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A Jenkins-Serrin theorem in $M^2 \times \mathbb{R}$

Ana Lucia Pinheiro*

Abstract. In this paper we study minimal surfaces in $M \times \mathbb{R}$, where M is a complete surface. Our main result is a Jenkins-Serrin type theorem which establishes necessary and sufficient conditions for the existence of certain minimal vertical graphs in $M \times \mathbb{R}$. We also prove that there exists a unique solution of the Plateau's problem in $M \times \mathbb{R}$ whose boundary is a Nitsche graph and we construct a Scherk-type surface in this space.

Keywords: minimal graphs, product spaces.

Mathematical subject classification: 53A10, 53C42.

1 Introduction

Let M be a complete Riemannian surface and $D \subset M$ be a geodesically convex (open) domain with $\bar{D}$ compact. We call such a domain an *admissible domain*. Suppose that ∂D contains two sets of open geodesic arcs $A_1, \dots, A_k$ and $B_1, \dots, B_l$, with the property that neither two A_i nor two B_j have a common endpoint. The remaining part of ∂D is the union of open convex arcs $C_1, \dots, C_h$ and all endpoints. Let $f_s : C_s \to \mathbb{R}$, $1 \leq s \leq n$, be continuous functions. Moreover let $\mathcal{P}$ be an *admissible polygon*, i.e., a polygon inscribed in ∂D whose vertices are chosen among the vertices of A_i, B_j, and let

$$\alpha := \sum_{A_i \subset \mathcal{P}} \|A_i\|, \quad \beta := \sum_{B_j \subset \mathcal{P}} \|B_j\|, \quad \gamma := \text{perimeter } (\mathcal{P}),$$

where $\| \ \|$ denotes the length of the arc.

We prove the following result:

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Theorem 1.1. *With the above notation assume further that* $\{C_s\} \neq \emptyset$. *Then there exists a function* $u\colon D \to \mathbb{R}$ *whose graph is a minimal surface in* $D \times \mathbb{R}$ *with*

$$u\Big|_{A_i} = +\infty, \quad u\Big|_{B_j} = -\infty, \quad u\Big|_{C_s} = f_s,$$

if and only if

$$2\alpha < \gamma, \quad 2\beta < \gamma \tag{1}$$

for each admissible polygon $\mathcal{P}$. *Moreover, if the function u exists, it is unique.*

Assume now that $\{C_s\} = \emptyset$. *Then the function u exists if and only if*

$$\alpha = \beta$$

for $\mathcal{P} = \partial D$ *and condition (1) holds for all other admissible polygons* $\mathcal{P}$. *In this case, if the function u exists, it is unique up to an additive constant.*

This theorem is analogous to that of Jenkins and Serrin [JS] for minimal graphs in $\mathbb{R}^3$ and generalizes the similar result of Nelli and Rosenberg [NR1] in $\mathbb{H}^2 \times \mathbb{R}$.

A Jordan curve $\Gamma \subset M \times \mathbb{R}$ is a *Nitsche graph* if it admits a parametrization $\{(\alpha(s), t(s)),\ s \in \mathbb{S}^1\}$, whose orthogonal projection on M is a monotone parametrizatiom $\alpha(s)$ of the boundary ∂D of a domain $D \subset M$. By monotone parametrization of ∂D we mean that $\alpha : \mathbb{S}^1 \to \partial D$ is continuous and there exist disjoint closed intervals $J_1, \ldots, J_l \subset \mathbb{S}^1$ such that $\alpha_{|_{J_k}}$ is constant for all k, and $\alpha_{|_{\mathbb{S}^1 - (\bigcup_k J_k)}}$ is injective and regular. See Figure 1.

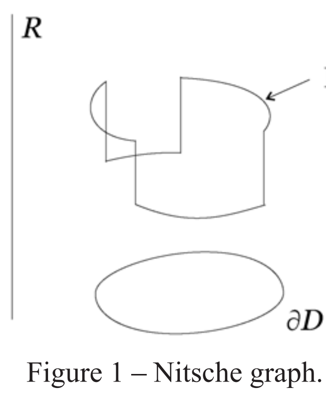

Figure 1 – Nitsche graph.

Given a Nitsche graph Γ on the boundary of a domain D, by a *minimal graph* with boundary Γ we mean a minimal surface contained in $\bar{D} \times \mathbb{R}$ which is a graph on D. The next result assures the existence and uniqueness of such a surface when D is an admissible domain.

Theorem 1.2. *If Γ is a Nitsche graph on the boundary of an admissible domain, then there exists a unique minimal graph with boundary Γ. Hence it is a disk.*

The proof of this theorem asserts that the Dirichlet problem for the minimal equation in $M \times \mathbb{R}$, on an admissible domain, has a unique solution, called *minimal solution* even if the boundary data is continuous except in a finite set of points. In $\mathbb{R}^3$, this Dirichlet problem was considered by Finn [F] and, in a more general case by Nitsche [N].

The uniqueness in our case is a consequence of the following general maximum principle.

Theorem 1.3. *Let $D \subset M$ be an admissible domain and $E = \{P_1, \ldots, P_k\} \subset \partial D$. Let $\Gamma_n \subset \partial D \times \mathbb{R}$, for $n = 1, 2$, be Nitsche graphs, u_n minimal graphs with boundary Γ_n and $\pi_n \colon \Gamma_n \to \partial D$ the vertical projections. Suppose that $\pi_n^{-1}(P_i) \subset \Gamma_n$ is a vertical segment, for all $i = 1, \ldots, k$. If $u_1 \leq u_2$ on $\partial D - E$, then $u_1 \leq u_2$ on D.*

To define a Scherk surface in $M \times \mathbb{R}$ consider $\Delta \subset M$ an embedded geodesic triangle, with open sides a, b, c opposite to the vertices A, B, C, respectively, and such that interior angles are smaller than π. Suppose $\overline{\Delta} \subset D$, where D is an admissible domain. We have the following result:

Theorem 1.4. *There exists a minimal function u defined on $\overline{\Delta} - \bar{a}$ that satisfies*

$$u\Big|_b = u\Big|_c = 0 \quad \text{and} \quad \lim_{x \to \operatorname{int}(a)} u(x) = +\infty.$$

Moreover, $|\nabla u(x)| \to +\infty$ when x approaches the side a. We denote by ∇ the gradient on M.

We call the graph of u a *Scherk surface in $M \times \mathbb{R}$*. This minimal surface plays an important role along the proof of the Jenkins-Serrin type theorem.

For example, let $M = \mathbb{S}^2$ be the round sphere. This theorem assures that, given a geodesic triangle Δ contained in an open hemisphere of $\mathbb{S}^2$, there exists a function $u \colon \Delta \to \mathbb{R}$ whose graph is a Scherk surface in $\mathbb{S}^2 \times \mathbb{R}$. This particular case was proved by H. Rosenberg [Ro].

This work is part of my doctoral thesis [P] at Universidade Federal do Rio de Janeiro. The author would like to thank Professors Harold Rosenberg and Walcy Santos for their advice and encouragement during the preparation of this paper and Professor Enaldo Vergasta for his help on writing.

2 Proof of Theorem 1.2

As $\overline{D} \times \mathbb{R}$ is a homogeneously regular manifold and $\partial(\overline{D} \times \mathbb{R})$ is mean-convex, by Morrey [Mo], Meeks and Yau [MY1], [MY2] there exists an embedded minimal disk $\Sigma \subset D \times \mathbb{R}$ that is a solution for the Plateau problem with boundary Γ. Our effort is to prove that Σ is a graph on D.

Assertion 2.1. *For all $p \in \operatorname{int} \Sigma$, $T_p\Sigma$ is not a vertical plane in $D \times \mathbb{R}$.*

To prove the assertion assume, by contradiction, that there exists a point $p \in \operatorname{int} \Sigma$ such that $p \in M(c) := M \times \{c\}$ for some $c \in \mathbb{R}$, and the tangent plane π to Σ at p is vertical in $D \times \mathbb{R}$. This means that there exists a basis $\{\partial/(\partial t), v\}$ of π, where $\partial/(\partial t)$ is the tangent vector to Σ in the $\mathbb{R}$-direction and v is tangent to $M(c)$ at p. We take $\|v\| = 1$.

As vertical translations are isometries in $M \times \mathbb{R}$, we can assume $c = 0$. So, there exists a unique geodesic $\gamma \subset M(0)$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

The geodesic γ intersects ∂D exactly in two points. In fact, if γ accumulates on D, then there exist a sequence of points $p_n \in \gamma$ and a point $p \in \gamma$ such that p_n converges to p on D, but p_n does not converge to p on γ. This means that on D the distance between p_n and p goes to 0 when n goes to infinity, and for n sufficiently large there exists another curve $\beta \subset D$ joining p_N to p, for all $N \geq n$, whose length is smaller than the length of the arc of γ joining p_n to p. This contradicts the hypothesis on D.

On the other hand, $\gamma \times \mathbb{R}$ is a totally geodesic surface in $D \times \mathbb{R}$, in particular, it is minimal. Moreover, $T_p(\gamma \times \mathbb{R}) = \pi$. Therefore, near p, $I = \Sigma \cap (\gamma \times \mathbb{R})$ is a set of at least two curves, which intersect transversally at p. If there exists a cycle α in $I - \partial\Sigma$, then α is the boundary of a minimal disk in Σ. Thus we could touch this disk at an interior point with another minimal surface $\beta \times \mathbb{R}$, where β is a geodesic curve of D, but this can not happen by the maximum principle.

So each branch of these curves leaving p must go to $\partial\Sigma$ and, as $\gamma \cap \partial D$ has exactly two points, at least two of the branches go to the same point or vertical segment of $\partial\Sigma$. This yields a compact cycle α in I and, by the same previous argument, we have a contradiction. This concludes the proof of the Assertion 2.1.

With the next argument we prove that the tangent planes on vertical segments are not vertical. In fact, as Σ is an embedded disk, it separates the space $D \times \mathbb{R}$ in two connected components. As Σ is orientable, by the Assertion 2.1 we can assume that the normal vector field N points up at every point of $\operatorname{int} \Sigma$. Suppose that there exist two consecutive points P and Q in the intersection of $\operatorname{int} \Sigma$ with the same vertical line of $D \times \mathbb{R}$. By hypotheses, $N(P)$ and $N(Q)$ point up in $D \times \mathbb{R}$.

In particular, the vector $-N(P)$ must point down. But, as P and Q are consecutive points of int Σ in the same vertical line, the vectors $-N(P)$ and $N(Q)$ should point to the same component of $D \times \mathbb{R}$. This is a contradiction and proves that Σ is a graph on D. See [ADR] for a similar argument.

The uniqueness is a consequence of the Theorem 1.3. $\square$

3 Proof of Theorem 1.3

Denote by ϕ the function $u_1 - u_2$ and assume by contradiction that the set $\mathcal{U} = \{p \in D;\ \phi(p) > 0\}$ is non empty. After a vertical translation of the graph u_1, if necessary, we can assume $u_1 < u_2$ on $\partial D - E$ and the curves contained in $\partial\mathcal{U} = \{p \in D; \phi(p) = 0\}$ have no singularities, i.e., the vector $\nabla\phi(p)$ is not null, for all $p \in \partial\mathcal{U}$.

