BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 40(1), 107-115

Hermitian decomposition of continuous functions on a fractal surface

Ricardo Abreu Blaya, Juan Bory Reyes and Tania Moreno García

Abstract. In this paper the Théodoresco transform is used to show that, under additional assumptions, each Hölder continuous function f defined on the boundary Γ of a fractal domain $\Omega \subset \mathbb{R}^{2n}$ can be expressed as $f = \Psi^+ - \Psi^-$, where $\Psi^\pm$ are Hölder continuous functions on Γ and Hermitian monogenically extendable to Ω and to $\mathbb{R}^{2n} \setminus (\Omega \cup \Gamma)$ respectively.

Keywords: Hermitian Clifford analysis, Théodoreco transform, Fractal geometry.

Mathematical subject classification: 30G35.

1 Introduction

In its original setting, Clifford analysis studies the notion of a monogenic function, i.e. a null solution of the Dirac operator $\partial_{\underline{x}}$ in the Euclidean space $\mathbb{R}^m$. It provides a generalization to higher dimension of the theory of holomorphic functions in the complex plane and at the same time a refinement of the theory of harmonic functions. For a standard account of this function theory along classical line we refer e.g. to the monographs [7, 11, 14].

Subsequently developments with remarkable new ideas and striking works put the Clifford analysis into a much more great extent of this function theory. However, touch here all modern aspects of that exceed the scope of this paper.

Results describing monogenic decompositions of Hölder continuous functions defined on the boundary of bounded domain of $\mathbb{R}^m$ are explicitly provided for Ahlfors David regular surfaces in [2]. Concretely, let Γ be a compact topological Ahlfors David regular surface following [10] and dividing the Euclidean space $\mathbb{R}^m$ into two simply connected domains Ω_+, and Ω_- so that $\infty \in \Omega_-$. Let f

Received 19 May 2008.

stands for a real Clifford algebra valued function defined on Γ satisfying a Hölder condition with order ν ($f \in C^{0,\nu}(\Gamma)$ for short), $0 < \nu \leq 1$. Then

$$\Phi(\underline{x}) = \int_\Gamma E(\underline{y} - \underline{x})\underline{\mathbf{n}}(\underline{y}) f(\underline{y}) d\mathcal{H}^{m-1}(\underline{y}), \quad \underline{x} \in \mathbb{R}^m \setminus \Gamma, \tag{1}$$

exists and define monogenic functions $\Phi^{\pm}$ in $\Omega_{\pm}$ respectively and vanishes at the point ∞, where

$$E(\underline{x}) = -\frac{1}{\sigma_m}\frac{\underline{x}}{|\underline{x}|^m},$$

with σ_m the surface area of the unit sphere in $\mathbb{R}^m$, being the fundamental solution of the operator $\partial_{\underline{x}}$. $\mathcal{H}^{m-1}$ denotes the $(m-1)$-dimensional Hausdorff measure (see [12, 15]) and $\mathbf{n}(\underline{y})$ stands for the unit normal vector on Γ at the point $\underline{y}$ introduced by Federer (see [13]). Φ could has been extended continuously to Γ, and by Sohotskii-Plemelj formulae

$$f(\underline{x}) = \Phi^+(\underline{x}) - \Phi^-(\underline{x}) \;\text{ on }\; \Gamma. \tag{2}$$

The expression (1) is known as the Cliffordian Cauchy transform.

It should be noticed that, by the Painlevé and Liouville theorems in the Clifford analysis setting (see [1, 7]), the decomposition described in the previous paragraph is unique up to an additive constant.

For a brief review of the recent account of the problem of decomposition of monogenic real Clifford algebras valued functions on the boundary of general domain in Euclidean spaces involving special measurement (e.g. reasonable rectifiability) or some kind of regularity we refer the reader to [3].

2 Decomposition on fractal domain

If none measurable nature characteristic, among those considered in [3], is provided to the surface Γ, serious difficulties appear, because the Cliffordian Cauchy transform loses its meaning and becomes necessary to use a new method which does not use boundary integration and can thus be carried over fractal domains.

The meaning of fractal domain is considered in terms of the Hausdorff dimension of the boundary Γ, denoted by $\alpha_H(\Gamma)$, requiring $\alpha_H(\Gamma) > m - 1$.

