (see [3]): Let X be a complex space and p, q two points in X; we consider a *chain of holomorphic discs* from p to q; i.e. a chain of points $p = p_0, p_1, \ldots, p_k = q$ of X, and pairs of points $a_1, b_1, \ldots, a_k, b_k$ in the unit disc $D \subset \mathbb{C}$, and holomorphic functions $f_1, \ldots, f_k \in \mathrm{Hol}(D, X)$ such that

$$f_i(a_i) = p_{i-1} \quad \text{and} \quad f_i(b_i) = p_i \quad \text{for } i = 1, \ldots, k.$$

Let us denote this chain by α; we define its length $l(\alpha)$ by

$$l(\alpha) = \rho(a_1, b_1) + \ldots + \rho(a_k, b_k),$$

where ρ is the Poincaré-Bergman metric in the unit disc D. The pseudodistance d_X is defined by

$$d_X(p, q) = \inf_{\alpha} l(\alpha),$$

where the infimum is taken over all chains α of holomorphic discs from p to q. When the pseudodistance d_X is actually a distance, the space X is called

Kobayashi hyperbolic and d_X is called the *Kobayashi metric*. Moreover, if X is complete respect to the distance d_X, then X is called *complete Kobayashi hyperbolic*.

The Kobayashi distance can be considered as a generalization of the Poincaré distance because both agree on the unit disc $D \subset \mathbb{C}$.

Now we state our main theorem.

Theorem 1.1. *If $G \subset PU(2,1)$ is an infinite discrete group acting on $P^2_{\mathbb{C}}$ without invariant complex projective lines, then the connected component of the domain of discontinuity containing $\mathbb{H}^2_{\mathbb{C}}$ is G-invariant and complete Kobayashi hyperbolic.*

The following two theorems will be useful to prove Theorem 1.1. The reader may find the proofs in [3].

Theorem 1.2. *The complement of $2n+1$ or more hyperplanes in general position in $P^n_{\mathbb{C}}$ is complete Kobayashi hyperbolic.*

Theorem 1.3. *Let X and X_i, $i \in I$, be complex subspaces of a complex space Y such that $X = \cap_i X_i$. If all X_i are complete Kobayashi hyperbolic, so is X.*

2 Proof of the Main Theorem

Below, we consider (unless otherwise specified) that $G \subset PU(2,1)$ is a discrete subgroup acting on $P^2_{\mathbb{C}}$ without invariant complex projective lines.

Lemma 2.1. *If p is a point in $L(G) \subset \partial\mathbb{H}^2_{\mathbb{C}}$, then there are two different points $p_1, p_2 \in L(G)$ such that the three complex projective lines l_p, l_{p_1}, l_{p_2} (tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at the points p, p_1, p_2) are in general position.*

Proof. Since the group G is infinite and it has no invariant complex projective line, G is non-elementary; so there is $p_1 \neq p$ in $L(G)$. We denote by $q \in P^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}}$ the intersection point of l_p and l_{p_1}. We proceed by contradiction and we assume that every complex projective line in $\Lambda(G) = \bigcup_{x \in L(G)} l_x$ contains q; then, q is a fixed point of G. Therefore, G leaves the polar line to q invariant, a contradiction. $\square$

Lemma 2.2. *If p is a point in $L(G)$, then there are three different points $p_1, p_2, p_3 \in L(G)$ such that the complex projective lines $l_p, l_{p_1}, l_{p_2}, l_{p_3}$ (tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at the corresponding points) are in general position.*

Proof. By Lemma 2.1, there exist points p_1, p_2 such that the complex projective lines l_p, l_{p_1}, l_{p_2} are in general position. We denote by q_0, q_1, q_2 the intersection points of l_p and l_{p_1}, l_{p_1} and l_{p_2}, l_{p_2} and l_p (see Fig. 1).

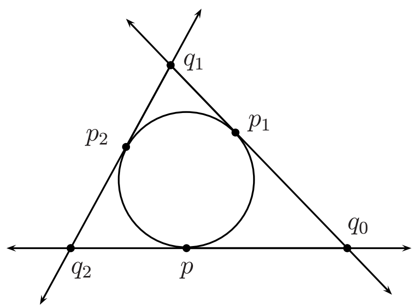

Figure 1 – Three lines in general position.