Consider a curve $\gamma \subset \partial\mathcal{U}$. Then γ is a proper curve in D. In fact, if γ accumulates on D, then one has a point $p = \gamma(t_0) \in D$ and a sequence of points $p_n \in \gamma$ such that p_n converges to p on D, but p_n does not converge to p on γ. So, there exists a curve $\beta \subset D$, such that $\beta(t_0) = p$, β joins p to p_n, for all n, and $\{\gamma'(t_0),\ \beta'(t_0)\}$ is a basis of T_pM. As $\phi_{|\gamma} \equiv 0$ and $\phi(p_n) = 0$, we have $(d\phi)_p(\gamma'(t_0)) = (d\phi)_p(\beta'(t_0)) = 0$ and, consequently $(d\phi)_p \equiv 0$. Now the equality $\langle\nabla\phi(p), v\rangle = (d\phi)_p(v)$, for all $v \in T_pM$, implies that $\nabla\phi(p) = 0$, what can not happen.

By the classical maximum principle, γ can not be closed in D. So γ goes to the boundary of D. As ϕ is a continuous function and we have supposed $u_1 < u_2$ on $\partial D - E$, then γ must go to E. So there exists a connected domain $\tilde{\mathcal{U}} \subset D$, with $\partial\tilde{\mathcal{U}} \subset \{\phi \equiv 0\} \cup E$.

Take $\epsilon > 0$ small. Let $\tilde{\mathcal{U}}_\epsilon \subset \tilde{\mathcal{U}}$ be the domain such that $\partial\tilde{\mathcal{U}}_\epsilon$ is the union of the set of all points in $\partial\tilde{\mathcal{U}}$ whose distance from $P_i \in \partial\tilde{\mathcal{U}} \cap E$ is greater than ϵ with the circular arcs C^i_ϵ with center at each $P_i \in \partial\tilde{\mathcal{U}} \cap E$ and radius ϵ, see Figure 2.

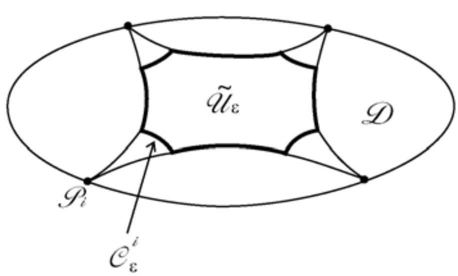

Figure 2 – Domain $\tilde{\mathcal{U}}_\epsilon$.

As u_i, $i = 1, 2$, satisfies the minimal equation div $\frac{\nabla u_i}{W_i} = 0$ on D, where $W_i = \sqrt{1 + |\nabla u_i|^2}$ [Sp], we have

$$\int_{\tilde{\mathcal{U}}_\epsilon} \operatorname{div}\left(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}\right) = 0.$$

By Stokes' theorem, this implies

$$\int_{\partial\tilde{\mathcal{U}}_\epsilon} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle = 0, \tag{2}$$

where ν is the inward unit conormal to $\partial\tilde{\mathcal{U}}_\epsilon$.

Assertion 3.1.

$$\left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nabla\phi \right\rangle = \frac{1}{2}(W_1 + W_2)\|N_1 - N_2\|^2,$$

where $N_i = \left(-\frac{\nabla u_i}{W_i}, \frac{1}{W_i}\right)$, $i = 1, 2$, *is the unit normal vector to the graph of* u_i.

The assertion is a consequence of the following equality

$$\left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nabla u_1 - \nabla u_2 \right\rangle = \langle\langle W_1 N_1 - W_2 N_2, N_1 - N_2 \rangle\rangle,$$

where $\langle\langle\, , \rangle\rangle$ is the inner product in $M \times \mathbb{R}$.

Now, as $1/2(W_1 + W_2) \geq 1$ and $|\nabla\phi| \neq 0$ on α_ϵ, where $\alpha_\epsilon = \partial\tilde{\mathcal{U}}_\epsilon - (\cup_i C_\epsilon^i)$, the assertion implies that

$$\left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nabla\phi \right\rangle > 0 \ \text{ on } \ \alpha_\epsilon. \tag{3}$$

On the other hand, as $\nabla\phi \neq 0$ and $\phi \equiv 0$ on α_ϵ, and $\phi > 0$ on $\tilde{\mathcal{U}}_\epsilon$, the vector $\nabla\phi$ has to point to int $\tilde{\mathcal{U}}_\epsilon$ along α_ϵ and, consequently, $\nabla\phi$ is a positive multiple of ν on α_ϵ. Therefore, by (3),

$$\left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle > 0 \ \text{ on } \ \alpha_\epsilon,$$

and

$$\int_{\alpha_\epsilon} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle \geq \delta > 0.$$

On $\cup_i C^i_\epsilon$, one has

$$\left| \int_{\cup_i C^i_\epsilon} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle \right| \leq 2\, l(\epsilon),$$

where $l(\epsilon) = \text{length}(\cup_i C^i_\epsilon)$ goes to zero, when ϵ goes to 0. So, when ϵ is sufficiently small, one has

$$\int_{\partial \tilde{\mathcal{U}}_\epsilon} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle > 0.$$

This is a contradiction with (2) and we have that $u_1 \leq u_2$ on D. □

4 Proof of Thorem 1.4

Let t be a fixed positive number and $\Gamma_t \subset \bar{\Delta} \times \mathbb{R}$ the Nitsche graph on $\partial\Delta$ obtained by the union of the sides b, c, the curve $a(t)$ obtained by raising the side a to height t, and the vertical segments joining the endpoints of a and $a(t)$.

By Theorem 1.2, there exists a unique minimal graph, denoted by Σ_t, with boundary Γ_t. That is, there exists a continuous function $u_t \colon \Delta - \{B,\ C\} \to \mathbb{R}$ such that $\Sigma_t = \operatorname{graph} u_t$, and

$$u_t(A) = 0, \quad u_t\big|_b = u_t\big|_c = 0, \quad u_t\big|_a = t.$$

Let t_1 and t_2 be positive real numbers with $t_1 \geq t_2$. Consider the function

$$f \colon \Delta - \{B,\ C\} \to \mathbb{R};\ f(x) = (u_{t_1} - u_{t_2})(x),$$

where u_{t_i}, $i = 1, 2$, is defined as above. As $f \geq 0$ on $\partial\Delta - \{B,\ C\}$, Theorem 1.3 implies that $f \geq 0$ on $\Delta - \{B,\ C\}$. Then $\{u_t\}$ is a nondecreasing and nonnegative sequence on $\Delta - \{B,\ C\}$. In order to see that the function $u = \lim_{t\to+\infty} u_t$ exists, we prove that the sequence u_t is uniformly bounded on compact subsets $K \subset \Delta - a$. The idea to prove this is to construct a minimal surface in $\Delta \times \mathbb{R}$ which is over the graph of u_t, for all t. We call this minimal surface an upper barrier for the sequence u_t.

Let $\tilde{a} \subset D$ be the geodesic arc that contains the side a and whose endpoints $\tilde{B}$ and $\tilde{C}$ are at a small distance δ from a, then $\|\tilde{a}\| = \|a\| + 2\delta$. Let $\tilde{b}$ and $\tilde{c}$ be the minimizing geodesics joining $\tilde{C}$ and $\tilde{B}$ to A, respectively, and $\tilde{\Delta}$ be the triangle in $M(0)$ with sides $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$. Consider the points $\tilde{P} \in \tilde{b}$, $\tilde{Q} \in \tilde{c}$ at a same small distance ϵ from A and the geodesic curve $\alpha_\epsilon \subset \tilde{\Delta}$ joining $\tilde{P}$ to $\tilde{Q}$.

Denote by P and Q the intersect points of α_ϵ and b and c, respectively. Now, let b_ϵ the segment of b between C and P, c_ϵ be the segment of c between B and Q, $\tilde{b}_\epsilon$ be the segment of $\tilde{b}$ between $\tilde{C}$ and $\tilde{P}$ and $\tilde{c}_\epsilon$ be the segment of $\tilde{c}$ between $\tilde{B}$ and $\tilde{Q}$. See Figure 3.

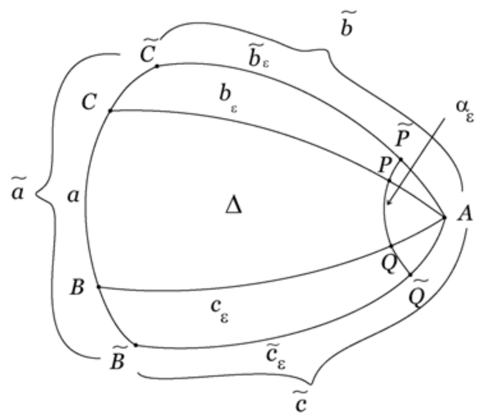

Figure 3 – Triangle $\tilde{\Delta}$.

Fix $\tau \in \mathbb{R}$, $\tau > 0$, and let $R(\tilde{b}_\epsilon, \tau)$ and $R(\tilde{c}_\epsilon, \tau)$ be the curves that are the boundary of $\tilde{b}_\epsilon \times [0, \tau]$ and $\tilde{c}_\epsilon \times [0, \tau]$, respectively. See Figure 4.

We use the Douglas criteria for the Plateau problem [J] to prove the existence of a least area oriented minimal annulus with boundary $R(\tilde{b}_\epsilon, \tau) \cup R(\tilde{c}_\epsilon, \tau)$.

Assertion 4.1. *Let $\mathcal{D}$ be a minimal disk with boundary $R(\tilde{b}_\epsilon, \tau)$. Then area$(\mathcal{D}) \geq \|\tilde{b}_\epsilon\|\tau$.*

In fact, by the *co-area formula*,

$$\text{area}(\mathcal{D}) = \int_{\min_{x\in\mathcal{D}} h(x)}^{\max_{x\in\mathcal{D}} h(x)} \left(\int_{h^{-1}(t)} \frac{ds_t}{|\nabla_{\mathcal{D}} h|} \right) dt,$$

where h is the height function in $M \times \mathbb{R}$ and ds_t is the volume form on $h^{-1}(t)$.

The function h is harmonic on $\mathcal{D}$ [Ro], thus the minimum and the maximum of h on D would be attained at the boundary of $\mathcal{D}$. Then

$$\text{area}(\mathcal{D}) \quad = \quad \int_0^\tau \left(\int_{h^{-1}(t)} \frac{ds_t}{|\nabla_{h^{-1}(t)} h|} \right) dt\,.$$

Denoting by $\tilde{\nabla}$ the gradient on $M \times \mathbb{R}$, we have $\nabla_{h^{-1}(t)} h = \tilde{\nabla} h - \langle \tilde{\nabla} h, N\rangle N$, where N is the unit normal vector to $h^{-1}(t)$. As $\tilde{\nabla} h = \partial/(\partial t)$, we have

$|\nabla_{h^{-1}(t)} h| \leq 1$ and

$$\begin{aligned} \operatorname{area}(\mathcal{D}) &\geq \int_0^\tau \left(\int_{h^{-1}(t)} ds_t \right) dt \\ &= \int_0^\tau \|h^{-1}(t)\| \, dt \\ &\geq \int_0^\tau \|\tilde{b}_\epsilon(t)\| \, dt \,, \end{aligned}$$

as asserted. At the last inequality we used that $\tilde{b}_\epsilon$ is the minimizing geodesic joining $\tilde{C}$ to $\tilde{P}$.