Using different technique the method of obtaining the desired decomposition for monogenic functions and for a wide class of domains is developed in [1, 6]. For the convenience of the reader we repeat the relevant material given there, thus making our exposition self-contained:

Suppose $\Omega \subset \mathbb{R}^m$ be a bounded simply connected domain with boundary Γ. Let $f \in C^{0,\nu}(\Gamma)$, and $\mathcal{E}_0 f$ be the Whitney extension of f from Γ on the whole

$\mathbb{R}^m$. As was shown in [1, 6], when the exponent ν and the Minkowski dimension $\alpha(\Gamma)$ of the surface Γ, see [15], satisfy the relation

$$1 > \nu > \frac{\alpha(\Gamma)}{m}, \tag{3}$$

then $\partial_{\underline{x}}\mathcal{E}_0 f$ is integrable in Ω with any exponent less than $\frac{m-\alpha(\Gamma)}{1-\nu}$, i.e., under condition (3) it is integrable with certain exponent greater than m. Note that this is bounded for $\nu = 1$.

If Γ is a $(m-1)$-rectifiable surface (i.e. $\mathcal{H}^{m-1}(\Gamma) < \infty$, and it is the Lipschitz image of some bounded set of $\mathbb{R}^{m-1}$) then $\alpha(\Gamma) = \alpha_H(\Gamma) = m-1$, following [1]. Condition (3) turns into $\nu > \frac{m-1}{m}$.

Moreover, the formula

$$\Psi(\underline{x}) = \mathcal{X}(\underline{x})\mathcal{E}_0 f(\underline{x}) + \mathcal{T}_\Omega \partial_{\underline{x}}\mathcal{E}_0 f(\underline{x}),$$

where $\mathcal{X}(\underline{x})$ is the characteristic function of the set Ω, gives monogenic functions in Ω and in $\mathbb{R}^m \setminus (\Omega \cup \Gamma)$ respectively, continuous in the corresponding closed domains and satisfy on Γ the jump relation:

$$f(\underline{x}) = \Psi^+(\underline{x}) - \Psi^-(\underline{x}). \tag{4}$$

Here and subsequently $\mathcal{T}_\Omega g$ denote the Théodoresco transform of an integrable in Ω function g defined by

$$\mathcal{T}_\Omega g(\underline{x}) := -\int_\Omega E(\underline{y}-\underline{x})g(\underline{y})d\underline{y}.$$

Consequently, the desired decomposition works equally well for those more general data strongly depending of the assumption (3). When such (3) is violated then some obstructions can be constructed as were proved in [4].

In the sequel we assume ν and $\alpha(\Gamma)$ to be connected by (3).

2.1 Uniqueness of the decomposition

To ensure uniqueness of the decomposition (4) we need to introduce some additional requirements.

The function Ψ, monogenic in $\mathbb{R}^m \setminus \Gamma$ must satisfy a Hölder condition with exponent μ, $0 < \mu < 1$ on each of the sets $\overline{\Omega}$ and $\overline{\mathbb{R}^m \setminus (\Omega \cup \Gamma)}$, i.e. the restrictions $\Psi\big|_\Omega$ and $\Psi\big|_{\mathbb{R}^m\setminus(\Omega\cup\Gamma)}$ must be μ-Hölder continuous in the closed domains respectively, and the boundary vales of these restrictions $\Psi^\pm$ are the usual continuous limit values. Ψ is required to be normalized $\Psi(\infty) = 0$.

It is essential to point out that, $\mathcal{T}_\Omega \partial_{\underline{x}} \mathcal{E}_0 f \in C^{0,\mu}(\mathbb{R}^m)$ with

$$\mu < \frac{m\nu - \alpha(\Gamma)}{m - \alpha(\Gamma)}, \tag{5}$$

which is due to the fact that the Théodoresco transform maps the space of $p-$integrable functions with compact support to $C^{0,\lambda}(\mathbb{R}^m)$ if $p > m$ and $\lambda = 1 - \frac{m}{p}$.