We proceed by contradiction and we assume that every line in $\Lambda(G) = \cup_{x\in L(G)} l_x$ contains q_0, q_1 or q_2. Since there are infinitely many lines in $\Lambda(G)$, we may assume that infinitely many lines in $\Lambda(G)$ contain q_0. Now, either q_1 or q_2 is contained in infinitely many lines in $\Lambda(G)$ because otherwise the line l_{p_2} is isolated in $\Lambda(G)$, and this cannot happen. So we assume that q_1 is contained in infinitely many lines of $\Lambda(G)$. If it happens that q_2 is contained only in finitely many lines in $\Lambda(G)$, then the set $\{q_0, q_1\}$ is G-invariant, which implies that l_{p_1} is G-invariant, a contradiction. Hence, the set of the points in $P^2_{\mathbb{C}}$ where infinitely many lines in $\Lambda(G)$ are concurrent, is precisely $\{q_0, q_1, q_2\}$; then, this set is G-invariant. If g is any element in G then $g^{3!}$ fixes the points q_0, q_1 and q_2; this implies that l_p, l_{p_1}, l_{p_2} are lines $g^{3!}$-invariant. Thus, the points p, p_1, and p_2 are fixed by $g^{3!}$; then, g is elliptic. We conclude that G is finite, a contradiction. $\square$

Lemma 2.3. *For every point $p \in L(G)$ there are four different points p_1, p_2, p_3, p_4 in $L(G)$ such that the complex projective lines l_p, l_{p_1}, l_{p_2}, l_{p_3}, l_{p_4} (tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at the corresponding points) are in general position.*

Proof. By Lemma 2.2 there are three different points p_1, p_2, p_3 in $L(G)$ such that the complex projective lines l_p, l_{p_1}, l_{p_2}, l_{p_3} (tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at the corresponding points) are in general position.

Let $q_0, q_1, q_2, q_3, q_4, q_5$ be the intersection points of l_{p_1} and l_{p_2}, l_p and l_{p_2}, l_p and l_{p_1}, l_{p_2} and l_{p_3}, l_{p_1} and l_{p_3}, l_p and l_{p_3}, respectively (see Fig. 2).

We proceed by contradiction and we assume that every line in $\Lambda(G)$ contains some point of the set $\{q_0, q_1, q_2, q_3, q_4, q_5\}$. We assume that there are infinitely

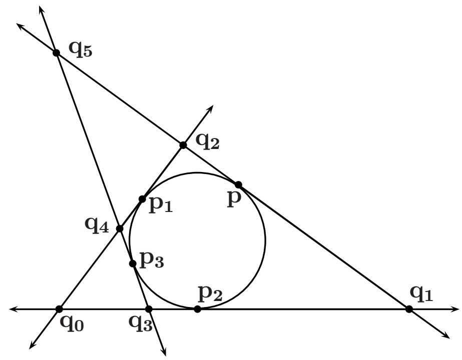

Figure 2 – Four lines in general position.

many lines in $\Lambda(G)$ through q_0. If it happens that through none of the points q_i, $\quad i = 1, \ldots, 5$ there are infinitely many lines of $\Lambda(G)$, then the line l_{p_0} is an isolated line in $\Lambda(G)$ and it cannot happen. So there is at least one point $q_i \neq q_0$ contained in an infinite number of lines in $\Lambda(G)$. There must be another point q_j, $q_0 \neq q_j \neq q_i$, contained in infinitely many lines in $\Lambda(G)$ because otherwise the line determined by q_0 and q_i is G-invariant, a contradiction. So there are at least three points of the set $\{q_0, q_1, q_2, q_3, q_4, q_5\}$, each contained in infinitely many lines in $\Lambda(G)$. If there are precisely three of such points, then they are not aligned because otherwise the line determined by them is G-invariant. An analogous reasoning to that of Lemma 2.2 shows that $g^{3!}$ is an elliptic element for any $g \in G$, which implies that G is finite. Hence, we can assume that each of the four points q_0, q_1, q_2, q_3 is contained in infinitely many lines in $\Lambda(G)$, and three out of these four points, e. g. q_0, q_1, q_2, are in general position. Also, there is a fixed natural number n_0 such that for any $g \in G$, g^{n_0} fixes the points q_0, q_1, q_2; then, the three lines determined by these points are g^{n_0}-invariant. If each of these three lines is tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$, then g^{n_0} has three different fixed points in $\partial\mathbb{H}^2_{\mathbb{C}}$; so, g is elliptic for all $g \in G$; hence, G is finite. On the other hand, if one of these three lines is not tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$, then its pole, denoted by z, is a fixed point of g^{n_0}, and we have two possibilities,