Based in this assertion we conclude that the area of the disks $\tilde{b}_\epsilon \times [0, \tau]$ and $\tilde{c}_\epsilon \times [0, \tau]$ are the minimum areas of the disks with boundary $R(\tilde{b}_\epsilon, \tau)$ and $R(\tilde{c}_\epsilon, \tau)$, respectively.

Now, consider the annulus

$$\mathcal{A} = \alpha_\epsilon \times [0,\ \tau] \cup \tilde{a} \times [0,\ \tau] \cup \tilde{B}\tilde{Q}\tilde{P}\tilde{C} \cup \tilde{B}\tilde{Q}\tilde{P}\tilde{C}(\tau),$$

where $\tilde{B}\tilde{Q}\tilde{P}\tilde{C}$ is the quadrilateral contained in $M(0)$, with sides $\tilde{a},\ \tilde{b}_\epsilon, \alpha_\epsilon$ and $\tilde{c}_\epsilon$, and $\tilde{B}\tilde{Q}\tilde{P}\tilde{C}(\tau) \subset M(\tau)$ is $\tilde{B}\tilde{Q}\tilde{P}\tilde{C}$ raised to height τ. See Figure 4. We claim that if τ is sufficiently large the annulus $\mathcal{A}$ has area smaller than the sum of the areas of the disks $\tilde{b}_\epsilon \times [0, \tau]$ and $\tilde{c}_\epsilon \times [0, \tau]$.

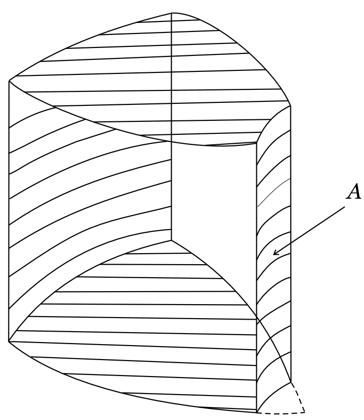

Figure 4 – Annulus $\mathcal{A}$ and Curves $R(\tilde{b}_\epsilon, \tau)$ and $R(\tilde{c}_\epsilon, \tau)$.

In fact, $\operatorname{area}(\mathcal{A}) \leq \operatorname{area}\Big((\tilde{b}_\epsilon \times [0, \tau]) \cup (\tilde{c}_\epsilon \times [0, \tau]) \Big)$ is equivalent to

$$\begin{aligned} \|\alpha_\epsilon\|\tau + [\ \|a\| + 2\delta\ ]\,\tau + 2\ \operatorname{area}(\tilde{B}\tilde{Q}\tilde{P}\tilde{C}) &\leq \Big(\|\tilde{c}\| + \|\tilde{b}\| - 2\epsilon \Big)\,\tau \\ \Leftrightarrow \Big(\ \|\tilde{c}\| + \|\tilde{b}\| - 2\epsilon - \|\alpha_\epsilon\| - \|a\| - 2\delta \Big)\,\tau &> 2\ \operatorname{area}(\tilde{B}\tilde{Q}\tilde{P}\tilde{C}), \end{aligned}$$

and area$(\tilde{B}\tilde{Q}\tilde{P}\tilde{C})$ does not depend of τ, when ϵ and δ are sufficiently small, and consequently $\|\alpha_\epsilon\|$ is small too. So, if τ is sufficiently large, to prove the assertion we have to show that

$$\|\tilde{c}\| + \|\tilde{b}\| - \|a\| > 0.$$

Observe that M is a metric space and $\tilde{b}$ and $\tilde{c}$ are geodesic curves, so the triangle inequality holds. Then

$$\begin{aligned} \|a\| < \|\tilde{a}\| &= \text{dist}\,(\tilde{B}, \tilde{C}) \\ &\leq \text{dist}\,(\tilde{B}, A) + \text{dist}\,(A, \tilde{C}) \\ &= \|\tilde{c}\| + \|\tilde{b}\|. \end{aligned}$$

Hence, area$(\mathcal{A})$ is smaller than the sum of the areas of the disks $\tilde{b}_\epsilon \times [0, \tau]$ and $\tilde{c}_\epsilon \times [0, \tau]$, and by the Douglas criteria, there exists a least area minimal annulus $A(\delta, \tau)$ with boundary $R(\tilde{b}_\epsilon, \tau) \cup R(\tilde{c}_\epsilon, \tau)$.

The annulus $A(\delta, \tau)$ is above Σ_t for all $t > 0$, that is, if a vertical geodesic in int$(\Delta \times \mathbb{R})$ meets both surfaces, then the point of $A(\delta, \tau)$ is above the points of Σ_t. To see this, we translate vertically $A(\delta, \tau)$ to height t, and then one lowers it continuously. By the classical maximum principle there does not exist interior points between the surfaces until $A(\delta, \tau)$ reaches the original position. Moreover, as δ goes to 0, the same argument shows that the annulus $A(\tau) := A(0, \tau)$ is above Σ_t. The boundary maximum principle assures that at each interior point of the vertical geodesics $B \times [0, \tau]$ and $C \times [0, \tau]$ the tangent planes to $A(\tau)$ and Σ_t are not parallel. As $A(\tau)$ is above Σ_t, we can say that in each of these points the angle between the tangent plane to $A(\tau)$ and the geodesic plane containing $\tilde{b}_\epsilon \times [0, \tau]$ or $\tilde{c}_\epsilon \times [0, \tau]$ is larger than the angle between this last plane and the tangent plane to Σ_t. Therefore the annulus $A(\tau)$ is an upper barrier for the sequence u_t.

We claim that the horizontal projections of the annulus $\mathcal{A}(\tau)$ is an exhaustion for $\bar{\Delta}$, when τ goes to $+\infty$. Consequently, for all compact set $K \subset \pi(A(\tau)) \subset \Delta$ and for all $t \in \mathbb{R}$, there exists an upper barrier for Σ_t. So there exists a function u defined on $\bar{\Delta} - \bar{a}$ such that graph u is minimal in $\Delta \times \mathbb{R}$,

$$u = \lim_{t \to \infty} u_t,$$

and

$$u\big|_b = u\big|_c = 0, \qquad \lim_{x \to \text{int}(a)} u(x) = +\infty\ .$$

Now, we prove that $\pi(A(\tau))$ is an exhaustion of $\bar{\Delta}$. Let Ω be the non-compact connected component of $\Delta \times \mathbb{R} - \text{int}(A(\tau))$. As, for all $k > \tau$, $\partial\Omega =$

$\partial(\bar{\Delta} \times \mathbb{R}) \cup A(\tau)$ is piecewise smooth mean-convex, there exists a least area connected minimal surface $A(k) \subset \Omega \subset \bar{\Delta} \times \mathbb{R}$ with boundary $R(b_\epsilon, k) \cup R(c_\epsilon, k)$.

Translating vertically $A(\tau)$ to height $\tau - k$, by the maximum principle, one guarantees that $A(\tau)$ and $A(k)$ have no common interior points, and they are not tangent at boundary points. So, when k goes to $+\infty$, the angle the tangent plane of $A(k)$ makes along the vertical boundary segments is controlled by that of $A(\tau)$.

For each $n > \tau$, denote by $N(n)$ the surface $A(2n)$ translated down a distance n. As each $N(n)$ is stable, one has uniform local area bounds, and uniform curvature estimates [Sc]. So, there exists a subsequence of $N(n)$, $n > \tau$, converging to a minimal surface $N(\infty) \subset \Delta \times \mathbb{R}$. By the classical maximum principle, $A(\tau)$ can be translated up to $+\infty$ and down to $-\infty$ without ever touching $N(\infty)$ in interior points. Then $N(\infty)$ has a connected component N whose boundary is the union of the vertical geodesics $B \times \mathbb{R}$ and $C \times \mathbb{R}$. We prove that $N = \tilde{a} \times \mathbb{R}$ which means that the compact sets in the vertical projection of $N(n)$ on Δ exhaust Δ.

At first, let us parametrize the sides $\tilde{b}$ and $\tilde{c}$ of $\tilde{\Delta}$ by the same parameter t, $t \in [0, 1]$ such that $\tilde{b}(0) = \tilde{C}$, $\tilde{c}(0) = \tilde{B}$ and $\tilde{b}(1) = \tilde{c}(1) = A$.

Consider $\{C_t\}_{0 \leq t \leq 1}$ a set of curves where $C_t = \tilde{b}[0, t] \cup \tilde{c}[0, t] \cup \tilde{\gamma}_t$ and $\tilde{\gamma}_t$ is the unique minimizing geodesic of $\tilde{\Delta}$ joining $\tilde{b}(t)$ and $\tilde{c}(t)$. This set is a foliation of $\tilde{\Delta}$ by geodesics, with $C_1 = A$, and $C_0 = \tilde{a}$ and, for each $t \in [0, 1]$, $C_t \times \mathbb{R}$ is the union of three minimal surfaces in $M \times \mathbb{R}$, and the angle between these surfaces is smaller than π. Moreover, $\partial(C_t \times \mathbb{R}) = (\tilde{B} \times \mathbb{R}) \cup (\tilde{C} \times \mathbb{R})$. Letting t goes to 0, these surfaces can not touch N, since N would be C_t by the maximum principle. Therefore, either $N = a \times \mathbb{R}$ or there is a largest positive $t_0 > 0$ such that N is asymptotic to $C_{t_0} \times \mathbb{R}$ at infinity.

Suppose, by contradiction, that the latter case happens, i.e., for some $0 < t_0 < 1$ there is a sequence $x_n \in N \cap M(n)$ such that $\operatorname{dist}(x_n, C_{t_0} \times \mathbb{R})$ goes to 0 when n goes to ∞. Denote by $S(n)$ the surface N vertically translated in order to the height of x_n becomes zero. By the same argument used for $N(n)$, $n > \tau$, we can claim that a subsequence of $S(n)$, $n \geq 1$, converges to a minimal surface S. Moreover, by the hypotheses, S touches $C_{t_0} \times \mathbb{R}$ at some interior point at height zero. Then $S = C_{t_0} \times \mathbb{R}$. Now, let $K \subset C_{t_0} \times \mathbb{R}$ be a compact domain such that the distance between K and $\partial(C_{t_0} \times \mathbb{R})$ is positive and the projection of K in $\tilde{\Delta}$ contains points of $\tilde{\Delta} - \Delta$. As $S = C_{t_0} \times \mathbb{R}$, we can say that there exists domains in $N(n)$ that converge uniformly to K when n goes to ∞. So there exist points of $N(n)$ with vertical projection in $\tilde{\Delta} - \Delta$. This is impossible since $N(n)$ is a vertical translation of $A(2n)$ whose vertical projection is contained in Δ. This shows that the horizontal projections of $\mathcal{A}(\tau)$ forms a exhaustion of Δ, as we claimed.