At the same time the uniqueness of the required decomposition follows from the removability of the surface Γ (according to the Dolzhenko type theorem, proved in [1]) under the condition that

$$\mu > \alpha_H(\Gamma) - m + 1. \tag{6}$$

A decomposition of type (4), the function Ψ being $\mu-$Hölder continuous on $\overline{\Omega}$ and $\overline{\mathbb{R}^m \setminus (\Omega \cup \Gamma)}$ whenever μ satisfies the conditions (5) and (6), is said to be of class $C_{0,\mu}$.

Summarizing, we have

Theorem 1. *Let $\alpha(\Gamma) < m$ and suppose valid* (3). *Then, for any f in $C^{0,\nu}(\Gamma)$, $0 < \nu \leq 1$ a unique decomposition* (4) *of class $C_{0,\mu}$ occurs.*

3 Hermitian case

Hermitian Clifford analysis deals with the simultaneous null solutions of the orthogonal Dirac operators $\partial_{\underline{x}}$ and its twisted counterpart $\partial_{\underline{x}|}$, introduced below. For a thorough treatment of this higher dimensional function theory we refer the reader to e.g. [8, 9, 16, 17].

Let $(e_1, \ldots, e_{2n})$ be an orthonormal basis of the Euclidean space $\mathbb{R}^{2n}$. Consider the complex Clifford algebra $\mathbb{C}_{2n}$ constructed over $\mathbb{R}^{2n}$. The non-commutative multiplication in $\mathbb{C}_{2n}$ is governed by

$$\begin{aligned} e_j^2 &= -1, & j &= 1, \ldots, 2n, \\ e_j e_k + e_k e_j &= 0, & 1 &\leq j \neq k \leq 2n. \end{aligned}$$

A basis for $\mathbb{C}_{2n}$ is obtained by considering for a set $A = \{j_1, \ldots, j_k\} \subset \{1, \ldots, 2n\}$ the element $e_A = e_{j_1} \ldots e_{j_k}$, with $j_1 < \cdots < j_k$. For the empty set $\emptyset$, we put $e_\emptyset = 1$, the latter being the identity element.

Any Clifford number $a \in \mathbb{C}_{2n}$ may thus be written as

$$a = \sum_A a_A e_A, \;\; a_A \in \mathbb{C},$$

and its Hermitian conjugate $\overline{a}$ is defined by

$$\overline{a} = \sum_A \bar{a}_A \overline{e}_A, \qquad \overline{e}_A = (-1)^{\frac{k(k+1)}{2}} e_A, \qquad |A| = k.$$

The Euclidean space $\mathbb{R}^{2n}$ is embedded in the Clifford algebra $\mathbb{C}_{2n}$ by identifying $(x_1, \ldots, x_{2n})$ with the real Clifford vector $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{n} \left(e_{2j-1} x_{2j-1} + e_{2j} x_{2j} \right).$$

We also introduce for each real Clifford vector $\underline{x}$ its twisted counterpart

$$\underline{x}| = \sum_{j=1}^{n} \left(e_{2j-1} x_{2j} - e_{2j} x_{2j-1} \right).$$

Observe that the Clifford vectors $\underline{x}$ and $\underline{x}|$ are orthogonal with respect to the standard Euclidean scalar product, which implies that $\underline{x}\,\underline{x}| = -\underline{x}|\,\underline{x}$.

The Fischer dual of the vector $\underline{x}$ is the first order differential operator

$$\partial_{\underline{x}} = \sum_{j=1}^{n} \left(e_{2j-1} \partial_{x_{2j-1}} + e_{2j} \partial_{x_{2j}} \right)$$

called Dirac operator.

A notion of monogenicity can be associated to the Fischer dual of the vector $\underline{x}|$ in a similar way to that of the vector $\underline{x}$ given by

$$\partial_{\underline{x}|} = \sum_{j=1}^{n} \left(e_{2j-1} \partial_{x_{2j}} - e_{2j} \partial_{x_{2j-1}} \right).$$

We notice that the Dirac operators $\partial_{\underline{x}}$ and $\partial_{\underline{x}|}$ anticommute and factorize the Laplacian, i.e.

$$-\partial_{\underline{x}}^2 = \Delta = -\partial_{\underline{x}|}^2.$$

Thus monogenicity with respect to $\partial_{\underline{x}}$ (respectively $\partial_{\underline{x}|}$) can be regarded as a refinement of harmonicity.