$$z \in \mathbb{H}^2_{\mathbb{C}} \quad \text{or} \quad z \in P^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}};$$

in any case, it is not hard to see that g^{n_0} is elliptic, therefore g, is elliptic; we conclude that G is finite. $\square$

An analogous reasoning to the proof of Lemma 2.3 may be used to prove (under the same hypothesis) that given a point $p \in L(G)$ and a number $n \in \mathbb{N}$, it is possible to find n different points $p_1, \ldots, p_n \in L(G)$ such that $l_p, l_{p_1}, \ldots, l_{p_n}$ are $n+1$ complex projective lines in $P^2_{\mathbb{C}}$ (tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at the corresponding points) in general position.

Proof of the main theorem. Let p be a point in $L(G)$. By Lemma 2.1, there are four points p_0, p_1, p_2, p_3 in $L(G)$ such that $l_p, l_{p_0}, l_{p_1}, l_{p_2}, l_{p_3}$ are in general position. We note that $X_p = P^2_{\mathbb{C}} \setminus (l_p \cup l_{p_0} \cup l_{p_1} \cup l_{p_2} \cup l_{p_3})$ is complete Kobayashi hyperbolic; moreover,

$$\Omega(G) = P^2_{\mathbb{C}} \setminus \bigcup_{p \in L(G)} l_p = \bigcap_{p \in L(G)} X_p.$$

Thus, $\Omega(G)$ is complete Kobayashi hyperbolic by theorem 1.2. Given that $\Omega_0(G) \subset \Omega(G)$ is closed in $\Omega(G)$, we conclude that $\Omega_0(G)$ is complete Kobayashi hyperbolic. $\square$

Finally, we give examples for both cases: when $\Omega(G) = P^2_{\mathbb{C}} \setminus \Lambda(G)$ is connected, and when it is not connected. For definitions and theorems about chains and $\mathbb{R}$-circles, see [2].

Examples. If $G \subset PO(2,1)$ is a discrete subgroup, then it can be considered a discrete subgroup of $PU(2,1)$ by means of the canonical inclusion $PO(2,1) \hookrightarrow PU(2,1)$. Since G is discrete in $PU(2,1)$, it acts properly and discontinuously on $\mathbb{H}^2_{\mathbb{C}}$ and leaves invariant the totally geodesic subspace

$$\mathbb{H}^2_{\mathbb{R}} = \left\{[r_1 : r_2 : r_3] \in \mathbb{H}^2_{\mathbb{C}} : r_1, r_2, r_3 \in \mathbb{R}\right\}.$$

We must mention that $\mathbb{H}^2_{\mathbb{R}}$ is not a complex geodesic. Moreover, the limit set according to Chen-Greenberg, L(G), is contained in the G-invariant closed set

$$\partial\mathbb{H}^2_{\mathbb{R}} = \left\{[r_1 : r_2 : 1] \in \partial\mathbb{H}^2_{\mathbb{C}} : r_1, r_2 \in \mathbb{R}\right\}$$

(the set $\partial\mathbb{H}^2_{\mathbb{R}}$ is called a $\mathbb{R}$-circle) (see [2]). We notice that the group G acts on $P^2_{\mathbb{C}}$ without invariant complex lines whenever G is non elementary.

i) Let G be a discrete subgroup of $PO(2,1)$ such that L(G) is a Cantor set. Let $p = [z_1 : z_2 : 1]$ be a point in $\partial\mathbb{H}^2_{\mathbb{C}} \setminus \partial\mathbb{H}^2_{\mathbb{R}}$; we define the function $\pi_p : \partial\mathbb{H}^2_{\mathbb{C}} \setminus \{p\} \to l_p$ as follows. If q is any point in $\partial\mathbb{H}^2_{\mathbb{C}} \setminus \{p\}$, then $\pi_p(q)$ is the intersection point of l_p and l_q.