To finish the proof of Theorem we need to show that given a sequence of points $z_n \in \operatorname{int}(\Delta)$ such that $z_n \to z \in \operatorname{int}(a)$ we have

$$|\nabla u(z_n)| \overset{n\to+\infty}{\longrightarrow} +\infty\,.$$

For each positive integer large enough n, let $\gamma_n = u(\Delta) \cap M(n)$ and $\varphi_n = \pi(\gamma_n)$, where π is the vertical projection of $\Delta \times \mathbb{R}$ on Δ. Denote by Δ_n the connected part of Δ bounded by φ_n, b and c. Consider a sequence of points $z_n \in \Delta$ with $z_n \in \varphi_n$, for all n.

Let $\partial/(\partial t)$ be the vertical vector in $\Delta \times \mathbb{R}$, ν_n be the outward unit conormal to the boundary $\Sigma_{\Delta_n} = \operatorname{graph} u_{|\Delta_n}$ and N_n be the unit normal to the Σ_{Δ_n}, such that $\langle N_n, \partial/(\partial t)\rangle \geq 0$.

At each point $p \in \partial\Sigma_{\Delta_n}$, we consider the basis $\beta = \{\gamma_n', \nu_n, N_n\}$ of $T_p(\Delta \times \mathbb{R})$, where γ_n' is a unit tangent vector to γ_n. See Figure 5.

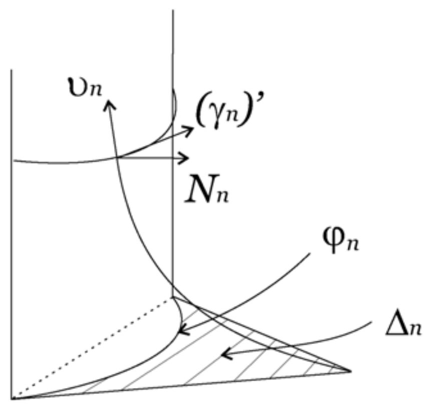

Figure 5 – Basis β.

At points of Σ_{Δ_n}, we have

$$\frac{\partial}{\partial t} = d\gamma_n' + e\nu_n + fN_n,$$

where $d,\ e,\ f \in \mathbb{R}$. The curve γ_n is horizontal, then $\left\langle \partial/(\partial t), \gamma_n' \right\rangle = 0$. Therefore $d = 0$ and $\partial/(\partial t) = e\nu_n + fN_n$. Moreover, as $\langle \nu_n, N_n\rangle = 0$ and $|\nu_n| = 1$, one has

$$\left\langle \frac{\partial}{\partial t}, \nu_n \right\rangle = e \quad \text{and} \quad \left\langle \frac{\partial}{\partial t}, N_n \right\rangle = f.$$

So

$$\frac{\partial}{\partial t} = \left\langle \frac{\partial}{\partial t}, \nu_n \right\rangle \nu_n + \left\langle \frac{\partial}{\partial t}, N_n \right\rangle N_n,$$

or

$$\left\langle \frac{\partial}{\partial t}, \nu_n \right\rangle \nu_n = \frac{\partial}{\partial t} - \left\langle \frac{\partial}{\partial t}, N_n \right\rangle N_n.$$

Finally, as $\left|\frac{\partial}{\partial t}\right|^2 = 1$,

$$\left\langle \frac{\partial}{\partial t}, \nu_n \right\rangle = \sqrt{1 - \left\langle \frac{\partial}{\partial t}, N_n \right\rangle^2}. \tag{4}$$

Assertion 4.2. *Let $u \colon \Delta \to \mathbb{R}$ be a minimal solution in $\Delta \times \mathbb{R}$ which converges to infinity as one approaches an open geodesic arc $a \in \partial\Delta$. Then the tangent plane to graph u approaches the vertical as one converges to a.*

Let $z_n \in \Delta$ be a sequence of points as before. To prove the assertion it is sufficient to show that the tangent plane at $p_n = (z_n, u(z_n))$ is almost vertical when n goes to ∞.

First, we extend ν_n to the interior points of Σ_{Δ_n}, that is we define on Σ_{Δ_n} the outward conormal ν_τ in $\gamma_\tau \subset \Sigma_{\Delta_n}$, $0 < \tau \leqslant n$. By the previous argument, we have

$$\left\langle \frac{\partial}{\partial t}, \nu_\tau \right\rangle = \sqrt{1 - \left\langle \frac{\partial}{\partial t}, N_\tau \right\rangle^2}.$$

Observe that at points of Σ_{Δ_n} where the tangent plane is almost vertical, the vertical projection of N_n must be almost zero. By (4), this means that $\langle \partial/(\partial t), \nu_n \rangle$ approaches 1. Then the tangent plane at $p_n \in \Sigma_{\Delta_n}$ is almost vertical when n goes to ∞ if, and only if, for each $\epsilon > 0$ and $q \in a$, there exists a neighborhood of a in Δ such that

$$\left\langle \frac{\partial}{\partial t}, \nu_n \right\rangle > 1 - \epsilon, \tag{5}$$

at each point of the neighborhood, for n sufficiently large.

Suppose, by contradiction, that (5) does not hold. Then $\exists\, q \in \operatorname{int}(a)$ and $\exists\, \delta > 0$ such that $\forall\, n \in \mathbb{N}$, $\exists\, z_n \in \Delta$, with $z_n \overset{n\to+\infty}{\longrightarrow} q$ and $\exists\, \tilde{n} > n$, such that

$$\left\langle \frac{\partial}{\partial t}, \nu_{\tilde{n}} \right\rangle \leq 1 - \delta.$$

Denote by $D(p_n, R)$ the open disk of center p_n and radius R in Σ_{Δ_n}. It is possible to choose a number $R > 0$, independent on n, such that $D(p_n, R) \subset \Sigma_{\Delta_n}$, where $p_n = (z_n, u(z_n))$, and R is the intrinsic radius. In fact, as $q \in \mathrm{int}(a)$, one has $\mathrm{dist}(p_n, \partial\Sigma_{\Delta_n}) >> 0$, for all n. We use again curvature estimates for stable minimal surfaces to guarantee that Σ_{Δ_n} is a graph on a disk $D(p_n, r) \subset T_{p_n}\Sigma_{\Delta_n}$ and this graph has bounded distance from $D(p_n, r)$. Moreover, r depends only on R and, consequently, it is independent of n. Hence, if z_n is close enough to a, the vertical projection of $D(p_n, r)$ is out of Δ. But, as the distance between Σ_{Δ_n} and $D(p_n, r)$ is bounded, the vertical projection of Σ_{Δ_n} is out of Δ too. This is a contradiction.

Therefore, the assertion holds and the proof of theorem is finished. □

5 Proof of Theorem 1.1

First we prove that the condition (1) is sufficient for the existence of u. This proof is divided in five cases. Our argument is analogous to [JS].

Case 1. ∂D contains just one geodesic arc A and one strictly convex arc C. The function $f : C \to \mathbb{R}$ is continuous and positive.

The surface constructed in this case is a generalization of the Scherk type surface given by Theorem 1.4. In fact, now the boundary of the domain contains a non-geodesic arc and on this arc the function f can take positive values. Later we prove that on strictly convex arcs infinite values are not possible. The argument in the proof of the first case is analogous to the proof of Theorem 1.4.

Proof of Case 1. Let $n \in \mathbb{R}$, $n > 0$. Consider $\Gamma_n \subset \partial(D \times \mathbb{R})$ the curve that is the union of the following arcs: the geodesic arc A raised to height n, the graph of the function $\min(n, f)$ and the vertical geodesic segments joining the endpoints of the curves just described.

Let Σ_n be the graph of the function $u_n : D \to \mathbb{R}$ that is a solution of the Plateau problem in $D \times \mathbb{R}$ with boundary Γ_n. By the general maximum principle (Theorem 1.3), $\{u_n\}$ is a nondecreasing sequence. Let us prove that $\{u_n\}$ is uniformly bounded on each compact set $K \subset D - A$. So we will construct an upper barrier for Σ_n, for all n, using the Douglas criteria.

Let $\tilde{A} \subset M(0)$ be a geodesic arc extending A, whose endpoints $\tilde{P}$ and $\tilde{Q}$ are at a small distance δ from A, so $\|\tilde{A}\| = \|A\| + 2\delta$. Let $\tilde{C}$ a strictly convex arc, parallel to C, joining $\tilde{P}$ to $\tilde{Q}$ (thus $\mathrm{dist}(\tilde{C}, C) = \delta$), M be the midpoint of $\tilde{C}$ and $\tilde{D}$ be the region bounded by $\tilde{A}$ and $\tilde{C}$. Consider $\tilde{E}$, $\tilde{F} \in \tilde{C}$ at a same small distance $\epsilon > 0$ from M. Now, consider the following curves: α_ϵ, β_ϵ be minimizing geodesics of $\tilde{D}$ joining $\tilde{P}$ to $\tilde{E}$ and $\tilde{Q}$ to $\tilde{F}$, respectively; $\tilde{\alpha}_\epsilon$, $\tilde{\beta}_\epsilon$ subarcs of $\tilde{C}$, bounded by $\tilde{P}$, $\tilde{E}$ and $\tilde{Q}$, $\tilde{F}$, respectively; $\tilde{\alpha}$, $\tilde{\beta}$ subarcs of $\tilde{C}$, such that $\tilde{C} = \tilde{\alpha} \cup \tilde{\beta}$, $\tilde{\alpha} \cap \tilde{\beta} = \{M\}$; α, β minimizing geodesic joining $\tilde{P}$ to M and $\tilde{Q}$ to M, respectively; and finally, $\tilde{\gamma}_\epsilon \subset \tilde{D}$ a minimizing geodesic joining $\tilde{E}$ to $\tilde{F}$. Figure 6 can help us.

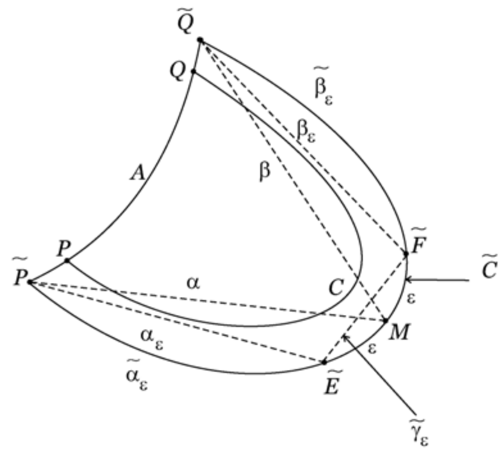

Figure 6 – Region $\tilde{D}$.

For fixed $t > 0$, denote by $\tilde{R}(\tilde{\alpha}_\epsilon, t)$ the boundary of $D_{\tilde{\alpha}_\epsilon} := \tilde{\alpha}_\epsilon \times [0, t]$ and by $\tilde{R}(\tilde{\beta}_\epsilon, t)$ the boundary of $D_{\tilde{\beta}_\epsilon} := \tilde{\beta}_\epsilon \times [0, t]$. Let $\tilde{D}_{\tilde{\alpha}_\epsilon}$ and $\tilde{D}_{\tilde{\beta}_\epsilon}$ be the disks that are solutions to the Plateau problem with boundary $\tilde{R}(\tilde{\alpha}_\epsilon, t)$ and $\tilde{R}(\tilde{\beta}_\epsilon, t)$, respectively.