Further, a continuously differentiable function f in an open set Ω of $\mathbb{R}^{2n}$ with values in $\mathbb{C}_{2n}$ is called a (left) Hermitian monogenic (or h-monogenic) function in Ω if and only if it satisfies in Ω the system

$$\partial_{\underline{x}} f = 0 = \partial_{\underline{x}|} f.$$

Hence, we are in a position to introduce the Théodoresco transform associated with the fundamental solution of $\partial_{\underline{x}}|$, denoted by

$$E|(\underline{x}) = -\frac{1}{\sigma_{2n}} \frac{\underline{x}|}{|\underline{x}|^{2n}},$$

in a similar way with that of $\partial_{\underline{x}}$. More precisely, for an integrable function g in Ω we have

$$\mathcal{T}_\Omega|g(\underline{x}) := -\int_\Omega E|(\underline{y}-\underline{x})g(\underline{y})d\underline{y}.$$

3.1 Hermitian monogenic decomposition

It is natural to try to extend the results describing monogenic decomposition to the Hermitian case.

Let Γ be a compact topological surface in $\mathbb{R}^{2n}$ which bounds a domain Ω, and suppose that f is any ν-Hölder continuous $\mathbb{C}_{2n}$-valued function defined on Γ.

Question. Can we decompose f additively into two part in the way of (4), where instead of monogenicity of Ψ is required the Hermitian ones?

For abbreviation we continue to say that a positive solution of this question under asserted convention before is named to be of class $C_{0,\mu}$.

In an early paper [5] we had derived necessary and sufficient conditions for an affirmative answer of this question making standing assumptions on the domains under consideration but we will not develop this point here.

This note is intended as an attempt to look more closely at the case of fractal domains.

It appears rather surprising and interesting that we have here a nice theoretical justification to suppose that the method described in Section 2 play a crucial role in solving the question at present.

We can now formulate our main result

Theorem 2. *Let $\alpha(\Gamma) < m$, and suppose that*

$$f \in C^{0,\nu}(\Gamma), \quad 0 < \nu \leq 1.$$

Then, assuming (3) *to hold there exist an unique decomposition* (4) *of type $C_{0,\mu}$ if and only if*

$$\mathcal{T}_\Omega(\partial_{\underline{x}}\mathcal{E}_0 f)\big|_\Gamma = \mathcal{T}_\Omega|(\partial_{\underline{x}}|\mathcal{E}_0 f)\big|_\Gamma. \tag{7}$$

Proof. Let us suppose Ψ gives the unique Hermitian monogenic decomposition of the function f. Hence, Theorem 1 shows that $\Psi = \Psi_1 = \Psi_2$, where

$$\Psi_1 = \mathcal{X}\mathcal{E}_0 f + \mathcal{T}_\Omega \partial_{\underline{x}} \mathcal{E}_0 f$$
$$\Psi_2 = \mathcal{X}\mathcal{E}_0 f + \mathcal{T}_\Omega |\partial_{\underline{x}}| \mathcal{E}_0 f.$$

Letting boundary value on Γ we obtain (7).

Conversely, assume (7) to hold. From (7) and on account of the last-mentioned expresions for $\Psi_{1,2}$, we obtain

$$\Psi_1^{\pm}\big|_{\Gamma} = \Psi_2^{\pm}\big|_{\Gamma}.$$

Note that $\Psi_1 - \Psi_2$ is harmonic in $\mathbb{R}^{2n} \setminus \Gamma$, and by the classical Dirichlet problem $\Psi_1 = \Psi_2$ on $\mathbb{R}^{2n} \setminus \Gamma$. Therefore, both functions Ψ_1 and Ψ_2 are Hermitian monogenic in $\mathbb{R}^{2n} \setminus \Gamma$. Using again Theorem 1 leads to the unique desired decomposition given by $\Psi := \Psi_1 = \Psi_2$. $\square$

Let us mention one straight consequence of the theorem.