The image of the $\mathbb{R}$-circle $\partial\mathbb{H}^2_{\mathbb{R}}$ under the map π_p is a curve parametrized as $c(t) = [\cos t - z_1 : \sin t - z_2 : \bar{z_1}\cos t + \bar{z_2}\sin t - 1]$, and a straightforward computation shows that this curve is regular.

Let z be a point in $(P^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}}) \cap \Omega(G)$; then there is a complex line l_p containing z and tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at a point $p \in \partial\mathbb{H}^2_{\mathbb{C}} \setminus \partial\mathbb{H}^2_{\mathbb{R}}$. Now, $l_p \cap \Lambda(G)$ is the image under π_p of the set $L(G)$ and it cannot disconnect this complex line; so, we can join z and p by a path completely contained in $\Omega(G)$. It follows that $\Omega(G)$ is connected and by Theorem 1.1 it is complete Kobayashi hyperbolic.

ii) On the other hand, when G is a discrete subgroup of $PO(2,1)$ with $L(G) = \partial\mathbb{H}^2_{\mathbb{R}}$ (see [2]), we claim that $\Omega(G)$ is not connected.

To prove this statement, let us consider a chain C with non-zero linking number with respect to the $\mathbb{R}$-circle $\partial\mathbb{H}^2_{\mathbb{R}}$. Let z be the polar point to C; then,

$$z \in \Omega(G) \setminus \overline{\mathbb{H}^2_{\mathbb{C}}}.$$

If the point z can be joined by a path γ_1 in $\Omega(G)$ to some point $p \in \partial\mathbb{H}^2_{\mathbb{C}} \setminus L(G)$, then z and p can be joined by a path $\gamma : [0,1] \to \Omega(G)$ so that the image is contained in

$$\Omega(G) \setminus \overline{\mathbb{H}^2_{\mathbb{C}}},$$

except for the point $\gamma(1) = p \in \partial\mathbb{H}^2_{\mathbb{C}}$. Thus the polar to $\gamma(t)$ induces a homotopy of chains $C_{\gamma(t)}$ contained in $\partial\mathbb{H}^2_{\mathbb{C}} \setminus L(G)$, where $C_{\gamma(0)} = C_z = C$ and $C_{\gamma(1)} = p$; this is a contradiction.

The main difference between the examples above is the following: in the example i), the connected component of $\Omega(G)$ containing $\mathbb{H}^2_{\mathbb{C}}$ is equal to $\Omega(G)$; hence, by Theorem 1.1, $\Omega(G)$ is complete Kobayashi hyperbolic. On the other hand, in the example ii) the connected component of $\Omega(G)$ containing $\mathbb{H}^2_{\mathbb{C}}$ is not equal to $\Omega(G)$, though it is a G-invariant connected component and, by Theorem 1.1, it is complete Kobayashi hyperbolic.

We remark that the groups in the examples above are $\mathbb{R}$-Fuchsian groups, and these groups have no invariant complex lines whenever they are not elementary. Moreover, the fact that the limit set $L(G)$ of these groups is contained in an $\mathbb{R}$-circle, points out to the existence of many complex lines in $\Lambda(G)$ in general position. The situation is different for $\mathbb{C}$-Fuchsian groups, because they have an invariant complex line, and this kind of groups does not satisfy the hypothesis of our main Theorem 1.1. In fact, the limit set $L(G)$ of a $\mathbb{C}$-Fuchsian group is contained in a chain, so all tangent complex lines to $\partial\mathbb{H}^2_{\mathbb{C}}$ at points in $L(G)$ are concurrent; therefore, the limit set $\Lambda(G)$ has at most two lines in general position.

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