We want to construct a minimal annulus $S(\delta, t) \subset \tilde{D} \times \mathbb{R}$ with $\partial S(\delta, t) = \tilde{R}(\tilde{\alpha}_\epsilon, t) \cup \tilde{R}(\tilde{\beta}_\epsilon, t)$.

Consider the annulus

$$\mathcal{N} = \tilde{A} \times [0, t] \cup \tilde{\gamma}_\epsilon \times [0, t] \cup \tilde{P}\tilde{E}\tilde{F}\tilde{Q} \cup \tilde{P}\tilde{E}\tilde{F}\tilde{Q}(t),$$

where $\tilde{P}\tilde{E}\tilde{F}\tilde{Q}$ is the quadrilateral, contained in $M(0)$, whose sides are the curves $\tilde{A}$, $\tilde{\alpha}_\epsilon$, $\tilde{\gamma}_\epsilon$ and $\tilde{\beta}_\epsilon$, and $\tilde{P}\tilde{E}\tilde{F}\tilde{Q}(t) \subset M(t)$ is $\tilde{P}\tilde{E}\tilde{F}\tilde{Q}$ raised to height t. One has $\partial\mathcal{N} = \tilde{R}(\tilde{\alpha}_\epsilon, T) \cup \tilde{R}(\tilde{\beta}_\epsilon, T)$.

The area of the annulus $\mathcal{N}$ is given by

$$\text{area}(\tilde{\mathcal{N}}) = \|\tilde{A}\|t + \|\tilde{\gamma}_\epsilon\|t + 2\,\text{area}(\tilde{P}\tilde{E}\tilde{F}\tilde{Q}).$$

By the Douglas criteria, we must show that $\text{area}(\mathcal{N}) < \text{area}(\tilde{D}_{\tilde{\alpha}_\epsilon} \cup \tilde{D}_{\tilde{\beta}_\epsilon})$ to assure that $S(\delta, T)$ exists.

Let $D_{\alpha_\epsilon} := \alpha_\epsilon \times [0, t]$. By the Co-area formula, we guarantee that

$$\text{area}\left(D_{\alpha_\epsilon} \cup D_{\beta_\epsilon}\right) \leq \text{area}\left(\tilde{D}_{\tilde{\alpha}_\epsilon} \cup \tilde{D}_{\tilde{\beta}_\epsilon}\right).$$

So, if we show that

$$\text{area}(\mathcal{N}) < \text{area}\left(D_{\alpha_\epsilon} \cup D_{\beta_\epsilon}\right),$$

then $S(\delta, t)$ exists. That is, we should prove

$$\left[\|\alpha_\epsilon\| + \|\beta_\epsilon\| - (\|A\| + 2\delta + \|\tilde{\gamma}_\epsilon\|)\right] t > 2\ \text{area}(\tilde{P}\tilde{E}\tilde{F}\tilde{Q}).$$

As $\text{area}(\tilde{P}\tilde{E}\tilde{F}\tilde{Q})$ is constant, then if t is large and ϵ, δ are small enough, it is sufficient to show that $\|\alpha\| + \|\beta\| - \|A\| > 0$. This follows from the triangle inequality in M, since α, β and A are geodesics. Hence there exists an annulus $S(\delta, t)$, above Σ_n for all $n > 0$, with boundary $\tilde{R}(\tilde{\alpha}_\epsilon, t) \cup \tilde{R}(\tilde{\beta}_\epsilon, t)$. As $\delta > 0$, one has $\partial S(\delta, t) \cap \partial\Sigma_n = \emptyset$. Letting δ go to 0, we obtain that the annulus $S(t) = S(0, t)$ is above Σ_n, for all n. In fact, by the boundary maximum principle, at each interior point of $\partial S(t) \cap \partial\Sigma_n$ the tangent planes to $S(t)$ and Σ_n are not parallel. This means that $S(t)$ is above Σ_n, for all n, and consequently u_n is uniformly bounded on each compact set $K \subset \pi(S(t))$. So, $u = \lim_{n\to\infty} u_n$ exists on each $K \subset \pi(S(t))$. By the same argument used in the proof of Theorem 1.4 these compact sets exhaust $D - A$, when $t \to +\infty$, and there exists a function $u\colon D \to \mathbb{R}$ such that

$$u_{|A} = +\infty, \quad u_{|C} = \lim_{n\to\infty} \min(f, n) = f$$

as we desired. It concludes the proof of Case 1.

Remark 5.1. Let $D \subset M$ be an admissible domain and let $C \subset \partial D$ be a strictly convex curve. Denote by $\mathcal{C}(C)$ the open convex-hull of C. If $g\colon \mathcal{C}(C) \cup C \to \mathbb{R}$ is a minimal solution whose values on C are bounded, the proof of Case 1 shows that g is bounded on all compact sets $K \subset \mathcal{C}(C)$. In fact, consider the geodesic arc A joining the endpoints of C. By the proof of Case 1 there exists a Scherk-type surface u_+ defined on $\mathcal{C}(C)$, such that $u_{+|A} = +\infty$, $u_{+|C} = g_{|C}$ and for all compact $K \subset \mathcal{C}(C)$, u_+ is above the graph of g.

Assertion 5.1. *Let $\mathcal{C}(C)$ be the open convex-hull of a strictly convex curve C, and let $g\colon \mathcal{C}(C) \to \mathbb{R}$ be a minimal solution. If g is unbounded on C, then g is unbounded on $\mathcal{C}(C)$.*

In fact, since $g_{|C} = +\infty$, we can assume that $g \geq 0$ on $\mathcal{C}(C)$. Suppose, by contradiction, that there exists a point $p \in \mathcal{C}(C)$ such that $g(p) < +\infty$. Let u_- be a Scherk surface on $\mathcal{C}(C)$ with $u_{-|C} = 0$ and $u_{-|A} = -\infty$, where A is the geodesic joining the endpoints of C. One has $g > u_-$ on $\mathcal{C}(C)$. As $u_{-|\mathcal{C}(C)}$ is bounded, we can translate vertically up u_- until it touches graph of g at $(p, g(p))$. This is impossible by the classical maximum principle and hence there not exist such a point $p \in \mathcal{C}(C)$.

In order to continue the proof of Theorem 1.1, we need some preliminary results. Let $g\colon \overline{D} \to \mathbb{R}$ be a minimal solution, where $D \subset M$ is an admissible domain. Denote the graph of g by Σ and suppose that $g_{|\partial D}$ is bounded. Let us define $\nu_g(p)$ as an outward unit conormal vector at $p \in \partial\Sigma$ in a classic way, i.e., $\nu_g(p) \in T_p\Sigma$ and $\nu_g(p) \perp T_p(\partial\Sigma)$. Here $(\nu_3)_g$ is the component of ν_g in the $\partial/(\partial t)$-direction. We will establish some results about $(\nu_3)_g$.

Assertion 5.2. *Let $A \subset \partial D$ be an open geodesic arc. Then $|(\nu_3)_g(p)| < 1$, for all $p = (z, g(z)) \in \partial\Sigma$, where $z \in A$.*

Suppose, by contradiction, that there exists a point $p = (z, g(z)) \in \partial\Sigma$, with $z \in A$, such that $|(\nu_3)_g(p)| = 1$. This means that the tangent plane to Σ is vertical at p. So $T_p\Sigma$ and $T_p(A \times \mathbb{R})$ are vertical and parallel and the surface Σ is in the same side of $A \times \mathbb{R}$. This is impossible by the boundary maximum principle and the assertion is proved.

Remark 5.2. By a similar argument, this assertion holds if $p = (z, g(z))$, when z belongs to a strictly convex arc $C \in \partial D$.

Lemma 5.1. *Let $C \subset \partial D$ be a strictly convex arc. Then*

$$\int_C (\nu_3)_g ds < \|C\|$$

Proof. This is a consequence of the Remark 5.2. □

Assertion 5.3. $\displaystyle\int_{\partial\Sigma} (\nu_3)_g ds = 0.$

As the height function is harmonic on Σ [Ro], using Stokes theorem we have

$$0 = \int_\Sigma \Delta h \, dV_\Sigma = \int_{\partial\Sigma} \langle \nabla_\Sigma h, \ \nu \rangle \, dV_{\partial\Sigma},$$

where ν is the outward unit conormal on $\partial\Sigma$.

As $\tilde{\nabla} h = \nabla_\Sigma h + \left\langle N, \tilde{\nabla} h \right\rangle N$ and $\tilde{\nabla} h = \dfrac{\partial}{\partial t}$, one has

$$\begin{aligned} 0 &= \int_{\partial\Sigma} \left\langle \frac{\partial}{\partial t} - \left\langle N, \frac{\partial}{\partial t} \right\rangle N, \nu \right\rangle dV_{\partial\Sigma} \\ &= \int_{\partial\Sigma} \left\langle \frac{\partial}{\partial t}, \nu \right\rangle dV_{\partial\Sigma}. \end{aligned}$$

This concludes the proof of the assertion.

Lemma 5.2. *Let $A \subset \partial D$ be a geodesic arc and let $\{g_n\}$ be a sequence of minimal solutions on an open domain which are continuous on $D \cup A$. Denote by ν_n the unit outward conormal vector to the boundary of the graph of g_n, for each n. Then*

(i) *If $\{g_n\}$ diverges uniformly to infinity on compact subsets of A and remains uniformly bounded on compact sets of D, then*

$$\lim_{n\to\infty} \int_A (\nu_3)_n ds = \|A\|.$$

(ii) *If $\{g_n\}$ diverges uniformly to infinity on compact subsets of D and remains uniformly bounded on sets of A, then*

$$\lim_{n\to\infty} \int_A (\nu_3)_n ds = -\|A\|.$$

Proof.

(i) Let $\delta > 0$ be a fixed small number and A_δ be a subarc of A whose distance from ∂A is at least δ.

As g_n goes to ∞ when n goes to ∞ on A_δ, one has, for each n large enough, $(\nu_3)_n > 0$ on A_δ. By Assertion 5.2, one has $(\nu_3)_n < 1$. So

$$\lim_{n\to\infty} \int_{A_\delta} (\nu_3)_n ds \;\leq\; \lim_{n\to\infty} \int_{A_\delta} 1\; ds = \|A_\delta\|. \tag{6}$$

On the other hand, for each n, the tangent plane to graph g_n at points whose vertical projection belongs to A_δ is almost vertical when n goes to $+\infty$. Hence for all $\epsilon > 0$ small and n large, we have $|(\nu_3)_n| > (1 - \epsilon)$ on A_δ.