Corollary 1. *Let Γ be a $(m-1)$-rectifiable surface. Then, the theorem is still true if it is just assumed that $1 > \nu > \frac{m-1}{m}$ with $0 < \mu < m(\nu - 1) + 1$.*

Acknowledgments. The third author wishes to thank the IMPA, Rio de Janeiro, where the final version of the paper was written, for the invitation and hospitality. She was supported by the CAPES-MES project 028/07.

References

[1] R. Abreu Blaya, J. Bory Reyes and D. Peña Peña. *Jump problem and removable singularities for monogenic functions.* J. Geom. Anal., **17**(1) (2007), 1–13.

[2] R. Abreu Blaya, D. Peña Peña and J. Bory Reyes. *Clifford Cauchy type integrals on Ahlfors-David regular surfaces in $\mathbb{R}^{m+1}$*. Adv. Appl. Clifford Algebras, **13**(2) (2003), 133–156.

[3] R. Abreu Blaya, J. Bory Reyes and T. Moreno García. *Cauchy Transform on non-rectifiable surfaces in Clifford Analysis.* J. Math. Anal. Appl., **339** (2008), 31–44.

[4] R. Abreu Blaya, J. Bory Reyes and T. Moreno García. *Teodorescu transform decomposition of multivector fields on fractal hypersurfaces.* Wavelets, multiscale systems and hypercomplex analysis, 1–16, Oper. Theory Adv. Appl., **167** (2006), Birkhäuser, Basel.

[5] R. Abreu Blaya, J. Bory Reyes, D. Peña Peña and F. Sommen. *A boundary value problem for Hermitian monogenic functions*. Bound. Value Probl., vol. 2008, Article ID 385874, 7 pages, (2008).

[6] R. Abreu Blaya, J. Bory Reyes and T. Moreno García. *Minkowski dimension and Cauchy transform in Clifford análisis*. Compl. Anal. Oper. Theory, **1**(3) (2007), 301–315.

[7] F. Brackx, R. Delanghe and F. Sommen. *Clifford analysis.* Research Notes in Mathematics, 76, Pitman (Advanced Publishing Program), Boston, (1982).

[8] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen and V. Souček. *Fundaments of Hermitean Clifford Analysis Part I: Complex Structure.* Complex Anal. Oper. Theory, **1**(3) (2007), 341–365.

[9] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen and V. Souček. *Fundaments of Hermitean Clifford Analysis Part II: Splitting of h-monogenic equations.* Complex Var. Elliptic Equ., **52**(10-11) (2007), 1063–1079.

[10] G. David and S. Semmes. *Analysis of and on uniformly rectifiable sets.* Mathematical Surveys and Monographs, 38. American Mathematical Society, Providence, RI, (1993).

[11] R. Delanghe, F. Sommen and V. Souček. Clifford algebra and spinor-valued functions, 53. Kluwer Academic Publishers Group, Dordrecht, (1992).

[12] K.J. Falconer. *The geometry of fractal sets.* Cambridge Tracts in Mathematics, 85. Cambridge University Press, Cambridge, (1986).

[13] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York (1969).

[14] J. Gilbert and M. Murray. Clifford algebras and Dirac operators in harmonic analysis, 26. Cambridge University Press, Cambridge, (1991).

[15] P. Mattila. *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.* Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, (1995).

[16] R. Rocha-Chávez, M. Shapiro and F. Sommen. *Integral theorems for functions and differential forms in* $\mathbb{C}^m$. Research Notes in Mathematics, 428. Chapman & Hall/CRC, Boca Raton, FL, (2002).

[17] I. Sabadini and F. Sommen. *Hermitian Clifford analysis and resolutions.* Clifford analysis in applications. Math. Methods Appl. Sci., **25**(16-18) (2002), 1395–1413.

Ricardo Abreu Blaya
Facultad de Informática y Matemática
Universidad de Holguín
Holguín 80100
CUBA

E-mail: rabreu@facinf.uho.edu.cu

Juan Bory Reyes
Departamento de Matemática
Universidad de Oriente
Santiago de Cuba 90500
CUBA

E-mail: jbory@rect.uo.edu.cu

Tania Moreno García
Facultad de Informática y Matemática
Universidad de Holguín
Holguín 80100
CUBA

E-mail: tmorenog@facinf.uho.edu.cu / taniamg@impa.br