Consequently,

$$\lim_{n\to\infty} \int_{A_\delta} (\nu_3)_n ds \;\geq\; \lim_{n\to\infty} \int_{A_\delta} (1-\epsilon)\; ds\;,$$

and for $\epsilon \to 0$,

$$\lim_{n\to\infty} \int_{A_\delta} (\nu_3)_n ds \;\geq\; \|A_\delta\|. \tag{7}$$

Now, when δ goes to 0 and A_δ goes to A, (6) and (7) imply that

$$\lim_{n\to\infty} \int_{A} (\nu_3)_n ds = \|A\|.$$

(ii) The proof is analogous to (i) if we observe that now $(\nu_3)_n < 0$ on A_δ. $\square$

Remark 5.3. By the same argument used in the proof of the previous lemma, we can prove the following fact: Let $\{g_n\}$ be a monotone sequence of minimal solutions on D, and let V be a compact subset of D. If $\{g_n\}$ diverges uniformly on V and converges uniformly on $D - V$, then

$$\lim_{n\to\infty} \int_{A} (\nu_3)_n ds = -\;\|A\|,$$

on each geodesic arc $A \subset \partial V$.

Case 2. ∂D contains geodesic arcs $A_1, \ldots, A_k$ and strictly convex arcs $C_1, \ldots, C_h$. We suppose $f_s : C_s \to \mathbb{R}$ are continuous and bounded below.

Proof of Case 2. For each $n \in \mathbb{R}$, let Γ_n be the closed curve obtained by the union of the curves $A_i(n)$, $i \in \{1, \ldots n\}$, graph $\{\min(n, f_s), s \in \{1, \ldots, h\}$, and the vertical segments on the vertices of D such that Γ_n is a Nitsche graph. By Theorem 1.2, there exists a function u_n whose graph, denoted by Σ_n, is minimal with boundary Γ_n.

The following result is an interesting one on its own. The notation is the same as above.

Lemma 5.3. *Let $p \in D$. If the sequence $u_n(p)$ is bounded, then $\left|\nabla u_n(p)\right|$ is bounded.*

Proof. Let $B = B(p, \epsilon)$ be a geodesic ball with center p and radius $\epsilon > 0$. For each $v \in V = \{v \in T_pD;\ |v| = 1\}$ consider $\gamma_v \subset D$ the geodesic curve with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Since B is an admissible domain, from the prove of Assertion 2.1 we have that γ_v intersects ∂B exactly in two points. So γ_v divides B in two connected components.

Consider $\{\gamma_{vt};\ 0 < t \leq 1\}$ a foliation, by geodesic arcs, of one of these components with $\gamma_{v1} = \gamma_v$. Again, for each $t \in (0, 1)$, γ_{vt} divides B in two connected components. Denote by Δ_{vt} the component of B bounded by γ_{vt} such that $\gamma_v \subset int\ \Delta_{vt}$. See Figure 7.

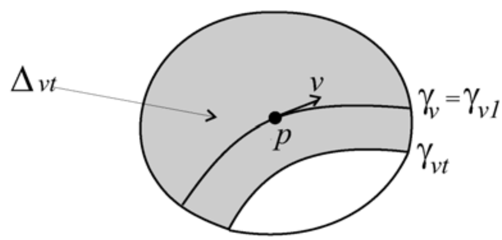

Figure 7 – Domain Δ_{vt}.

For $t \in (0, 1)$, let $\Sigma_{vt} = \text{graph}\,\phi_{vt}$ be a Scherk surface defined on $\Delta_{vt} \subset B$ with boundary values

$$\phi_{vt}|_{\gamma_{vt}} = \infty \quad \text{and} \quad \phi_{vt}|_{\partial D \cap \partial \Delta_{vt}} \equiv 0.$$

Letting δ small and fixed, we change $t \in [\delta, 1-\delta]$ and $v \in V$ continuously in order to have a 2-parameter compact continuous family $\mathcal{F}_{vt}$ of Scherk surfaces [RS].

Now, for each fixed $n \in \mathbb{R}$, let x and y be local coordinates in B, with p as origin,

$$\frac{\partial u_n}{\partial x}(p) > 0 \quad \text{and} \quad \frac{\partial u_n}{\partial y}(p) = 0.$$

So $\Sigma_n \cap M(u_n(p))$ is a horizontal curve tangent to the y direction.

Fix $v \in V$ and let δ be as before. The same argument used to prove Assertion 4.2 shows that the tangente plane π_{vt} at the point $(p, \phi_{vt}(p)) \in \Sigma_{vt}$, with p near to γ_{vt}, is almost vertical. Then there exists a smaller $t_0 \in (0, 1-\delta]$ such that $\forall\, t \in [t_0, 1-\delta]$ the intersection of $\Sigma_{vt} \cap (M(\phi_{vt}(p)))$ is a connected curve with endpoints contained in the vertical segments of $\Delta_{vt} \times \mathbb{R}$. Moreover this curve intersects $\gamma_v(\phi_{vt}(p))$ only at $(p, \phi_{vt}(p))$, where $\gamma_v(\phi_{vt}(p))$ denote the curve γ_v raised to height $\phi_{vt}(p)$. Choosing $\tilde{v} \in V$ such that $\tilde{v}$ corresponds to the y direction, one has

$$\frac{\partial u_n}{\partial y}(p) = \frac{\partial \phi_{\tilde{v}t}}{\partial y}(p) = 0,\ \forall\, t \in [t_0, 1-\delta]\ .$$

Suppose that $u_n(p) < \phi_{\tilde{v}t}(p)$ for all $t \in [t_0, 1-\delta]$,

Assertion 5.4. $\dfrac{\partial u_n}{\partial x}(p) < \dfrac{\partial \phi_{\tilde{v}t_0}}{\partial x}(p)$.

Suppose, by contradiction, that $\dfrac{\partial u_n}{\partial x}(p) \geq \dfrac{\partial \phi_{\tilde{v}t_0}}{\partial x}(p)$.

If $\dfrac{\partial u_n}{\partial x}(p) = \dfrac{\partial \phi_{\tilde{v}t_0}}{\partial x}(p)$, then $|\nabla u(p)| = |\nabla \phi_{\tilde{v}t_0}(p)|$.

We can translate vertically Σ_n up to $u_n(p)$ coincides with $\phi_{\tilde{v}t_0}(p)$. Now, $\Sigma_{\tilde{v}t_0}$ and Σ_n are tangent at the point $q = (p, \phi_{\tilde{v}t_0}(p)) = (p, u_n(p))$. So there exist at least two curves contained in the intersection of these graphs, which intersect transversally at q. If there exists a cycle α in $\Sigma_{\tilde{v}t_0} \cap \Sigma_n$, then α is the boundary of two minimal disks. By the classical maximum principle these disks are the same, what is impossible. As u_n is bounded and positive on B, the curves have bounded height and each branch must go to the vertical segments in $\Sigma_{\tilde{v}t_0}$. Because there are two segments and Σ_n is a graph on B, then two branches intersect a same vertical segment at the same point yielding again a cycle. This contradiction shows that $\frac{\partial u_n}{\partial x}(p) \neq \frac{\partial \phi_{\tilde{v}t_0}}{\partial x}(p)$.

Now, if $\frac{\partial u_n}{\partial x}(p) > \frac{\partial \phi_{\tilde{v}t_0}}{\partial x}(p)$, the angle between the tangent plane $T_{\tilde{v}t_0}$ and the horizontal direction is smaller than the angle between $T_p\Sigma_n$ and the horizontal direction.

By the Theorem 1.4, we have that $|\nabla \phi_{\tilde{v}t}(p)|$ goes to infinity when p approaches to $\gamma_{\tilde{v}}$, i.e., when t goes to 1. Then, for some $t_1 \in (t_0, 1)$, we have

$$\frac{\partial u_n}{\partial x}(p) < \frac{\partial \phi_{\tilde{v}t_1}}{\partial x}(p) \quad \text{and} \quad \frac{\partial u_n}{\partial y}(p) = 0 = \frac{\partial \phi_{\tilde{v}t_1}}{\partial y}(p).$$

Consequently, for some $\tilde{t} \in (t_0, t_1)$, one has

$$\frac{\partial u_n}{\partial x}(p) = \frac{\partial \phi_{\tilde{v}\tilde{t}}}{\partial x}(p) \quad \text{and} \quad \frac{\partial u_n}{\partial y}(p) = 0 = \frac{\partial \phi_{\tilde{v}\tilde{t}}}{\partial y}(p).$$

With the same argument used before, one obtains a contradiction and conclude the prove of the assertion.

Therefore we have

$$\frac{\partial u_n}{\partial x}(p) < \frac{\partial \phi_{\tilde{v}\tilde{t}}}{\partial x}(p) \quad \text{and} \quad |\nabla u_n(p)| < |\nabla \phi_{\tilde{v}\tilde{t}}(p)|.$$

For each function u_n, we constructed a Scherk surface $\phi_{\tilde{v}\tilde{t}}$ such that $|\nabla u_n(p)| < |\nabla \phi_{\tilde{v}\tilde{t}}(p)|$. As $\mathcal{F}_{vt}$ is a compact family and $|\nabla \phi_{vt}|$ is continuous, there is a Scherk surface $\phi \in \mathcal{F}_{vt}$ such that $|\nabla \phi_{vt}(p)| < |\nabla \phi(p)|$, for all $v \in V$ and $t \in (\delta, 1-\delta)$. So $|\nabla u_n(p)| < |\nabla\phi(p)|$ and the Lemma is proved. $\square$

Assertion 5.5. *The set $\mathcal{U} = \{p \in D;\ u_n(p)$ is a bounded sequence$\}$ is open.*

Let $p \in \mathcal{U}$. By curvature estimates for stable minimal surfaces [Sc], for each n, there is a neighborhood of $p_n = (p, u_n(p))$ where Σ_n is a graph with bounded gradient on a disk $D(p_n R) \subset T_{p_n}\Sigma_n$ whose radius R is independent of n. But the sequence $\nabla u_n(p)$ is bounded by the previous Lemma. Hence, there is a disk with fixed radius contained in the projection of each $D(p_n, R)$ over the horizontal plane and u_n is uniformly bounded on this disk. This shows that p is a interior point of $\mathcal{U}$, and the Assertion is proved.

By Remark 5.1, $\{u_n\}$ is uniformly bounded on compact sets contained in each open convex-hull $C(C_s)$, $s = 1, \ldots, h$. Hence, by the last assertion, a subsequence of $\{u_n\}$ (we will use the same notation) converges on the compact subsets of each open set $\mathcal{U} \subset D$ with $\bigcup_{s=1}^{h} C(C_s) \subset \mathcal{U}$. Moreover, $\{u_n\}$ diverges uniformly on the compact sets of the closed set $\mathcal{V} = \bar{D} - \mathcal{U}$. The next result shows that if $\mathcal{V}$ is not empty, it has special properties.

Lemma 5.4. *With the above notation, one has*

(i) *$\partial\mathcal{V}$ consists only of geodesic chords of D and parts of ∂D;*

(ii) *Two chords of $\partial\mathcal{V}$ can not have a common endpoint;*

(iii) *The endpoints of chords of $\partial\mathcal{V}$ are among the vertices of the geodesic arcs A_i;*

(iv) *A connected component of $\mathcal{V}$ can not consist only of an interior chord of D.*

Proof.

(i) Suppose, by contradiction, that there exists a strictly convex arc $C \subset \partial\mathcal{V}$. By Assertion 5.1, $\{u_n\}$ is unbounded on $C(C)$. On the other hand, as each connected component of $\mathcal{U}$ is convex, we have $C(C)$ contained in $\mathcal{U}$, and consequently, u_n is bounded in $C(C)$, what is a contradiction.

The same argument proves that vertices of $\partial\mathcal{V}$ can not be in D.

(ii) Suppose, by contradiction, that there exist arcs $L_1,\ L_2 \subset \partial\mathcal{V}$ with a common endpoint $q \in \partial D$. Let $Q_1 \in L_1$ and $Q_2 \in L_2$ be points such that the triangle T with vertices $Q,\ Q_1,\ Q_2$ belongs to D.

By Assertion 5.3,

$$\int_{\partial T} (\nu_3)_n ds = 0,$$

that is,

$$\lim_{n\to\infty}\left[\int_{\overline{Q_1 Q}} (\nu_3)_n ds + \int_{\overline{Q Q_2}} (\nu_3)_n ds\right] = -\lim_{n\to\infty}\int_{\overline{Q_2 Q_1}} (\nu_3)_n ds. \qquad (8)$$

We have either $T \subset \mathcal{U}$ or $T \subset \mathcal{V}$.

If $T \subset \mathcal{U}$, as $\overline{Q_1 Q}$ and $\overline{Q Q_2}$ are geodesic arcs, by Lemma 5.2(i) one has

$$\lim_{n\to\infty}\int_{\overline{Q_1 Q}\cup\overline{Q Q_2}} (\nu_3)_n ds = \|\overline{Q_1 Q}\| + \|\overline{Q Q_2}\|.$$

On the other hand, $-(\nu_3)_n < 1$ in $\overline{Q_2 Q_1}$. Then (8) implies that

$$\|\overline{Q_1 Q}\| + \|\overline{Q Q_2}\| < \|\overline{Q_2 Q_1}\|.$$

This is an absurd, because T is a triangle.

If $T \subset \mathcal{V}$, the equality (8) still holds. So an analogous argument works here.

(iii) Let $L \subset \partial\mathcal{V}$ be a geodesic arc of D with an endpoint $P \in \partial D$. Four situations are possible.

1. $P \in \operatorname{int} C_s$, for some s. In this case one has a subarc $L' \subset L \subset \partial\mathcal{V}$ such that $L' \subset C(C_s)$, where $\{u_n\}$ is bounded. Absurd, since $\{u_n\}$ is unbounded at $\partial\mathcal{V}$.
2. $P \in C_{s_1} \cap C_{s_2}$. Again, we have a subarc $L' \subset L \subset \partial\mathcal{V}$, where $\{u_n\}$ is bounded, with $L' \subset C(C_{s_1} \cup C_{s_2})$.
3. $P \in \operatorname{int} A_i$, for some i. Here we construct a triangle $T \subset D$ with vertices P, P_1, P_2, where $P_1 \in \partial\mathcal{V}$ and $P_2 \in A_i$. By a similar argument as used in (i), we obtain a contradiction.

(iv) By (iii), the endpoints of $\partial\mathcal{V}$ are among the endpoints of $\{A_i\}$. So, if $\mathcal{V}$ is the chord of D, with $P \in \partial\mathcal{V} \cap \partial\mathcal{D}$, we construct a triangle T with vertex P and the reasoning is the same as before. □

Assertion 5.6. $\mathcal{V} = \emptyset$.

Suppose, by contradiction, that for all admissible polygons $\mathcal{P}$ we have $2 \cdot \alpha < \gamma$, but there is no a function u as in the statement of the Theorem.

For all $s = 1, \ldots, h$, $C(C_s)$ is contained in $\mathcal{U}$ and, by the above Lemma, each component of $\mathcal{V}$ is bounded by a geodesic polygon $\mathcal{P}_{\mathcal{V}}$ with vertices among the endpoints of $\{A_i\}$. For this polygon, denote by $A_{i\mathcal{V}}$ the arcs of $A_i \subset \partial D$ that belong to $\mathcal{P}_{\mathcal{V}}$, $\gamma_{\mathcal{V}} =$ perimeter $\mathcal{P}_{\mathcal{V}}$ and $\alpha_{\mathcal{V}} = \sum_i \|A_{i\mathcal{V}}\|$.

By Assertion 5.3,

$$\int_{\mathcal{P}_{\mathcal{V}}} (\nu_3)_n ds = 0,$$

that is,

$$\int_{\bigcup_i A_{i\mathcal{V}}} (\nu_3)_n ds + \int_{\mathcal{P}_{\mathcal{V}} - \bigcup_i A_{i\mathcal{V}}} (\nu_3)_n ds \quad = \quad 0. \tag{9}$$

By Remark 5.3,

$$\lim_{n\to\infty} \int_{\mathcal{P}_{\mathcal{V}} - \bigcup_i A_{i\mathcal{V}}} (\nu_3)_n ds = -(\gamma_{\mathcal{V}} - \alpha_{\mathcal{V}}).$$

On the other hand, for all n, $|(\nu_3)_n| < 1$ on geodesic arcs. Then

$$\lim_{n\to\infty} \int_{\bigcup_i A_{i\mathcal{V}}} (\nu_3)_n ds \ \le \ \lim_{n\to\infty} \int_{\bigcup_i A_{i\mathcal{V}}} |(\nu_3)_n| ds \ \le \ \sum_i \|A_{i\mathcal{V}}\| = \alpha_{\mathcal{V}}.$$

Using (9), one has

$$\alpha_{\mathcal{V}} \ge \lim_{n\to\infty} \int_{\bigcup_i A_{i\mathcal{V}}} (\nu_3)_n ds = - \lim_{n\to\infty} \int_{\mathcal{P}_{\mathcal{V}} - \bigcup_i A_{i\mathcal{V}}} (\nu_3)_n ds = \gamma_{\mathcal{V}} - \alpha_{\mathcal{V}},$$

that is, $2\alpha_{\mathcal{V}} \ge \gamma_{\mathcal{V}}$, a contradiction.

Hence $\mathcal{V} = \emptyset$ and $\{u_n\}$ is uniformly bounded on each compact set $K \subset D$. Therefore $\{u_n\}$ converges to a function u defined on D, with boundary values as desired, and the proof of Case 2 is complete.

Remark 5.4. Two convex arcs C_s and $C_{\tilde{s}}$ contained in ∂D can have a common endpoint p. When this happens, it is clear by the proof of Case 2 that the minimal graph contains a vertical segment whose extreme points are the limit values of the continuous functions f_s and $f_{\tilde{s}}$ at p. The same argument used in Remark 5.1 assures that the function u is bounded on $C(C_1 \cup C_2)$.

Case 3. ∂D contains geodesic arcs $A_1, \ldots, A_k$ and convex arcs (not strictly convex) $C_1, \ldots, C_h$. Again, $f_s : C_s \to \mathbb{R}$ are continuous and positive.

The fundamental difference between Cases 2 and 3 is that now C_s may be a geodesic arc and $C(C_s) = C_s$. Consequently, it is not clear that $\{u_n\}$ is bounded on some open set $\mathcal{U} \subset D$. Moreover, ∂D is a polygon and using the same notation $\alpha = \Sigma_i \|A_i\|$ and $\gamma = \|\partial D\|$, we suppose that the condiction (1) holds for it. But, for this polygon we do not demand that the set of its vertices be contained in the set of endpoints of $\{A_i\}$.

Proof of Case 3. Consider $u_n : D \to \mathbb{R}$ a minimal solution with

$$u_n\Big|_{A_i} = n, \quad u_n\Big|_{C_s} = \min(n,\ f_s).$$

As before, let $\mathcal{U} = \big\{p \in D;\ u_n(p) < c, \forall\, n \in \mathbb{N},\ \text{for some constante c}\big\}$ and suppose that $\mathcal{U} = \emptyset$.

By Assertion 5.3,

$$\int_{\partial D} (\nu_3)_n ds \quad = \quad 0, \tag{10}$$

for each u_n.

Now, Remark 5.3 implies that

$$\lim_{n\to\infty} \int_{\cup_s C_s} (\nu_3)_n ds \quad = \quad -(\gamma - \alpha).$$

As $|(\nu_3)_n| < 1$ on each A_i, then

$$\lim_{n\to\infty} \int_{\cup_i A_i} (\nu_3)_n ds \quad \le \quad \alpha.$$

Using (10), we have $\alpha \ge \gamma - \alpha$ that is an absurd, because ∂D is an admissible polygon. So $\mathcal{U} \neq \emptyset$ and using the same argument of the proof of Case 2, we guarantee that $\mathcal{U} = D$ and conclude the proof of Case 3.

Case 4. ∂D contains geodesic arcs $A_1, \dots, A_k, B_1, \dots, B_l$ and convex arcs $C_1, \dots, C_h,\ h \ge 1$. The functions $f_s : C_s \to \mathbb{R}$ are continuous.

Proof of Case 4. By Case 3, we can find minimal solutions

$$u^+,\ u^- : D \to \mathbb{R},$$

such that

$$u^+\Big|_{A_i} = +\infty, \quad u^+\Big|_{B_j} = 0, \quad u^+\Big|_{C_s} = \max\{0,\ f_s\},$$

$$u^-\Big|_{A_i} = 0, \quad u^-\Big|_{B_j} = -\infty, \quad u^-\Big|_{C_s} = \min\{0,\ f_s\}.$$

On each C_s, let us define

$$(f_s)_n = \begin{cases} -n, & \text{if } f_s < -n, \\ f_s, & \text{if } |f_s| \leq n, \\ n, & \text{if } f_s > n, \end{cases}$$

and let $u_n : D \to \mathbb{R}$ be the minimal solution with boundary values

$$u_n\Big|_{A_i} = n, \quad u_n\Big|_{B_j} = -n, \quad u_n\Big|_{C_s} = (f_s)_n \ .$$

By the general maximum principle, one has

$$u^- \leq u_n \leq u^+ \text{ on } D.$$

Hence $\{u_n\}$ is uniformly bounded on compact sets of D, that is, there is a subsequence converging to a minimal solution u with the desired values on the boundary.

Case 5. ∂D contains only geodesic arcs $A_1, \ldots, A_k, B_1, \ldots, B_l$.

Proof of Case 5. Now ∂D is a geodesic polygon, thus $k = l$. For this polygon we have, by hypothesis, $\alpha = \beta$.

We need to construct some auxiliary sets and minimal solutions.

By Case 1, there exists a minimal solution $v_n : D \to \mathbb{R}$ such that

$$v_n\Big|_{A_i} = n, \quad v_n\Big|_{B_j} = 0.$$

For each $c \in (0, n)$, consider the following open subsets of D:

$$E_c = \{v_n > c\} \cap D, \quad F_c = \{v_n < c\} \cap D.$$

Let E_c^i be the component of E_c whose closure contains the edge A_i and let F_c^i be the component of F_c whose closure contains the edge B_j. By the maximum principle

$$E_c = \bigcup_{i=1}^{k} E_c^i \quad \text{and} \quad F_c = \bigcup_{i=1}^{k} F_c^i.$$

We choose c close enough to n such that the E_c^i are disjoint and we define

$$\mu(n) = \limsup \left\{c \in (0, n)\,;\ E_c^i \cap E_c^j = \emptyset,\ i \neq j\right\}.$$

There is at least one pair $i,\ j$ such that

$$\bar{E}^i_{\mu(n)} \cap \bar{E}^j_{\mu(n)} \neq \emptyset.$$

Then, for each i there exists a j such that $F^i_{\mu(n)} \cap F^j_{\mu(n)} = \emptyset$.

For each n, we define the following minimal solution on D:

$$u_n = v_n - \mu(n)\ .$$

In order to prove that the sequence $\{u_n\}$ is uniformly bounded on compact subsets of D, let us define two auxiliary minimal solutions on D. Let u_i^+ and u_i^- be the minimal solutions on D with the boundary values

$$u_i^+\Big|_{A_i} = \infty\ ,\quad u_i^+\Big|_{\partial D - A_i} = 0,$$

$$u_i^-\Big|_{B_j} = -\infty\ ,\ i \neq j\ ,\quad u_i^-\Big|_{\partial D - \bigcup_{j\neq i} B_j} = 0,$$

The existence of u_i^+ and u_i^-, for each $i \in \{1, \ldots, k\}$, is assured by previous cases.

Finally, for any $z \in D$ we define

$$u^+(z) = \max_{1\leq i\leq k} \left\{u_i^+(z)\right\},\ u^-(z) = \min_{1\leq i\leq k} \left\{u_i^-(z)\right\}.$$

At any point of D holds

$$u^- \leq\ u_n\ \leq u^+. \tag{11}$$

To prove this, first choose $p \in D$ such that $u_n(p) > 0$. Then p belongs to $E^i_{\mu(n)}$, for some i. As on $\partial E^i_{\mu(n)}$ one has $u_n \leq u_i^+$, then this inequality holds in $E^i_{\mu(n)}$ and

$$u_n(p) \leq u_i^+(p) \leq u^+(p).$$

The left inequality in (11) is obvious at the point p, since u^- is non positive.

The proof of (11) at points where u_n is negative is analogous, using the set $F^i_{\mu(n)}$.

Hence $\{u_n\}$ has a subsequence converging to a minimal solution $u\colon D \to \mathbb{R}$. Let us prove that u takes the right boundary values.

As we have

$$u_n\Big|_{A_i} = n - \mu(n) \quad \text{and} \quad u_n\Big|_{B_i} = -\mu(n), \tag{12}$$

we must prove that the sequences $\{n - \mu(n)\}$ and $\{\mu(n)\}$ both diverge to infinity. We prove it for the sequence $\{\mu(n)\}$; the proof for the other sequence is analogous.

By contradiction, take a subsequence such that $\mu(n)$ goes to a finite limit μ_0. Then, by (12),

$$u_n \overset{n\to\infty}{\longrightarrow} \infty \quad \text{on} \quad A_i,$$

and

$$u_n \overset{n\to\infty}{\longrightarrow} -\mu_0 \quad \text{on} \quad B_i.$$

So, for the limit function u we have

$$u\Big|_{A_i} = \infty, \quad u\Big|_{B_i} = -\mu_0.$$

Let $(\nu_3)_n$ be the unit inward conormal to the boundary of graph u. Using Lemma 5.2, we obtain

$$\begin{aligned}
\alpha = \lim_{n\to\infty} \int_{\bigcup_i A_i} (\nu_3)_n \, ds \quad &= \quad -\lim_{n\to\infty} \int_{\bigcup_i B_i} (\nu_3)_n \, ds \\
&\geq \quad -\lim_{n\to\infty} \int_{\bigcup_i B_i} |(\nu_3)_n| \, ds \\
&> \quad -\beta.
\end{aligned}$$

This is a contradiction with the hypothesis $\alpha = \beta$ and Case 5 is proved.

Now we prove that the condition (1) is necessary to the existence of the function u.

We fix the following notations: $\Sigma = \operatorname{graph} u$, $\Sigma_n = \Sigma \cap (M \times [-n, n])$, $D_n =$ is a vertical projection of Σ_n over D and $u_n = u_{|D_n}$.

Assertion 5.7. *When $n \to +\infty$, one has $u_n \to u$, $D_n \to D$ and $\Sigma_n \to \Sigma$ uniformly.*

For n large enough, the convex arcs C_s belongs to ∂D_n. The remaining part of ∂D_n is the union of non-geodesic arcs A_i^n, $B_j^n \subset D$ which endpoints approach to the endpoints of A_i, B_j, respectively, when n goes to $+\infty$.

Fixing $\delta > 0$ small, for each n, let $D_{n\delta} \subset D_n$ be a domain such that $\partial D_{n\delta}$ is the set of points in ∂D_n whose distance from the vertices of D_n is greater than δ and circular arcs with center in a vertex of D and radius δ. See Figure 8.

Denote by $A_i^{n\delta}$ the subarc of A_i^n contained in $\partial D_{n\delta}$.

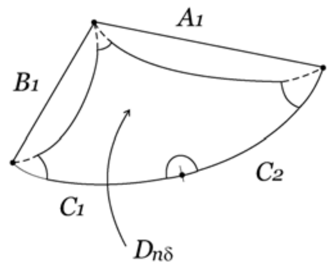

Figure 8 – Domain $D_{n\delta}$.

On every compact set $K \subset A_i^{n\delta}$, one has

$$A_i^{n\delta} \overset{n\to\infty}{\longrightarrow} A_i$$

because where p is near to A_i the tangent plane to Σ_n at points $z_n = (p, u_n(p))$, converges C^∞ to the tangent plane to $A_i \times \mathbb{R}$, when n goes to ∞. This last assertion is a consequence of Σ being a stable surface, i.e., Σ has bounded geometry. The same argument holds on the subarcs $B_j^{n\delta}$ of B_j contained on $\partial D_{n\delta}$, i.e., on every compact $K \subset B_j^{n\delta}$ one has

$$B_j^{n\delta} \overset{n\to\infty}{\longrightarrow} B_j.$$

Letting $\delta \to 0$,

$$u_n \to u, \quad D_n \to D \quad \text{and} \quad \Sigma_n \to \Sigma,$$

as we asserted.

Let $\mathcal{P} \subset D$ be an admissible polygon. Denote by $\hat{A}_i$, $\hat{B}_j$ the edges A_i, $B_j \subset \partial D$ which belong to $\mathcal{P}$.

It is clear that if $\{\hat{A}_i,\ \hat{B}_j\} = \emptyset$, (1) holds. Let us suppose that $\{\hat{A}_i\} \neq \emptyset$ and $\{\hat{B}_j\} = \emptyset$. The other cases are similar. Denote by $\mathcal{P}_n^\delta$ the curve constructed changing the sides $\hat{A}_i \subset \mathcal{P}$ by $\hat{A}_i^{n\delta}$ and putting circular arcs contained in $\partial D_{n\delta}$ in way that $\mathcal{P}_n$ is closed.

Denote by $\mathcal{P}^\delta$ the limit curve of $\mathcal{P}_n^\delta$, when $n \to \infty$.

By Assertion 5.3, one has

$$\int_{\mathcal{P}_n^\delta} (\nu_3)_n \, ds = 0,$$

where $(\nu_3)_n$ is the unit exterior conormal to the boundary of the graph of u restricted to the domain bounded by $\mathcal{P}_n^\delta$.

This is equivalent to

$$\int_{\bigcup_i \hat{A}_i^{n\delta}} (\nu_3)_n \, ds \quad = \quad -\int_{\mathcal{P}_n^\delta - \bigcup_i \hat{A}_i^{n\delta}} (\nu_3)_n \, ds. \tag{13}$$

When n goes to ∞, using Assertion 5.7, we have that the limit of the first integral is $\alpha - (2\delta)i$ and the limit of the second one is smaller than $\|\mathcal{P}^\delta\| - [\alpha - (2\delta)i]$. Letting δ goes to 0, this implies that $2\alpha < \gamma$.

Thus the existence of u implies that (1) holds.

To finish the proof of Theorem 1.1, we need to prove the uniqueness of the solution. Consider u_1 and u_2 two different minimal solutions assuming values $+\infty$ on each A_i, $-\infty$ on each B_j, and the same continuous data on each convex arc C_s. If $\{C_s\} = \emptyset$, suppose $\phi := u_1 - u_2$ is not constant.

First we suppose that $\{p \in D, u_1(p) < u_2(p)\}$ and $\{p \in D, u_1(p) > u_2(p)\}$ are not empty. Let $\epsilon > 0$ be sufficiently small such that $D_\epsilon = \{\phi(p) > \epsilon\} \neq \emptyset$ and ∂D_ϵ is regular.

A similar argument used in the proof of the general maximum principle works here.

In fact, as u_1 and u_2 are minimal solutions, one has

$$\int_{\partial D_\epsilon} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle \quad = \quad 0, \tag{14}$$

where ν is the outward conormal to ∂D_ϵ.

On the other hand, as $\phi = 0$ on $\{C_s\}$, ∂D_ϵ is composed of three parts. The first one is included in D, where $\nabla\phi \neq 0$, by hypothesis. Then, by Assertion 3.1,

$$\left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle$$

is no zero and it does not change the sign, so the integral in (14) is no zero on this part.

The second part is included in $\cup_{ij}\{A_i,\ B_j\}$. Now ν is the horizontal outward unit conormal to $\partial(D_\epsilon \times \{t\})$ and

$$N_n = \left(-\frac{\nabla u_n}{W_n}, \frac{1}{W_n}\right), \quad n = 1, 2,$$

is the unit normal vector to the graph of u_n. So we have that

$$\langle N_n, \nu\rangle = \left\langle -\frac{\nabla u_n}{W_n}, \nu \right\rangle$$

on $\partial(D_\epsilon \times \{t\})$, for each t. But, on each horizontal arc $A_i \times \{t\}$, one has

$$\langle N_n, \nu \rangle = \left\langle \nu_n, \frac{\partial}{\partial t} \right\rangle,$$

where ν_n is the outward conormal vector of $\partial(\operatorname{graph} u_n)$, $n = 1, 2$. Now, the Lemma 5.2 implies that the integral in (14) is zero on $\cup_{ij}\{A_i,\ B_j\}$.

The remaining part of ∂D_ϵ is composed of some vertices of ∂D; and its contribution to the integral is zero.

So we have

$$\int_{\partial D_\epsilon} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle \neq 0,$$

a contradiction.

If either $\{p \in D, u_1(p) < u_2(p)\}$ or $\{p \in D, u_1(p) > u_2(p)\}$ are empty, we translate vertically the graph of u so that the set $\tilde{\mathcal{U}} = \{p \in D; \phi(p) = 0\}$ is no empty and $\partial\tilde{\mathcal{U}}$ is regular. Now $\partial\tilde{\mathcal{U}} \cap \{C_s\} = \emptyset$ and the above argument works on $\partial\tilde{\mathcal{U}}$. $\square$

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Ana Lucia Pinheiro
Departamento de Matemática
Universidade Federal da Bahia
Av. Ademar de Barros s/n
40170-110 Salvador, BA
BRASIL

E-mail: anapinhe@ufba.br